The universe evolution and modified gravity: an overview

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Motivation

1. Quantum field theory calculations lead to higher-derivative corrections.
2. QFT in curved spacetime changes gravity at the early-epoch.
3. Then, why GR not any other theory?
4. Solar system tests maybe passed by number of modified gravity.

Fundamental approach

The choice of variables

- Fundamental approach: the choice of variables.
- Metric description.
- Palatini description.
- Pure connections choice as fundamental variables.
- Or some variables/structures of theory are hidden?
- Say, non-metricity. Or torsion should be added. (Gauge approach to gravity, recent F(T) gravity).
- Or still we are missing the most convenient choice of variables?

The choice of theory

- Eventually, non-linear and higher-derivative one.
- Local or non-local?

The universe evolution

1. Early-time inflation. R^2 (Starobinsky) inflation.
2. Dark energy from modified gravity.Dark energy from power-law F(R) gravity:S. Capozziello, Curvature quintessence, Int. J. Mod. Phys. D 11 (2002) 483.
3. Unification of inflation with late-time acceleration. First example:S. Nojiri and S. D. Odintsov, Modied gravity with negative and positive powers of the curvature: Unication of the inflation and of the cosmic acceleration, Phys. Rev. D 68 (2003) 123512,[hep-th/0307288]. No extra scalars, fluids, etc.
4. The better unification: inflation, radiation/matter dominance and dark energy. Example: S. Nojiri and S. D. Odintsov, Modied f(R) gravity consistent with realistic cosmology: From matter dominated epoch to dark energy universe, Phys. Rev. D 74 (2006) 086005,[hep-th/0608008]. One can include QG effects!
5. The possibility to include DM. Example: S. Capozziello, V. F. Cardone and A. -Troisi, Low surface brightness galaxies rotation curves in the low energy limit of r**n gravity: no need for dark matter?, Mon. Not. Roy. Astron. Soc. 375, 1423 (2007); S. Nojiri and S. D. Odintsov, Dark energy, in ation and dark matter from modied F(R) gravity, TSPU Bulletin N 8(110) (2011) 7 [arXiv:0807.0685 [hep-th]].

Proposals for modified gravity

1. Modified F(G) gravity:S. Nojiri and S. D. Odintsov, Modied Gauss-Bonnet theory as gravitational alternative for dark energy, Phys. Lett. B 631 (2005) 1; [hep-th/0508049].
2. string-inspired Gauss-Bonnet gravity with possibility to unify inflation with DE: S. Nojiri, S. D. Odintsov and M. Sasaki, Gauss-Bonnet dark energy, Phys. Rev. D 71 (2005) 123509,[hep-th/0504052].
3. Realistic non-local gravity:S. Deser and R. P. Woodard, Nonlocal Cosmology, Phys. Rev. Lett. 99 (2007) 111301
4. Born-Infeld gravity as BI electrodynamics.
5. non-minimal gravity:non-minimal scalar-curvature coupling which is predicted by renormalizability or F(R,T):T. Harko, F. S. N. Lobo, S. Nojiri and S. D. Odintsov, f (R;T) gravity, Phys. Rev. D 84 (2011) 024020 or coupling of matter lagrangian with gravity: S. Nojiri and S. D. Odintsov, Gravity assisted dark energy dominance and cosmic acceleration, Phys. Lett. B 599 (2004) 137,astro-ph/0403622.
6. Gravity with torsion, massive gravity, HL gravity

Possible unification with GUTs (example of HD gravity)

- Consistent gravitational physics in Solar System.
- Applications: relativistic stars at strong gravitational regime,wormholes without phantoms, new black holes thermodynamics (negative entropy due to HD terms?).
- Fresh review: Nojiri, S. D. Odintsov and V. K. Oikonomou, Modied Gravity Theories on a Nutshell: Inflation, Bounce and Late-time Evolution, arXiv:1705.11098 [gr-qc], Phys.Repts.2018.

Overview of modified gravity and FRW cosmology.

The action:

$$S = \int d^4x \sqrt{-g} \left[\frac{F(R)}{2\kappa^2} + \mathcal{L}^{(\mathrm{matter})} \right] , \qquad (1)$$

where g is the determinant of the metric tensor $g_{\mu\nu}$, $\mathcal{L}^{(\mathrm{matter})}$ is the matter Lagrangian and $F(R)$ a generic function of the Ricci scalar, R.
We shall write

$$F(R) = R + f(R) \,. \qquad (2)$$

Field eqs:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa^2 \left(T^{\mathrm{MG}}_{\mu\nu} + \tilde{T}^{(\mathrm{matter})}_{\mu\nu} \right) \,. \qquad (3)$$

Here, $R_{\mu\nu}$ is the Ricci tensor and the part of modified gravity is formally included into the 'modified gravity' stress-energy tensor $T^{\mathrm{MG}}_{\mu\nu}$, given by

$$T^{\mathrm{MG}}_{\mu\nu} = \frac{1}{\kappa^2 F'(R)} \left\{ \frac{1}{2} g_{\mu\nu} [F(R) - RF'(R)] + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F'(R) \right\} \,. \qquad (4)$$

$\tilde{T}^{(\mathrm{matter})}_{\mu\nu}$ is given by the non-minimal coupling of the ordinary matter stress-energy tensor $T^{(\mathrm{matter})}_{\mu\nu}$ with geometry, namely,

$$\tilde{T}^{(\mathrm{matter})}_{\mu\nu} = \frac{1}{F'(R)} T^{(\mathrm{matter})}_{\mu\nu} \,. \qquad (5)$$

Overview of modified gravity and FRW cosmology.

The trace of Eq. (3) reads

$$3\Box F'(R) + RF'(R) - 2F(R) = \kappa^2 T^{(\mathrm{matter})}\,, \tag{6}$$

with $T^{(\mathrm{matter})}$ the trace of the matter stress-energy tensor. We can rewrite this equation as

$$\Box F'(R) = \frac{\partial V_{\mathrm{eff}}}{\partial F'(R)}\,, \tag{7}$$

where

$$\frac{\partial V_{\mathrm{eff}}}{\partial F'(R)} = \frac{1}{3}\left[2F(R) - RF'(R) + \kappa^2 T^{(\mathrm{matter})}\right]\,, \tag{8}$$

$F'(R)$ being the so-called 'scalaron' or the effective scalar degree of freedom. On the critical points of the theory, the effective potential V_{eff} has a maximum (or minimum), so that

$$\Box F'(R_{\mathrm{CP}}) = 0\,, \tag{9}$$

and

$$2F(R_{\mathrm{CP}}) - R_{\mathrm{CP}} F'(R_{\mathrm{CP}}) = -\kappa^2 T^{(\mathrm{matter})}\,. \tag{10}$$

For example, in absence of matter, i.e. $T^{(\mathrm{matter})} = 0$, one has the de Sitter critical point associated with a constant scalar curvature R_{dS}, such that

$$2F(R_{\mathrm{dS}}) - R_{\mathrm{dS}} F'(R_{\mathrm{dS}}) = 0\,. \tag{11}$$

Performing the variation of Eq. (6) with respect to R, by evaluating $\Box F'(R)$ as

$$\Box F'(R) = F''(R)\Box R + F'''\nabla^{\mu} R \nabla_{\nu} R \,, \tag{12}$$

we find, to first order in δR,

$$\begin{aligned} &\Box R + \frac{F'''(R)}{F''(R)} g^{\mu\nu}\nabla_{\mu} R \nabla_{\nu} R - \frac{1}{3F''(R)}\left[2F(R) - RF'(R) + \kappa^2 T^{\mathrm{matter}}\right] \\ &+\Box\delta R + \left\{\left[\frac{F'''(R)}{F''(R)} - \left(\frac{F'''(R)}{F''(R)}\right)^2\right] g^{\mu\nu}\nabla_{\mu} R \nabla_{\nu} R + \frac{R}{3} - \frac{F'(R)}{3F''(R)}\right. \\ &\left. + \frac{F'''(R)}{3(F''(R))^2}\left[2F(R) - RF'(R) + \kappa^2 T^{\mathrm{matter}}\right] - \frac{\kappa^2}{3F''(R)} \frac{dT^{\mathrm{matter}}}{dR}\right\}\delta R \\ &+ 2\frac{F'''(R)}{F''(R)} g^{\mu\nu}\nabla_{\mu} R \nabla_{\nu}\delta R + \mathcal{O}(\delta R^2) \simeq 0 \,. \end{aligned} \tag{13}$$

This equation can be used to study perturbations around critical points. By assuming $R = R_0 \simeq \mathrm{const}$ (local approximation), and $\delta R / R_0 \ll 1$, we get

$$\Box\delta R \simeq m^2 \delta R + \mathcal{O}(\delta R^2) \,, \tag{14}$$

where

$$m^2 = \frac{1}{3}\left[\frac{F'(R_0)}{F''(R_0)} - R_0 + \frac{\kappa^2}{F''(R_0)} \frac{dT^{\mathrm{matter}}}{dR}\Big|_{R_0}\right] \,. \tag{15}$$

Note that

$$m^2 = \frac{\partial^2 V_{\rm eff}}{\partial F'(R)^2}\Big|_{R_0} . \quad (16)$$

The second derivative of the effective potential represents the effective mass of the scalaron. Thus, if $m^2 > 0$ one gets a stable solution. For the case of the de Sitter solution, m^2 is positive provided

$$\frac{F'(R_{\rm dS})}{R_{\rm dS}F''(R_{\rm dS})} > 1 . \quad (17)$$

Modified FRW dynamics.

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 , \quad (18)$$

where $a(t)$ is the scale factor of the universe. In the FRW background, from $(\mu, \nu) = (0,0)$ and the trace part of the $(\mu, \nu) = (i,j)$ $(i,j = 1, \ldots, 3)$ components in Eq. (3), we obtain the equations of motion:

$$\rho_{\rm eff} = \frac{3}{\kappa^2}H^2 , \quad (19)$$

$$p_{\rm eff} = -\frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) , \quad (20)$$

where $\rho_{\rm eff}$ and $p_{\rm eff}$ are the total effective energy density and pressure of matter and geometry, respectively,

$$\rho_{\rm eff} = \frac{1}{F'(R)}\left\{\rho + \frac{1}{2\kappa^2}\left[(F'(R)R - F(R)) - 6H\dot{F}'(R)\right]\right\} , \quad (21)$$

$$p_{\rm eff} = \frac{1}{F'(R)}\left\{p + \frac{1}{2\kappa^2}\left[-(F'(R)R - F(R)) + 4H\dot{F}'(R) + 2\ddot{F}'(R)\right]\right\} . \quad (22)$$

The standard matter conservation law is

$$\dot{\rho} + 3H(\rho + p) = 0 . \quad (23)$$

For a perfect fluid,

$$p = \omega\rho , \quad (24)$$

ω being the thermodynamical EoS-parameter of matter.

The standard matter conservation law is

$$\dot{\rho} + 3H(\rho + p) = 0\,. \tag{25}$$

For a perfect fluid,

$$p = \omega\rho\,, \tag{26}$$

ω being the thermodynamical EoS-parameter of matter. We also introduce the effective EoS by using the corresponding parameter $\omega_{\rm eff}$

$$\omega_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}}\,, \tag{27}$$

and get

$$\omega_{\rm eff} = -1 - \frac{2\dot{H}}{3H^2}\,. \tag{28}$$

If the strong energy condition (SEC) is satisfied ($\omega_{\rm eff} > -1/3$), the universe expands in a decelerated way, and vice-versa. Viability: Minkowski solution, observable cosmology, positive grav. constant. Local tests:spherical body solution,correct newtonian limit.

One can rewrite $F(R)$ gravity as the scalar-tensor theory. By introducing the auxiliary field A, the action (**??**) of the $F(R)$ gravity is rewritten in the following form:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{F'(A)\left(R-A\right)+F(A)\right\} \ . \tag{29}$$

By the variation of A, one obtains $A = R$. Substituting $A = R$ into the action (29), one can reproduce the action in (**??**). Furthermore, by rescaling the metric as $g_{\mu\nu} \to \mathrm{e}^{\sigma} g_{\mu\nu}$ $\left(\sigma = -\ln F'(A)\right)$, we obtain the Einstein frame action:

$$\begin{aligned} S_E =& \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma)\right) \ , \\ V(\sigma) =& \mathrm{e}^{\sigma} g\left(\mathrm{e}^{-\sigma}\right) - \mathrm{e}^{2\sigma} f\left(g\left(\mathrm{e}^{-\sigma}\right)\right) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \ . \end{aligned} \tag{30}$$

Here $g\left(\mathrm{e}^{-\sigma}\right)$ is given by solving the equation $\sigma = -\ln\left(1+f'(A)\right) = -\ln F'(A)$ as $A = g\left(\mathrm{e}^{-\sigma}\right)$. Due to the conformal transformation, a coupling of the scalar field σ with usual matter arises. Since the mass of σ is given by

$$m_\sigma^2 \equiv \frac{3}{2}\frac{d^2V(\sigma)}{d\sigma^2} = \frac{3}{2}\left\{\frac{A}{F'(A)} - \frac{4F(A)}{\left(F'(A)\right)^2} + \frac{1}{F''(A)}\right\} \ , \tag{31}$$

unless m_σ is very large, the large correction to the Newton law appears.

A natural possibility is

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{-\frac{R}{R_0}}\right) - \Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right) + \gamma R^\alpha\,. \tag{32}$$

For simplicity, we call

$$f_i = -\Lambda_i\left(1 - \mathrm{e}^{-\left(\frac{R}{R_i}\right)^n}\right)\,, \tag{33}$$

where R_i and Λ_i assume the typical values of the curvature and expected cosmological constant during inflation, namely $R_i,\ \Lambda_i \simeq 10^{20-38}\mathrm{eV}^2$, while n is a natural number larger than one. The presence of this additional parameter is motivated by the necessity to avoid the effects of inflation during the matter era, when $R \ll R_i$, so that, for $n > 1$, one gets

$$R \gg |f_i(R)| \simeq \frac{R^n}{R_i^{n-1}}\,. \tag{34}$$

The last term in Eq. (32), namely γR^α, where γ is a positive dimensional constant and α a real number, is necessary to obtain the exit from inflation. If $\gamma \sim 1/R_i^{\alpha-1}$ and $\alpha > 1$, the effects of this term vanish in the small curvature regime.

By taking into account the viability conditions the simplest choice of parameters to introduce in the function of Eq. (32) is:

$$n = 4 \,, \qquad \alpha = \frac{5}{2} \,, \tag{35}$$

while the curvature R_i is set as

$$R_i = 2\Lambda_i \,. \tag{36}$$

In this way, $n > \alpha$ and we avoid undesirable instability effects in the small-curvature regime. ..also no anti-gravity effects. From Eq. (**??**) one recovers the unstable de Sitter solution describing inflation as

$$R_{\mathrm{dS}} = 4\Lambda_i \,. \tag{37}$$

We note that, due to the large value of n, R_{dS} is sufficiently large with respect to R_i, and $f_i(R_{\mathrm{dS}}) \simeq -\Lambda_i$. One can also expect that, on top of this graceful exit from inflation, the effective scalar degree of freedom may also give rise to reheating.
Efective energy density $\rho_{\mathrm{DE}} = \rho_{\mathrm{eff}} - \rho/F'(R)$ in the case of the of Eq. (32), near the late-time acceleration era describing current universe.
The variable

$$y_H \equiv \frac{\rho_{\mathrm{DE}}}{\rho_m^{(0)}} = \frac{H^2}{\tilde{m}^2} - a^{-3} - \chi a^{-4} \,. \tag{38}$$

Here, $\rho_m^{(0)}$ is the energy density of matter at present time, $\tilde{m}^2$ is the mass scale

$$\tilde{m}^2 \equiv \frac{\kappa^2 \rho_m^{(0)}}{3} \simeq 1.5 \times 10^{-67} \mathrm{eV}^2 \,, \tag{39}$$

and χ is defined as

$$\chi \equiv \frac{\rho_r^{(0)}}{\rho_m^{(0)}} \simeq 3.1 \times 10^{-4} \,, \tag{40}$$

where $\rho_r^{(0)}$ is the energy density of radiation at present (the contribution from radiation is also taken into consideration).

The EoS-parameter $\omega_{\rm DE}$ for dark energy is

$$\omega_{\rm DE} = -1 - \frac{1}{3}\frac{1}{y_H}\frac{dy_H}{d(\ln a)} \,. \tag{41}$$

By combining Eq. (19) with Eq. (**??**) and using Eq. (195), one gets

$$\frac{d^2 y_H}{d(\ln a)^2} + J_1 \frac{dy_H}{d(\ln a)} + J_2 y_H + J_3 = 0 \,, \tag{42}$$

where

$$J_1 = 4 + \frac{1}{y_H + a^{-3} + \chi a^{-4}}\frac{1-F'(R)}{6\tilde{m}^2 F''(R)} \,, \tag{43}$$

$$J_2 = \frac{1}{y_H + a^{-3} + \chi a^{-4}}\frac{2-F'(R)}{3\tilde{m}^2 F''(R)} \,, \tag{44}$$

$$J_3 = -3a^{-3} - \frac{(1-F'(R))(a^{-3}+2\chi a^{-4}) + (R-F(R))/(3\tilde{m}^2)}{y_H + a^{-3} + \chi a^{-4}}\frac{1}{6\tilde{m}^2 F''(R)} \,, \tag{45}$$

and thus, we have

$$R = 3\tilde{m}^2 \left(\frac{dy_H}{d\ln a} + 4y_H + a^{-3} \right) \,. \tag{46}$$

The parameters of Eq. (32) are chosen as follows:

$$\begin{gathered} \Lambda = (7.93)\tilde{m}^2 \,, \\ \Lambda_i = 10^{100}\Lambda \,, \\ R_i = 2\Lambda_i \,, \qquad n = 4 \,, \\ \alpha = \frac{5}{2} \,, \qquad \gamma = \frac{1}{(4\Lambda_i)^{\alpha-1}} \,, \\ R_0 = 0.6\Lambda \,, \quad 0.8\Lambda \,, \quad \Lambda \,. \end{gathered} \tag{47}$$

Eq. (198) can be solved in a numerical way, in the range of $R_0 \ll R \ll R_i$ (matter era/current acceleration). y_H is then found as a function of the red shift z,

$$z = \frac{1}{a} - 1\,. \tag{48}$$

In solving Eq. (198) numerically, we have taken the following initial conditions at $z = z_i$

$$\frac{dy_H}{d(z)}\Big|_{z_i} = 0\,,$$

$$y_H\Big|_{z_i} = \frac{\Lambda}{3\tilde{m}^2}\,, \tag{49}$$

which correspond to the ones of the ΛCDM model. This choice obeys to the fact that in the high red shift regime the exponential model is very close to the ΛCDM Model. The values of z_i have been chosen so that $RF''(z = z_i) \sim 10^{-5}$, assuming $R = 3\tilde{m}^2(z+1)^3$. We have $z_i = 1.5,\ 2.2,\ 2.5$ for $R_0 = 0.6\Lambda,\ 0.8\Lambda,\ \Lambda$, respectively. In setting the parameters, we have used the last results of the WMAP, BAO and SN surveys.

Using Eq. (41), one derives $\omega_{\rm DE}$ from y_H. In the present universe ($z = 0$), one has $\omega_{\rm DE} = -0.994,\ -0.975,\ -0.950$ for $R_0 = 0.6\Lambda,\ 0.8\Lambda,\ \Lambda$. The smaller R_0 is, our model becomes more indistinguishable from the ΛCDM model, where $\omega_{\rm DE} = -1$.

A viable exponential $F(R)$ model

S. D. Odintsov, D. Sáez-Gómez and G. S. Sharov, Eur. Phys. J. C. **77** (2017) 862, arXiv:1709.06800
with the action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g}\, F(R) + S^m,$$

where

$$F(R) = R - 2\Lambda \left[1 - \exp\left(-\beta \frac{R}{2\Lambda} \right) \right] - \Lambda_i \left[1 - \exp\left(-\left(\frac{R}{R_i} \right)^n \right) \right] + \gamma R^\alpha. \tag{50}$$

reproduces early time inflation and late-time acceleration in concordance with observational constraints.

A viable exponential $F(R)$ model: Inflation

The (last) inflationary terms support the slow-roll inflation scenario at early times:

$$R > R_i , \qquad R_i / \Lambda = 10^{86} - 10^{104} . \tag{51}$$

Under the conditions

$$2 < \alpha < 3 , \qquad n > \alpha , \qquad R_i = 2\Lambda_i , \qquad \gamma \simeq \Lambda_i^{1-\alpha} . \tag{52}$$

- at early times (51) an unstable (inflationary) de Sitter point $R = R_{dS}$ arises under the equality $G(R_{dS}) = 0$ $\left(\text{here } G = 2F(R) - RF_R\right)$ or

$$R_{dS} - (\alpha - 2)\gamma R_{dS}^{\alpha} - 2\Lambda_i = 0 ;$$

- a successful exit from inflation appears;
- we avoid the effects of inflation during the matter era when $R \ll R_i$ (the inflationary terms become negligible);
- we avoid anti-gravity effects and instabilities during the matter era.

We express the action via an additional scalar mode ϕ

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,[\phi R - V(\phi)] + S^m\,, \quad \text{where} \quad \phi = F_R, \quad V(\phi) = RF_R - F\,,$$

conformally transform it into the Einstein frame $\tilde{g}_{\mu\nu} = \phi \cdot g_{\mu\nu}$ and redefine

$$\phi = \mathrm{e}^{\sqrt{\frac{2}{3}}\kappa\tilde{\phi}}\,, \qquad V = 2\kappa^2\phi^2 \cdot \tilde{V}\,.$$

The calculated slow-roll parameters ϵ, η, the spectral index of the perturbations n_s and the tensor-to-scalar ratio r,

$$\epsilon = \frac{1}{2\kappa^2}\left(\frac{\tilde{V}'(\tilde{\phi})}{\tilde{V}(\tilde{\phi})}\right)^2, \quad \eta = \frac{1}{\kappa^2}\frac{\tilde{V}''(\tilde{\phi})}{\tilde{V}(\tilde{\phi})}, \qquad n_\mathrm{s} - 1 = -6\epsilon + 2\eta\,, \quad r = 16\epsilon$$

under the conditions (52) obey the Planck and Bicep2 constraints

$$n_\mathrm{s} = 0.968 \pm 0.006\,, \qquad r < 0.07\,.$$

The corresponding number of e-folds $N \simeq 58$ lies in the range $55 \leq N \leq 65$.

A viable exponential $F(R)$ model: Late-time acceleration and observations

At the late-time epoch ($R \ll R_i$ and $z < 10^4$) the inflationary terms are negligible and the Lagrangian (50) becomes

$$F(R) = R - 2\Lambda\Big[1 - \exp\Big(-\beta\frac{R}{2\Lambda}\Big)\Big]. \qquad (53)$$

The dynamical equations

$$F_R R_{\mu\nu} - \frac{F}{2}g_{\mu\nu} + \big(g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\nabla_\beta - \nabla_\mu\nabla_\nu\big)F_R = \kappa^2 T_{\mu\nu}$$

in the flat FLRW space-time with the metric $ds^2 = -dt^2 + a^2(t)\,d\mathbf{x}^2$ are reduced to the system for the Ricci scalar R and the Hubble parameter $H = \dot{a}/a$:

$$\begin{aligned} \frac{dH}{dN} &= \frac{R}{6H} - 2H, \qquad (N = \log a) & (54)\\ \frac{dR}{dN} &= \frac{1}{F_{RR}}\Big(\frac{\kappa^2\rho}{3H^2} - F_R + \frac{RF_R - F}{6H^2}\Big), & (55)\\ \rho &= \rho_m^0 a^{-3} + \rho_r^0 a^{-4} = \rho_m^0\big(a^{-3} + X^* a^{-4}\big). \end{aligned}$$

During the early universe (for $z \geq 10^4$ in practice) when curvature R is large, the model (53) transforms into the ΛCDM model with $F(R) = R - 2\Lambda$ and its viable solutions tend asymptotically to ΛCDM solutions with parameters

$$H_0^* \equiv H_0^{\Lambda CDM}, \qquad \Omega_m^* \equiv \Omega_m^{\Lambda CDM}, \qquad \Omega_\Lambda^* \equiv \Omega_\Lambda^{\Lambda CDM}. \qquad (56)$$

Starting from the ΛCDM asymptotical behaviour at $a < 10^{-4}$ we integrate the system (54), (55) and compare its solutions at the matter-dominated epoch $z \leq 10^3$ (for 4 free parameters of the model β, Ω_m^*, Ω_Λ^*, H_0^*) with the available observational constraints.

A viable exponential $F(R)$ model: Late-time acceleration and observations

The observational constraints include:

- The Union 2.1 Supernovae Ia data with $N_{SN} = 580$ data points (the observed SNe Ia distance moduli μ_i^{obs} for redshifts z_i at $0 \le z_i \le 1.41$). We compare μ_i^{obs} with $\mu^{th}(z_i)$ and calculate the χ^2 function:

$$\mu^{th}(z) = 5\log_{10}\frac{D_L(z)}{10\text{pc}}, \qquad D_L(z) = (1+z)D_M(z), \qquad D_M(z) = c\int_0^z \frac{d\tilde{z}}{H(\tilde{z})}$$

$$\chi^2_{SN}(\beta, \Omega_m^*, \Omega_\Lambda^*) = \min_{H_0^*} \sum_{i,j=1}^{N_{SN}} \Delta\mu_i \left(C_{SN}^{-1}\right)_{ij} \Delta\mu_j, \qquad \Delta\mu_i = \mu^{th}(z_i) - \mu_i^{obs}.$$

- Baryon acoustic oscillations (BAO) data include 17 data points for $d_z(z) = r_s(z_d)/D_V(z)$ and 7 data points for $A(z) = H_0\sqrt{\Omega_m^0}D_V(z)/(cz)$, where $r_s(z_d)$ is the sound horizon scale at the end of the baryon drag epoch,

$$D_V(z) = \left[czD_M^2(z)/H(z)\right]^{1/3}.$$

- We use $N_H = 30$ values $H(z_i)$ estimated from differential ages of galaxies and $\chi^2_H = \min_{H_0} \sum_{i=1}^{N_H} \left[\frac{H^{obs}(z_i) - H^{th}(z_i, p_j)}{\sigma_{H,i}}\right]^2$.

- The CMB parameters $\mathbf{x} = (R, \ell_A, \omega_b) = \left(\sqrt{\Omega_m^0}\frac{H_0 D_M(z_*)}{c}, \frac{\pi D_M(z_*)}{r_s(z_*)}, \Omega_b^0 h^2\right)$ are compared with the estimations from Ref. Q.-G. Huang, K. Wang, S. Wang, JCAP, 1512 (2015) 022:

$$R^{Pl} = 1.7448 \pm 0.0054, \quad \ell_A^{Pl} = 301.46 \pm 0.094, \quad \omega_b^{Pl} = 0.0224 \pm 0.00017.$$

For the $F(R)$ model (53) we calculated the optimal values, 1σ errors for the model parameters and $\min\chi^2$, which are compared in Table 1 with the predictions of the ΛCDM model.

Model	data	Ω_m^*	Ω_Λ^*	β	$\min\chi^2/d.o.f$
$F(R)$	SNe+BAO+$H(z)$	$0.282^{+0.010}_{-0.009}$	$0.696^{+0.025}_{-0.037}$	$3.36^{+\infty}_{-2.16}$	572.07 / 631
$F(R)$	SNe+BAO+$H(z)$+CMB	$0.280^{+0.001}_{-0.001}$	$0.637^{+0.047}_{-0.062}$	$2.38^{+\infty}_{-0.80}$	575.51 / 634
ΛCDM	SNe+BAO+$H(z)$	$0.282^{+0.010}_{-0.009}$	$0.718^{+0.009}_{-0.010}$	∞	572.93 / 633
ΛCDM	SNe+BAO+$H(z)$+CMB	$0.2772^{+0.0003}_{-0.0004}$	$0.7228^{+0.0004}_{-0.0003}$	∞	583.24 / 636

Table: Predictions of the exponential $F(R)$ model (53) and the ΛCDM for different data sets.

One may conclude that the considered exponential $F(R)$ model with the full Lagrangian (50) is capable to provides the right predictions for the inflationary epoch and for late-time acceleration in such a way that no other fields are required. The model satisfies the observational constraints, demonstrates better results in $\min\chi^2$ than the ΛCDM model, but it has the extra parameter β. Thus, the statistical difference between the $F(R)$ model (53) and the ΛCDM model is not significant.

Ghost-free Generalized Lagrange Multiplier $F(R)$ gravity

The action

$$S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\left\{F(R) + \lambda\left(\partial_\mu\phi\partial^\mu\phi + G(R)\right)\right\} , \tag{57}$$

where $G(R)$ is an differentiable function of the scalar curvature R. We rewrite the action (57) as follows,

$$S = \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\left\{\left(F'(A) + \lambda G'(A)\right)(R - A) + F(A) + \lambda\left(\partial_\mu\phi\partial^\mu\phi + G(A)\right)\right\} . \tag{58}$$

By using the following conformal transformation,

$$g_{\mu\nu} \to \mathrm{e}^\sigma g_{\mu\nu} , \quad \sigma = -\ln\left(F'(A) + \lambda G'(A)\right) , \tag{59}$$

we obtain the following Einstein frame action,

$$\begin{aligned} S_E =& \frac{1}{2\kappa^2} \int d^4x\sqrt{-g}\left\{R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma) + \lambda\left(\mathrm{e}^\sigma\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\right)\right\} , \\ V(\sigma) =& \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} . \end{aligned} \tag{60}$$

By using the second equation in (59), we may eliminate the function λ as long as the condition $G'(A) \neq 0$ holds true, and we obtain,

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(A,\sigma) + \frac{\mathrm{e}^{-\sigma} - F'(A)}{G'(A)}\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\right)\right.$$
$$V(A,\sigma) = A\mathrm{e}^{\sigma} - F(A)\mathrm{e}^{2\sigma}\,. \qquad (61)$$

We should note that the model (**??**) corresponds to $G(A) = 1$ and therefore $G'(A) = 0$. By using the equation obtained when the action is varied with respect to A,

$$0 = \left(-\frac{F''(A)}{G'(A)} - \frac{\left(\mathrm{e}^{-\sigma} - F'(A)\right)G''(A)}{G'(A)^2}\right)\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\right)\,, \qquad (62)$$

we can find the function A as a function of σ and $\partial_\mu\phi\partial^\mu\phi$ as $A = A(\sigma, \partial_\mu\phi\partial^\mu\phi)$. In effect, Eq. (61) can be written as follows,

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left[R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V\left(A\left(\sigma,\partial_\mu\phi\partial^\mu\phi\right),\sigma\right)\right.$$
$$\left. + \frac{\mathrm{e}^{-\sigma} - F'\left(A\left(\sigma,\partial_\mu\phi\partial^\mu\phi\right)\right)}{G'\left(A\left(\sigma,\partial_\mu\phi\partial^\mu\phi\right)\right)}\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G\left(A\left(\sigma,\partial_\mu\phi\partial^\mu\phi\right)\right)\right)\right]\,. \qquad (63)$$

Although it is difficult to find the explicit form of $A\left(\sigma,\partial_\mu\phi\partial^\mu\phi\right)$, the action (63) does not include any higher derivative terms. Therefore the model of Eq. (57) is ghost free.

If $-\frac{F''(A)}{G'(A)} - \frac{\left(\mathrm{e}^{-\sigma} - F'(A)\right)G''(A)}{G'(A)^2} \neq 0$, Eq. (62) gives,

$$0 = \mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\,, \tag{64}$$

which is nothing but the constraint equation in the Einstein frame given by the variation of λ in the Jordan frame action (58). The variations of the action (61) with respect to σ, ϕ, and $g_{\mu\nu}$ give

$$0 = \frac{3}{2}\nabla_\mu\nabla^\mu\sigma - A\mathrm{e}^{\sigma} + 2F(A)\mathrm{e}^{2\sigma} - \frac{\mathrm{e}^{-\sigma}}{G'(A)}\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\right) + \frac{\mathrm{e}^{-\sigma} - F'(A)}{G'(A)}\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + 2\right. \tag{65}$$

$$0 = \nabla^\mu\left(\frac{\mathrm{e}^{-\sigma} - F'(A)}{G'(A)}\mathrm{e}^{\sigma}\partial_\mu\phi\right)\,, \tag{66}$$

$$\begin{aligned}0 = & - R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}R + \frac{1}{2}\left\{-\frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(A,\sigma) + \frac{\mathrm{e}^{-\sigma} - F'(A)}{G'(A)}\left(\mathrm{e}^{\sigma}\partial_\mu\phi\partial^\mu\phi + \mathrm{e}^{2\sigma}G(A)\right)\right\} g_{\mu} \\ & + \frac{3}{2}\partial_\mu\sigma\partial_\nu\sigma - \frac{\left(\mathrm{e}^{-\sigma} - F'(A)\right)\mathrm{e}^{\sigma}}{G'(A)}\partial_\mu\phi\partial_\nu\phi\,, \end{aligned} \tag{67}$$

(68)

We now consider the condition that the flat Minkowski space-time becomes a solution. Because A is nothing but the scalar curvature in the Jordan frame, we require $A = 0$ and we also assume that σ is a constant and ϕ only depends on time t,

$$A = 0 \,, \quad \sigma = \sigma_0 \,, \quad \phi = \phi(t) \,. \tag{69}$$

Then Eq. (66) is trivially satisfied and Eqs. (64), (65), and (67) reduce to the following forms,

$$0 = - \mathrm{e}^{\sigma_0} \dot\phi^2 + \mathrm{e}^{2\sigma_0} G(0) \,, \tag{70}$$

$$0 = 2F(0)\mathrm{e}^{2\sigma_0} + \frac{\mathrm{e}^{-\sigma_0} - F'(0)}{G'(0)} \left(-\mathrm{e}^{\sigma_0} \dot\phi^2 + 2\mathrm{e}^{2\sigma_0} G(0) \right) \,, \tag{71}$$

$$0 = - \frac{1}{2} F(0)\mathrm{e}^{2\sigma_0} - \frac{\mathrm{e}^{-\sigma_0} - F'(0)}{G'(0)} \mathrm{e}^{\sigma_0} \dot\phi^2 \tag{72}$$

$$0 = \frac{1}{2} F(0)\mathrm{e}^{2\sigma_0} \,. \tag{73}$$

Then by using (70), we find

$$0 = F(0) = \left(\mathrm{e}^{-\sigma_0} - F'(0) \right) G(0) \,, \tag{74}$$

Eq. (70) can be solved to give

$$\phi = \phi_0 \pm t \mathrm{e}^{\frac{\sigma_0}{2}} \sqrt{G_0} \,. \tag{75}$$

Here ϕ_0 is a constant. In order to investigate if there is a ghost or not, we consider the perturbation from the flat Minkowski space-time.

By using (74), we consider the case that $F(0) = G(0) = 0$ and the perturbation from the solution given by (69) and (75),

$$A = \delta A \,, \quad \sigma = \sigma_0 + \delta\sigma \,, \quad \phi = \phi_0 + \delta\phi \,. \tag{76}$$

Then the scalar part in the action (61) has the following form,

$$\begin{aligned} S_E =& \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ R - \frac{3}{2} \partial_\mu \delta\sigma \partial^\mu \delta\sigma - \mathrm{e}^{\sigma_0} \delta A \delta\sigma + 2\mathrm{e}^{2\sigma_0} F'(0) \delta A \delta\sigma \right. \\ & + \frac{\left(\mathrm{e}^{-\sigma_0} - F'(0)\right) \mathrm{e}^{\sigma_0}}{G'(0)} \left(\partial_\mu \delta\phi \partial^\mu \delta\phi + \frac{1}{2} \mathrm{e}^{2\sigma_0} G''(0) \delta A^2 + 2\mathrm{e}^{2\sigma_0} G'(0) \delta\sigma \delta A \right) \\ & \left. + \left(-\mathrm{e}^{-\sigma_0} \delta\sigma - F''(0) \delta A - \frac{\left(\mathrm{e}^{-\sigma_0} - F'(0)\right) G''(0)}{G'(0)} \delta A \right) \mathrm{e}^{2\sigma_0} \delta A \right\} . \end{aligned} \tag{77}$$

The equation given by the variation with respect to δA gives δA in terms of σ. Then by substituting the expression $\delta A = C\delta\sigma$ with a constant C, we obtain the mass term for $\delta\sigma$. The action (77) tells that as long as the following relation holds true,

$$\frac{\mathrm{e}^{-\sigma_0} - F'(0)}{G'(0)} < 0 \,, \tag{78}$$

the ghost does not appear.

By varying the action (57) with respect to the function λ and with respect to the scalar field ϕ, we obtain the following equations,

$$0 = \partial_\mu \phi \partial^\mu \phi + G(R)\,, \tag{79}$$

$$0 = \nabla^\mu \left(\lambda \partial_\mu \phi \right)\,, \tag{80}$$

On the other hand, upon variation of the action with respect to the metric $g_{\mu\nu}$, we obtain,

$$0 = \frac{F(R)}{2} g_{\mu\nu} - \left(F'(R) + \lambda G'(R) \right) R_{\mu\nu} - \lambda \partial_\mu \phi \partial_\nu \phi + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2 \right) \left(F'(R) + \lambda G'(R) \right)\,. \tag{81}$$

We assume that the geometric background is flat FRW metric with line element of the form of Eq. (**??**), and also that the function λ and also the scalar field ϕ depend only on the cosmic time t. In effect, the Eqs. (79) and (80), take the following form,

$$0 = -\dot{\phi}^2 + G(R)\,, \quad 0 = \frac{d}{dt} \left(a^3 \lambda \dot{\phi} \right)\,, \tag{82}$$

which can be rewritten as follows,

$$\dot{\phi} = \pm \sqrt{G(R)}\,, \quad a^3 \lambda \dot{\phi} = C\,, \tag{83}$$

where C is an integration constant. Also, the (t, t) and (i, j) components of Eq. (81) yield the following equations,

$$0 = -\frac{F(R)}{2} + 3\left(\dot{H} + H^2 \right) \left(F'(R) \pm \frac{C G'(R)}{a^3 \sqrt{G(R)}} \right) \mp \frac{C\sqrt{G(R)}}{a^3} - 3H \frac{d}{dt} \left(F'(R) \pm \frac{C G'(R)}{a^3 \sqrt{G(R)}} \right)\,, \tag{84}$$

$$0 = \frac{F(R)}{2} - \left(\dot{H} + 3H^2\right)\left(F'(R) \pm \frac{CG'(R)}{a^3\sqrt{G(R)}}\right) + \left(\frac{d^2}{dt^2} + 2H\frac{d}{dt}\right)\left(F'(R) \pm \frac{CG'(R)}{a^3\sqrt{G(R)}}\right) . \tag{85}$$

If we define a new quantity $J(R,a)$ as follows,

$$J(R,a) \equiv F(R) \pm \frac{2C\sqrt{G(R)}}{a^3} , \tag{86}$$

Eq. (84) can be rewritten in the following form,

$$0 = -\frac{J(R,A)}{2} + 3\left(\dot{H} + H^2 - 3H\frac{d}{dt}\right)\frac{\partial J(R,a)}{\partial R} . \tag{87}$$

We should note that when $C = 0$, Eqs. (84) and (85) become identical to the equations of the standard $F(R)$ gravity, which indicates that any solution of the standard $F(R)$ gravity is also a solution of the model (57).
An analytic form for the $F(R)$ and $G(R)$ gravity, can be given if the de Sitter spacetime is considered, in which case $H = H_0$ and $a = \mathrm{e}^{H_0 t}$. Then Eqs. (84) and (85) can be cast in the following form,

$$0 = -\frac{F(R_0)}{2} + 3H_0^2F'(R_0) \pm \frac{C}{a^3\sqrt{G(R_0)}}\left(12H_0^2G'(R_0) - G(R_0)\right) , \tag{88}$$

$$0 = \frac{F(R_0)}{2} - 3H_0^2F'(R_0) , \tag{89}$$

where $R_0 = 12H_0^2$. Then in order for the solution describing the de Sitter space-time to exist, the functions $F(R)$ and $G(R)$ must simultaneously satisfy the following differential equations,

$$0 = 2F(R_0) - R_0 F'(R_0)\,, \quad 0 = R_0 G'(R_0) - G(R_0)\,. \tag{90}$$

A special solution to the differential equations (90) is the following,

$$F(R) = \alpha R^2\,, \quad G(R) = \beta R\,, \tag{91}$$

and both the differential equations(90) are satisfied. Note that other examples of such theory leading to de Sitter space maybe found.
Another ghost-free model of generalized $F(R)$ gravity, can be obtained in the Einstein frame, if the scalar fields $\tilde{\lambda}$ and ϕ are introduced in the Lagrangian as follows [**?**],

$$S_E = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{R - \frac{3}{2}g^{\rho\sigma}\partial_\rho\sigma\partial_\sigma\sigma - V(\sigma) + \tilde{\lambda}\left(\partial_\mu\phi\partial^\mu\phi + 1\right)\right\}\,. \tag{92}$$

By applying the inverse of the transformation (**??**), we obtain,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F'(A) \left(R - A \right) + F(A) + \lambda \left(\partial_\mu \phi \partial^\mu \phi + F'(A) \right) \right\} \,, \tag{93}$$

where $\lambda = F'(A)\tilde{\lambda}$. Upon varying the action with respect to A, we obtain the following equation,

$$A = R + \lambda \,. \tag{94}$$

Then by substituting Eq. (94) in the action (93), we obtain the following action,

$$S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F(R + \lambda) + \lambda \partial_\mu \phi \partial^\mu \phi \right\} \,, \tag{95}$$

which is the action of the mimetic $F(R)$ gravity without ghost. If we further redefine λ as follows $\lambda \to \lambda - R$, we obtain the following action,

$$S_{F(R)} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left\{ F(\lambda) + \left(\lambda - R \right) \partial_\mu \phi \partial^\mu \phi \right\} \,, \tag{96}$$

If we assume that the leading order of $F(\lambda)$ is linear,

$$F(\lambda) = \lambda + \mathcal{O}\left(\lambda^2\right) \,, \tag{97}$$

or equivalently,

$$F(R+\lambda) = R + \lambda + \mathcal{O}\left(R+\lambda^2\right) \,, \tag{98}$$

the leading order in the action (95) is effectively the standard Einstein action with the mimetic constraint,

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\left\{R + \lambda\left(\partial_\mu\phi\partial^\mu\phi + 1\right) + \mathcal{O}\left(R+\lambda^2\right)\right\} \,. \tag{99}$$

Hence, the proposed models may serve for unification of inflation, dark energy and dark matter.

Reconstruction of slow-roll F(R) from inflationary indices. S. Odintsov and V.Oikonomou, Annals Phys. 388 (2018) 267-275

By using a bottom-up approach, we shall investigate how a viable set of the observational indices n_s and r can be realized by an $F(R)$ gravity in the context of the slow-roll approximation, where n_s is the power spectrum of the primordial curvature perturbations and r is the scalar-to-tensor ratio. It is important to note that the slow-roll approximation shall be considered to hold true during our calculations. In this case, the dynamics of inflation is quantified perfectly by the generalized slow-roll indices ϵ_1 ,ϵ_2, ϵ_3, ϵ_4. The first slow-roll parameter ϵ_1 controls the duration of the inflationary era and more importantly if it occurs in the first place, and it is equal to $\epsilon_1 = -\frac{\dot{H}}{H^2}$. In the case of vacuum $F(R)$ gravity in the context of the slow-roll approximation, the slow-roll parameters can be approximated as follows,

$$\epsilon_2 = 0 \, , \quad \epsilon_1 \simeq -\epsilon_3 \, , \quad \epsilon_4 \simeq \frac{F_{RRR}}{F_R}\left(24\dot{H} + 6\frac{\ddot{H}}{H}\right) - 3\epsilon_1 + \frac{\dot{\epsilon}_1}{H\epsilon_1} \, , \tag{100}$$

where $F_R = \frac{\mathrm{d}F}{\mathrm{d}R}$, and $F_{RRR} = \frac{\mathrm{d}^3F}{\mathrm{d}R^3}$. In addition, the spectral index of the primordial curvature perturbations of the vacuum $F(R)$ gravity, and the corresponding scalar-to-tensor ratio, are equal to,

$$n_s \simeq 1 - 6\epsilon_1 - 2\epsilon_4 , \quad r = 48\epsilon_1^2 \, . \tag{101}$$

Reconstruction of slow-roll F(R) from inflationary indices.

At this point, let us exemplify our bottom-up reconstruction method by using a characteristic example, and to this end, let us assume that the scalar-to-tensor ratio r is equal to,

$$r = \frac{c^2}{(q+N)^2}\,, \tag{102}$$

where N is the e-foldings number and c, q are arbitrary parameters for the moment. As we now demonstrate, the choice (102) can lead to a viable inflationary cosmology. By using the expression in Eq. (101) for the scalar-to-tensor ratio r, we obtain that,

$$r = \frac{48\dot{H}(t)^2}{H(t)^4} \tag{103}$$

and by expressing the above expression in terms of the e-foldings number N, by using the following,

$$\frac{\mathrm{d}}{\mathrm{d}t} = H\frac{\mathrm{d}}{\mathrm{d}N}\,, \tag{104}$$

the scalar-to-tensor ratio in terms of $H(N)$ is,

$$r = \frac{48H'(N)^2}{H(N)^2}\,, \tag{105}$$

where the prime now indicates differentiation with respect to N. By combing Eqs. (102) and (105), we obtain the differential equation,

$$\frac{\sqrt{48}H'(N)}{H(N)} = \frac{c}{(q+N)}\,, \tag{106}$$

which can be solved and the solution is,

$$H(N)=\gamma(N+q)^{\frac{c}{4\sqrt{3}}}\,. \tag{107}$$

The spectral index n_s can be calculated in terms of N, however it is worth providing the expression in terms of the cosmic time, which is,

$$n_s\simeq 1+\frac{4\dot{H}(t)}{H(t)^2}-\frac{2\ddot{H}(t)}{H(t)\dot{H}(t)}+\frac{F_{RRR}}{F_R}\left(24\dot{H}+6\frac{\ddot{H}}{H}\right)\,, \tag{108}$$

so by using (107) and also the following expression,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}=H^2\frac{\mathrm{d}^2}{\mathrm{d}N^2}+H\frac{\mathrm{d}H}{\mathrm{d}N}\frac{\mathrm{d}}{\mathrm{d}N}\,, \tag{109}$$

the spectral index in terms of the e-foldings number is equal to,

$$n_s \simeq 1+\frac{4H'(N)}{H(N)}-\frac{2\left(H(N)H''(N)+H'(N)^2\right)}{H(N)H'(N)}+\frac{F_{RRR}}{F_R}\left(24H(N)H'(N)+6H(N)H''(N)+6H'(N)^2\right), \tag{110}$$

where the prime indicates differentiation with respect to the e-foldings number. Finally, by substituting Eq. (107), the spectral index becomes equal to,

$$n_s = 1+\frac{c}{\sqrt{3}(N+q)}-\frac{cN}{\sqrt{3}(N+q)^2}-\frac{cq}{\sqrt{3}(N+q)^2}+\frac{2N}{(N+q)^2}+\frac{2q}{(N+q)^2}+ \tag{111}$$
$$\frac{c^2\gamma^2F_{RRR}(N+q)^{\frac{c}{2\sqrt{3}}-2}}{8F_R}+\frac{5\sqrt{3}c\gamma^2F_{RRR}(N+q)^{\frac{c}{2\sqrt{3}}-1}}{2F_R}.$$

We need first to investigate which $F(R)$ gravity can produce the inflationary era quantified by Eqs. (107) and (111), in order to find the analytic form of the last two terms in Eq. (111). As we shall see, if the parameter c is appropriately chosen, an analytic expression for $F(R)$ can be obtained. In order to find the $F(R)$ gravity which realizes the observational indices (107) and (111), so the cosmological equation appearing in Eq. (136), can be rewritten in the form,

$$-18\left(4H(t)^2\dot{H}(t)+H(t)\ddot{H}(t)\right)F_{RR}(R)+3\left(H^2(t)+\dot{H}(t)\right)F_R(R)-\frac{F(R)}{2}=0, \tag{112}$$

where $F'(R)=\frac{\mathrm{d}F(R)}{\mathrm{d}R}$. The e-folding number N, which in terms of the scale factor a is,

$$e^{-N}=\frac{a_0}{a}, \tag{113}$$

and in the following we set $a_0 = 1$. By writing the FRW equation of Eq. (302) in terms of the e-foldings number N, we obtain,

$$-18\left(4H^3(N)H'(N) + H^2(N)(H')^2 + H^3(N)H''(N)\right)F_{RR}(R) \tag{114}$$
$$+3\left(H^2(N) + H(N)H'(N)\right)F_R(R) - \frac{F(R)}{2} = 0\,,$$

where the primes stand for $H' = \mathrm{d}H/\mathrm{d}N$ and $H'' = \mathrm{d}^2H/\mathrm{d}N^2$. By using the function $G(N) = H^2(N)$, the differential equation (114) can be cast as follows,

$$-9G(N(R))\left(4G'(N(R)) + G''(N(R))\right)F_{RR}(R) + \left(3G(N) + \frac{3}{2}G'(N(R))\right)F_R(R) - \frac{F(R)}{2} = 0\,, \tag{115}$$

where $G'(N) = \mathrm{d}G(N)/\mathrm{d}N$ and $G''(N) = \mathrm{d}^2G(N)/\mathrm{d}N^2$. Also the Ricci scalar can be expressed in terms of the function $G(N)$ as follows,

$$R = 3G'(N) + 12G(N)\,. \tag{116}$$

Thus, by solving the differential equation (115), we can find the $F(R)$ gravity which may realize a cosmological evolution. Now we shall make use of the reconstruction technique we just presented in order to find the $F(R)$ gravity which realizes the observational indices (107) and (111). In our case, the function $G(N)$ is,

$$G(N) = \gamma^2(N+q)^{\frac{c}{2\sqrt{3}}}\,, \tag{117}$$

and consequently, the algebraic equation (116) takes the following form,

$$12\gamma^2(N+q)^{\frac{c}{2\sqrt{3}}} + \frac{1}{2}\sqrt{3}c\gamma^2(N+q)^{\frac{c}{2\sqrt{3}}-1} = R\,. \tag{118}$$

Reconstruction of slow-roll F(R) from inflationary indices.

In general it is quite difficult to obtain a general solution to this equation, however is c is chosen appropriately, it is possible to obtain even full analytic results. For example if $c = \sqrt{12}$, the results have a fully analytic form. In the following we shall investigate only the case with $c = \sqrt{12}$, in which case the algebraic equation (118) becomes,

$$3\gamma^2 + 12\gamma^2 N + 12\gamma^2 q = R\,, \tag{119}$$

so the function $N(R)$ is equal to,

$$N(R) = \frac{-3\gamma^2 - 12\gamma^2 q + R}{12\gamma^2}\,. \tag{120}$$

By combining Eqs. (117) and (120) the differential equation (115) in this case becomes,

$$-36\gamma^4 \left(\frac{-3\gamma^2 - 12\gamma^2 q + R}{12\gamma^2} + q \right) F''(R) + \frac{1}{4}\left(3\gamma^2 + R\right) F'(R) - \frac{F(R)}{2} = 0\,, \tag{121}$$

which can be solved analytically, and the solution is,

$$F(R) = \frac{3}{2}\sqrt{3}\gamma^3\delta + \frac{\delta R^2}{2\sqrt{3}\gamma} - 3\sqrt{3}\gamma\delta R + \mu\left(R - 3\gamma^2\right)^{3/2} L_{\frac{1}{2}}^{\frac{3}{2}}\left(\frac{1}{12}\left(\frac{R}{\gamma^2} - 3\right)\right)\,, \tag{122}$$

where the function $L_n^{\alpha}(x)$ is the generalized Laguerre Polynomial and also δ and μ are arbitrary integration constants. The existence of the Laguerre polynomial term, imposes the constraint $R < 3\gamma^2$, however in this case the term containing the root becomes complex. Hence in order to avoid inconsistencies, we set $\mu = 0$, and hence the resulting $F(R)$ gravity is,

$$F(R) = \frac{3}{2}\sqrt{3}\gamma^3\delta + \frac{\delta R^2}{2\sqrt{3}\gamma} - 3\sqrt{3}\gamma\delta R\,, \tag{123}$$

which is a variant form of the Starobinsky model. By requiring the coefficient of R to be equal to one, δ must be equal to $\delta=-\frac{1}{3\sqrt{3}\gamma}$, hence the resulting $F(R)$ gravity during the slow-roll era is,

$$F(R)=R-\frac{\gamma^2}{2}-\frac{R^2}{18\gamma^2}\,. \qquad (124)$$

We can find the Hubble rate as a function of the cosmic time, by solving the differential equation,

$$\dot{N}=H(N(t))\,, \qquad (125)$$

where $H(N)$ is given in Eq. (107), and the resulting evolution is,

$$N(t)=\frac{1}{4}\left(\Lambda^2-4q+\gamma^2t^2-2\gamma\Lambda t\right)\,, \qquad (126)$$

where $\Lambda>0$ is an integration constant. Then we easily find by combining Eqs. (126) and (107) that the Hubble rate as a function of the cosmic time is (recall that $c=\sqrt{12}$),

$$H(t)=\frac{\gamma\Lambda}{2}-\frac{\gamma^2t}{2}\,. \qquad (127)$$

Hence, the resulting evolution is a quasi-de Sitter evolution, if Λ is chosen to be quite large so that it dominates the evolution at the early-time era, in which case $H(t)\simeq\frac{\gamma\Lambda}{2}$. Also it is trivial to see that $\ddot{a}>0$, so the solution (127) describes an inflationary era. Finally, let us now demonstrate if the resulting cosmology is compatible with the Planck data. Firstly, let us see how the spectral index becomes in view of Eq. (124) and due to the fact that $F_{RRR}=0$, the spectral index becomes,

$$n_s=1+\frac{c}{\sqrt{3}(N+q)}-\frac{cN}{\sqrt{3}(N+q)^2}-\frac{cq}{\sqrt{3}(N+q)^2}+\frac{2N}{(N+q)^2}+\frac{2q}{(N+q)^2}\,. \qquad (128)$$

Reconstruction of slow-roll F(R) from inflationary indices.

By using the value of c, namely $c = \sqrt{12}$, and also for $N = 60$ and $q = -118$, the observational indices become,

$$n_s \simeq 0.9658, \quad r \simeq 0.00346842 \,. \tag{129}$$

Recall that the 2015 Planck data constrain the observational indices as follows,

$$n_s = 0.9644 \pm 0.0049 \,, \quad r < 0.10 \,, \tag{130}$$

and also, the latest BICEP2/Keck-Array data constrain the scalar-to-tensor ratio as follows,

$$r < 0.07 \,, \tag{131}$$

at 95% confidence level. Hence, the observational indices (129) are compatible to both the Planck and the BICEP2/Keck-Array data.
Hence, by using a bottom-up approach, we found in an analytic way the $F(R)$ gravity which may realize a viable set of observational indices (n_s, r). In principle, more choices for the observational indices are possible, although in most of the cases, semi-analytic results will be obtained, due to the complexity of the differential equation (115).

Based on **S.D. Odintsov and V.K. Oikonomou**, arXiv:1711.03389, Phys. Rev. D accepted
Motivation
Why to look for an autonomous dynamical system approach for $F(R)$ gravity?
Non-linear dynamical systems, even the autonomous ones, can be studied by using the Hartman-Grobman theorem, only in the case that the fixed points are hyperbolic, and only in this case serious information regarding the stability of the fixed points can be obtained.
A convincing non-autonomous example is the following:
Consider the one dimensional dynamical system $\dot{x} = -x + t$. The solution can be easily found to be $x(t) = t - 1 + e^{-t}(x_0 + 1)$, from which it is obvious that all the solutions asymptotically approach $t - 1$ for $t \to \infty$. Also it is easy to see that the only fixed point is the time-dependent solution $x = t$, which however is not a solution to the dynamical system.
In addition, a standard analysis by using the fixed point theorems, shows that the vector field actually move away from the attractor $x(t) = t - 1$, which is simply wrong.
Therefore, for $F(R)$ gravity, a way to obtain an autonomous dynamical system is needed. With regards to the inflationary era, this study will reveal:

- The existence of de Sitter fixed points.
- Their stability, either studied numerically, or analytically.

The stability of a fixed point can reveal important properties of the phase space, for example one could argue that the graceful exit from the inflationary era is a feature related to the existence of unstable de Sitter attractors

The vacuum $F(R)$ gravity autonomous dynamical system

The vacuum $f(R)$ gravity action is,

$$\mathcal{S}=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}f(R)\,, \tag{132}$$

where $\kappa^2=8\pi G=\frac{1}{M_p^2}$ and also M_p is the Planck mass scale.

The equations of motion are:

$$F(R)R_{\mu\nu}(g)-\frac{1}{2}f(R)g_{\mu\nu}-\nabla_\mu\nabla_\nu f(R)+g_{\mu\nu}\Box F(R)=0\,, \tag{133}$$

which can be written as follows,

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}=\frac{\kappa^2}{F(R)}\Big(T_{\mu\nu}+\frac{1}{\kappa^2}\Big(\frac{f(R)-RF(R)}{2}g_{\mu\nu}+\nabla_\mu\nabla_\nu F(R)-g_{\mu\nu}\Box F(R)\Big)\Big)\,, \tag{134}$$

with the prime indicating differentiation with respect to the Ricci scalar.

For the FRW metric,

$$ds^2=-dt^2+a(t)^2\sum_{i=1,2,3}\left(dx^i\right)^2\,, \tag{135}$$

where $a(t)$ is the scale factor, the cosmological equations of motion become,

$$0=-\frac{f(R)}{2}+3\left(H^2+\dot{H}\right)F(R)-18\left(4H^2\dot{H}+H\ddot{H}\right)F'(R)\,, \tag{136}$$

$$0=\frac{f(R)}{2}-\left(\dot{H}+3H^2\right)F(R)+6\left(8H^2\dot{H}+4\dot{H}^2+6H\ddot{H}+\dddot{H}\right)F'(R)+36\left(4H\dot{H}+\ddot{H}\right)^2F'(R)\,, \tag{137}$$

where $F(R)=\frac{\partial f}{\partial R}$, $F'(R)=\frac{\partial F}{\partial R}$, and $F''(R)=\frac{\partial^2 F}{\partial R^2}$.

What is now needed is to find suitable variables in order to construct the autonomous dynamical system.

Choice of the dynamical variables
we shall introduce the following variables,

$$x_1 = -\frac{\dot{F}(R)}{F(R)H}, \quad x_2 = -\frac{f(R)}{6F(R)H^2}, \quad x_3 = \frac{R}{6H^2}. \tag{138}$$

In the following we shall use the e-foldings number N, instead of the cosmic time, so the derivative with respect to the e-foldings number can be expressed as follows,

$$\frac{\mathrm{d}}{\mathrm{d}N} = \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}, \tag{139}$$

which shall be useful. Hence, by using the variables (138) we obtain the following dynamical system,

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}N} &= -4 - 3x_1 + 2x_3 - x_1x_3 + x_1^2, \\
\frac{\mathrm{d}x_2}{\mathrm{d}N} &= 8 + m - 4x_3 + x_2x_1 - 2x_2x_3 + 4x_2, \\
\frac{\mathrm{d}x_3}{\mathrm{d}N} &= -8 - m + 8x_3 - 2x_3^2,
\end{aligned} \tag{140}$$

where the parameter m is equal to,

$$m = -\frac{\ddot{H}}{H^3}. \tag{141}$$

By looking the dynamical system (140), it is obvious that the only N-dependence (or time dependence) is contained in the parameter m. Also we did not expressed m as a function of N, since we shall assume that this parameter will take constant values.
The effective equation of state (EoS) for a general $f(R)$ gravity theory is,

$$w_{eff} = -1 - \frac{2\dot{H}}{3H^2} \, , \tag{142}$$

and it can be written in terms of the variable x_3 as follows,

$$w_{eff} = -\frac{1}{3}(2x_3 - 1) \, . \tag{143}$$

By using the dynamical system (140) and the EoS (143), given the value of the parameter m, we shall investigate the structure of the phase space corresponding to the vacuum $f(R)$ gravity, and we shall discuss in detail the physical significance and implications of the results.

The parameter m appearing in the non-linear dynamical system (140) plays an important role, since it is the only source of time-dependence in the dynamical system. Let us note that for certain cosmological evolutions this parameter is constant. For example, a quasi de Sitter evolution, in which case the scale factor is,

$$a(t) = e^{H_0 t - H_i t^2}\,, \tag{144}$$

the parameter m is equal to zero, and the same applies for a de Sitter evolution.
However, in this section we shall not assume that the scale factor has a specific form, but we shall study in general the cases $m \simeq 0$.

With regard to the $m \simeq 0$ case, this is easy to check, since if we solve the differential equation $\frac{\ddot{H}}{H^3} = 0$, this yields the solution,

$$H(t) = H_0 - H_i t \,, \tag{145}$$

This means that we focus on cosmologies for which the approximate solution for the evolution is a quasi de Sitter evolution. This does not mean that the exact Hubble rate is a quasi-de Sitter evolution, but the approximate $f(R)$ gravity which drives the evolution, leads to an approximate quasi-de Sitter evolution. Interestingly enough, for the quasi-de Sitter evolution (145), the following conditions hold true,

$$H\dot{H} \gg \ddot{H}, \;\; \dot{H} \ll H^2 \,, \tag{146}$$

which are the slow-roll conditions. Hence the $m \simeq 0$ case is related to the slow-roll condition on the inflationary era.

de Sitter Inflationary Attractors and their Stability

We study the case $m \simeq 0$, which may possibly describe a quasi de Sitter evolution, however we shall analyze the dynamics of the system (140), for $m \simeq 0$ without specifying the Hubble rate.

In the case $m \simeq 0$, the fixed points are,

$$\phi_*^1 = (-1, 0, 2), \quad \phi_*^2 = (0, -1, 2) \,. \tag{147}$$

The eigenvalues for the fixed point ϕ_*^1 are $(-1, -1, 0)$, while for the fixed point ϕ_*^2 these are $(1, 0, 0)$. Hence both equilibria are non-hyperbolic, but as we show the fixed point ϕ_*^1 is stable and ϕ_*^2 is unstable.

Before we proceed let us discuss the physical significance of the two fixed points, and this can easily be revealed by observing that in both the equilibria (147), we have $x_3 = 2$. By substituting $x_3 = 2$ in Eq. (143), we get $w_{eff} = -1$, so effectively we have two de Sitter equilibria.

Also it is worth to have a concrete idea on how the dynamical system behaves analytically. Actually, the third equation of the dynamical system (140) is decoupled, and the solution of it reads,

$$x_3(N) = \frac{4N - 2\omega + 1}{2N - \omega}\,, \tag{148}$$

where ω is an integration constant which can be fixed by the initial conditions. The asymptotic behavior of the solution (148), that is for large N, is $x_3 \to 2$, which is exactly the behavior we indicated earlier.

Now let us analyze the dynamics of the cosmological system, and for starters we numerically solve the dynamical system (140) for various initial conditions and with the e-foldings number belonging to the interval $N = (0, 60)$. In Fig. (1) we present the numerical solutions for the dynamical system (140), for the initial conditions $x_1(0) = -8$, $x_2(0) = 5$ and $x_3(0) = 2.6$.

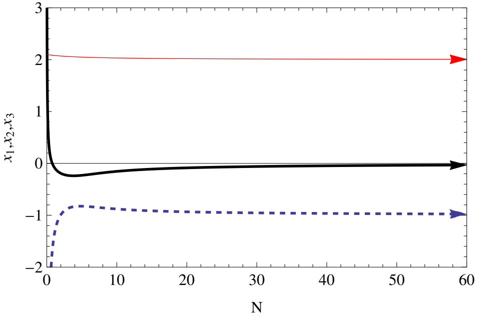

Figure: *Numerical solutions $x_1(N)$, $x_2(N)$ and $x_3(N)$ for the dynamical system (140), for the initial conditions $x_1(0) = -8$, $x_2(0) = 5$ and $x_3(0) = 2.6$, and for $m \simeq 0$.*

Approximate Form of the $f(R)$ Gravities Near the de Sitter Attractors
Effectively what we seek for is the behavior of the $f(R)$ gravities near the fixed points and with the slow-roll approximation holding true. Let us start with the first fixed point, namely $\phi_*^1 = (-1, 0, 2)$, so the following differential equations must hold true simultaneously at the fixed point,

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}R^2}\frac{\dot{R}}{H\frac{\mathrm{d}f}{\mathrm{d}R}} \simeq -1, \quad \frac{f}{H^2\frac{\mathrm{d}f}{\mathrm{d}R}6} \simeq 0\,, \tag{149}$$

which stem from the conditions $x_1 \simeq -1$ and $x_2 \simeq 0$. Since $m \simeq 0$ (or equivalently since the slow-roll approximation holds true), the left differential equation can be written as follows,

$$-24H_i\frac{\mathrm{d}^2 f}{\mathrm{d}R^2} - \frac{\mathrm{d}f}{\mathrm{d}R} = 0\,, \tag{150}$$

which can easily be solved and it yields,

$$f(R) \simeq \Lambda_1 - 24\Lambda_2 e^{-\frac{R}{24H_i}}\,. \tag{151}$$

The $f(R)$ gravity solution (151) is nothing but the approximate form of the $f(R)$ gravity in the large curvature era, which generates the quasi-de Sitter evolution of Eq. (145) or equivalently, that yields $m \simeq 0$.

Now let us consider the case of the second de Sitter fixed point, namely $\phi_*^2 = (0, -1, 2)$, and in this case the conditions $x_1 \simeq 0$ and $x_2 \simeq -1$ become,

$$- \frac{\mathrm{d}^2 f}{\mathrm{d}R^2} \frac{\dot{R}}{H \frac{\mathrm{d}f}{\mathrm{d}R}} \simeq 0, \quad - \frac{f}{H^2 \frac{\mathrm{d}f}{\mathrm{d}R} 6} \simeq -1 \, . \tag{152}$$

By using the fact that $R \simeq 12H^2$, when the quasi-de Sitter evolution is taken into account, the second differential equation can be written,

$$f \simeq \frac{\mathrm{d}f}{\mathrm{d}R} \frac{R}{2} \, , \tag{153}$$

which can be solved to yield,

$$f(R) \simeq \alpha R^2 \, . \tag{154}$$

The solution (154) is not the exact form of the $f(R)$ gravity which leads the cosmological system to the fixed point, but it is the approximate form of the $f(R)$ gravity near the fixed point ϕ_*^1 which corresponds to the case $m \simeq 0$. The approximate $f(R)$ gravity of Eq. (154) is very similar to the R^2 model.

This result is interesting, since it is well known (K.Bamba, R.Myrzakulov, S.D.Odintsov and L.Sebastiani, Phys. Rev. D 90 (2014) 043505) that R^2 corrections to viable $f(R)$ gravities, like the exponential, always trigger graceful exit from inflation, see the well-known viable Starobinsky inflation model.

R^2-corrected Logarithmic $F(R)$ gravity

The first model

$$I = \int_{\mathcal{M}} d^4 \sqrt{-g} \left[\frac{R}{\kappa^2} + \gamma(R) R^2 + f_{\rm DE}(R) + \mathcal{L}_m \right] , \tag{155}$$

The first Friedmann equation

$$\begin{aligned} 0 \quad = \quad & \frac{6H^2}{\kappa^2} - \gamma(R) \left[6R\dot{H} - 12H\dot{R} \right] + \gamma'(R) \left[24HR\dot{R} - 6R^2 \left(H^2 + \dot{H} \right) \right] + \gamma''(R) \left[6HR^2 \dot{R} \right] + \\ & f_{\rm DE} - (6H^2 + 6\dot{H}) f'_{\rm DE}(R) + 6H\dot{f}'_{\rm DE}(R) - \rho_m , \end{aligned} \tag{156}$$

In order to reproduce the early-time acceleration

$$\gamma(R) = \gamma_0 \left(1 + \gamma_1 \log \left[\frac{R}{R_0} \right] \right) , \quad 0 < \gamma_0 , \gamma_1 , \tag{157}$$

where R_0 is the curvature of the Universe at the end of inflation and γ_0 , γ_1 are positive dimensional constants. Since we would like to avoid the effects of R^2-gravity in the limit of small curvature

$$\gamma_1 \ll \frac{1}{\log \left[\frac{R_0}{4\Lambda} \right]} \ll 1 , \tag{158}$$

where $R = 4\Lambda$ is the curvature of the Universe when the dark energy is dominant, and Λ is the Cosmological constant. In the following, we will assume that $f_{\rm DE}(R)$ and $\mathcal{L}_m$ in (326) are negligible in the limit of high curvatures. The de Sitter solution with constant curvature $R_{\rm dS} = 12H_{\rm dS}$ follows from (156) and it reads,

$$H_{\rm dS}^2 \kappa^2 = \frac{1}{12\gamma_0\gamma_1} , \quad R_{\rm dS} \kappa^2 = \frac{1}{\gamma_0\gamma_1} . \tag{159}$$

If we perturb the de Sitter solution as follows,

$$H = H_{\rm dS} + \delta H(t)\,, \quad |\delta H(t)/H_{\rm dS}| \ll 1\,, \tag{160}$$

by keeping first order terms with respect to $\delta H(t)$,

$$\frac{12H_{\rm dS}}{\kappa^2}\left[\left(1-24H_{\rm dS}^2\gamma_0\gamma_1\kappa^2\right)\delta H(t) + 3\gamma_0\kappa^2\left(2+3\gamma_1+2\gamma_1\log\left[\frac{R_{\rm dS}}{R_0}\right]\right)\left(3H_{\rm dS}\delta\dot{H}(t)+\delta\ddot{H}(t)\right)\right] \simeq 0\,. \tag{161}$$

In the limit $R_0 \ll R_{\rm dS}$ the solution of this equation reads,

$$\delta H(t) \simeq h_\pm \mathrm{e}^{\Delta_\pm t}\,, \quad \Delta_\pm = \frac{H_{\rm dS}}{2}\left(-3 \pm \frac{\sqrt{\log\left[\frac{R_{\rm dS}}{R_0}\right]\left(16+9\log\left[\frac{R_{\rm dS}}{R_0}\right]\right)}}{\log\left[\frac{R_{\rm dS}}{R_0}\right]}\right)\,, \tag{162}$$

where $h_\pm$ are constants depending on the sign of $\Delta_\pm$. When the plus sign, the de Sitter expansion is unstable. We obtain,

$$H \simeq H_{\rm dS}\left(1 - h_0 \mathrm{e}^{\frac{H_{\rm dS}(t-t_0)}{\mathcal{N}}}\right)\,, \tag{163}$$

where t_0 is the time at the end of inflation when $R \simeq R_0$ and also h_0, R_0 and $\mathcal{N}$ stand for,

$$h_0 = \frac{(H_{\rm dS} - H_0)}{H_{\rm dS}} \,, \quad \mathcal{N} = \frac{3}{4} \log \left[\frac{R_{\rm dS}}{R_0} \right] \,, \quad R_0 = 12 H_0^2 \,. \tag{164}$$

In order to study the behavior of the solution during the exit from inflation, we introduce the e-foldings number,

$$N = \log \left[\frac{a(t_0)}{a(t)} \right] \equiv \int_t^{t_0} H(t) dt \,. \tag{165}$$

By using Eq. (163) we have,

$$N \simeq H_{\rm dS} (t_0 - t) \,, \tag{166}$$

where we have assumed that $\mathcal{N} \ll H_{\rm dS}(t - t_0)$, or equivalently $\mathcal{N} \ll N$. Thus, the Hubble parameter may be expressed as follows,

$$H \simeq H_{\rm dS} \left(1 - h_0 e^{-\frac{N}{\mathcal{N}}} \right) \,. \tag{167}$$

At the beginning of inflation we have $\mathcal{N} \ll N$ and $H \simeq H_{\rm dS}$, while at the end of the early-time acceleration, when $N = 0$, one recovers $H = H_0$.

During the quasi de Sitter expansion of inflation the Hubble parameter slowly decreases. The slow-roll parameters are defined as follows,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{H}\frac{dH}{dN}\,, \quad -\eta = \beta = \frac{\ddot{H}}{2H\dot{H}}\,, \tag{168}$$

where we assumed that the constant-roll condition holds true. At the beginning of the early-time acceleration the first slow-roll parameter ϵ is small, in which case the slow-roll approximation regime is realized. For the solution (167) in the limit $\mathcal{N} \ll N$, we get,

$$\epsilon \simeq \frac{h_0 \mathrm{e}^{\frac{H_{\mathrm{dS}}(t-t_0)}{\mathcal{N}}}}{\mathcal{N}} = \frac{h_0 \mathrm{e}^{-\frac{N}{\mathcal{N}}}}{\mathcal{N}}\,. \tag{169}$$

On the other hand, for the β parameter we obtain a constant value, namely,

$$\beta = \frac{1}{2\mathcal{N}}\,. \tag{170}$$

This means that the model at hand satisfies the condition for constant-roll inflation.
In the case of $F(R)$-gravity, the inflationary indices have the following form,

$$(1-n_s) \simeq \frac{2\dot{\epsilon}}{H\epsilon} = -\frac{2}{\epsilon}\frac{d\epsilon}{dN}\,, \quad r \simeq 48\epsilon^2\,. \tag{171}$$

By calculating these, we obtain,

$$(1-n_s) \simeq 4\beta - 2\epsilon \simeq \frac{2}{\mathcal{N}}\,, \quad r \simeq 48\,\frac{h_0^2 \mathrm{e}^{-2\frac{N}{\mathcal{N}}}}{\mathcal{N}^2}\,. \tag{172}$$

We can see that in the computation of the spectral index n_s we can omit the contribution of ϵ which tends to vanish for $\mathcal{N} \ll N$. Since the constant-roll inflationary condition is assumed, it turns out that this index is in fact independent on the total e-foldings number. The latest Planck data constrain the spectral index and the scalar-to-tensor ratio as follows,

$$n_s = 0.9644 \pm 0.0049\,, \quad r < 0.10\,. \tag{173}$$

As a consequence, we must require $\mathcal{N} \simeq 60$ in order to obtain a viable inflationary scenario. This means that at the beginning of inflation we have $60 \ll N$, a condition which solves the problem of initial conditions of the Friedmann Universe model we study.

By imposing $\mathcal{N} \simeq 60$ in Eq. (164) we obtain,

$$R_{\mathrm{dS}} \simeq R_0 \mathrm{e}^{80}\,, \tag{174}$$

The characteristic curvature at the time of inflation is $R_{\mathrm{dS}} \simeq 10^{120}\Lambda$, in which case one has $R_0 \simeq 1.8 \times 10^{85}\Lambda$ and from Eq. (158) we must require $\gamma_1 \ll 0.005$. Finally, the relation between γ_0 and γ_1 is fixed by Eq. (159) and we obtain,

$$\gamma_0 \simeq \frac{\mathrm{e}^{-80}}{\gamma_1 R_0 \kappa^2}\,. \tag{175}$$

The most natural generalization of the constant-roll condition in the Jordan frame is the following,

$$\frac{\ddot{H}}{2H\dot{H}} \simeq \beta \,, \tag{176}$$

where β is some real parameter. The condition (176) is the most natural generalization of the constant-roll condition used in scalar-tensor approaches, which is,

$$\frac{\ddot{\phi}}{H\dot{\phi}} = \beta \,, \tag{177}$$

since the condition (177) is nothing else but the second slow-roll index η, which in the most general case is equal to $\eta \sim -\frac{\ddot{H}}{2H\dot{H}}$. Equations of motion,

$$3F_R H^2 = \frac{F_R R - F}{2} - 3H\dot{F}_R \,, \tag{178}$$

$$-2F_R\dot{H} = \ddot{F} - H\dot{F} \,, \tag{179}$$

where F_R stands for $F_R = \frac{\partial F}{\partial R}$ and also the "dot" denotes differentiation with respect to t. The dynamics of inflation in the context of $F(R)$ gravity are governed by four inflationary indices, ϵ_i, $i = 1, \ldots 4$, which are defined as follows

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \,, \quad \epsilon_2 = 0 \,, \quad \epsilon_3 = \frac{\dot{F}_R}{2HF_R} \,, \quad \epsilon_4 = \frac{\dot{E}}{2HE} \,, \tag{180}$$

with the function E being equal to,

$$E = \frac{3\dot{F}_R^2}{2\kappa^2} \,. \tag{181}$$

Also for the calculation of the scalar-to-tensor ratio r, the quantity Q_s is needed, which is defined as follows,

$$Q_s = \frac{E}{F_R H^2 (1+\epsilon_3)^2} \,. \tag{182}$$

The spectral index of primordial curvature perturbations n_s, in the case that $\dot{\epsilon}_i \simeq 0$, is equal to [?, ?, ?],

$$n_s = 4 - 2\nu_s \,, \tag{183}$$

with ν_s being equal to,

$$\nu_s = \sqrt{\frac{1}{4} + \frac{(1+\epsilon_1-\epsilon_3+\epsilon_4)(2-\epsilon_3+\epsilon_4)}{(1-\epsilon_1)^2}} \,. \tag{184}$$

The above relation is quite general and holds true not only in the case that $\epsilon_i \ll 1$, but also when $\epsilon_i \sim \mathcal{O}(1)$. With regard to the scalar-to-tensor ratio, in the context of vacuum $F(R)$ gravity theories, it is defined as follows,

$$r = \frac{8\kappa^2 Q_s}{F_R} \,, \tag{185}$$

where the quantity Q_s is given in Eq. (182) above, and for the specific case of a vacuum $F(R)$ gravity, the scalar-to-tensor ratio is equal to,

$$r = \frac{48\epsilon_3^2}{(1+\epsilon_3)^2} \,. \tag{186}$$

The constant-roll condition (176), affects the inflationary indices of inflation ϵ_i, $i = 1, \ldots, 4$ appearing in Eq. (180), which can be written as follows,

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \,, \quad \epsilon_2 = 0 \,, \quad \epsilon_3 = \frac{\dot{F}_{RR}}{2HF_R}\left(24H\dot{H} + \ddot{H}\right) \,, \quad \epsilon_4 = \frac{F_{RRR}}{HF_R}\dot{R} + \frac{\ddot{R}}{H\dot{R}} \,, \tag{187}$$

Action,

$$F(R) = R - 2\Lambda\left(1 - \mathrm{e}^{\frac{R}{b\Lambda}}\right) - \tilde{\gamma}\Lambda\left(\frac{R}{3m^2}\right)^n , \quad (188)$$

where $\Lambda = 7.93m^2$, $\tilde{\gamma} = 1/1000$, $m = 1.57 \times 10^{-67}$eV, b is an arbitrary parameter and n is a positive real parameter.
Spectral index

$$n_s = 4 - \sqrt{\frac{(6^n(n-1)(-3\beta + (\beta+2)n - 1) + 36n(-33\beta + 35(\beta+2)n - 71))^2}{(36n(-12\beta + 12(\beta+2)n - 25) + 6^n(n-1))^2}} . \quad (189)$$

scalar-to-tensor ratio

$$r = \frac{48\left(6^n - \left(6^n - 36\right)n\right)^2}{\left(6^n - \left(6^n + 828\right)n\right)^2} . \quad (190)$$

It is noteworthy that both the spectral index and the scalar-to-tensor ratio depend only on β or n. A detailed analysis reveals that there is a large range of parameter values that may render the model compatible with the observations. For example by choosing $(n, \beta) = (2.1, -8.7)$, the spectral index becomes $n_s = 0.966239$ and the corresponding scalar-to-tensor ratio becomes $r = 0.0119893$. Also for $(n, \beta) = (0.9, -1.08)$, the spectral index becomes $n_s = 0.96742$ and the corresponding scalar-to-tensor ratio becomes $r = 0.0936944$. Finally for $(n, \beta) = (1.5, -0.4)$, the spectral index becomes $n_s = 0.960444$ and the corresponding scalar-to-tensor ratio becomes $r = 0.0669277$.

The model I appearing in Eq. (326) during the late-time era. A modified version of exponential gravity,

$$f_{\rm DE}(R) = -\frac{2\Lambda g(R)(1 - e^{-bR/\Lambda})}{\kappa^2}\,, \quad 0 < b\,, \tag{191}$$

where b is a positive parameter and Λ is the cosmological constant. The function of the Ricci scalar $g(R)$ is necessary to stabilize the theory at large redshifts

$$g(R) = \left[1 - c\left(\frac{R}{4\Lambda}\right)\log\left[\frac{R}{4\Lambda}\right]\right]\,, \quad 0 < c\,, \tag{192}$$

where c is a real and positive parameter. As a general feature of the model, we immediately see that, at $R = 0$, one has $f_{\rm DE}(R) = 0$ and we recover the Minkowski spacetime solution of Special Relativity. When $4\Lambda \leq R$, $f_{\rm DE}(R) \simeq -2\Lambda/\kappa^2$ we obtain the standard evolution of the ΛCDM model. Moreover, since $|f_{\rm DE}(R)| \sim 10^{-120} M_{Pl}^4$, we have that the modification of gravity for the dark energy sector is completely negligible in the high curvature limit of the inflationary era, where $R/\kappa^2 \sim M_{Pl}^4$.
When $g(R) \simeq 1$, it is easy to see that the following conditions hold true,

$$|F_R(R) - 1| \ll 1\,, \quad 0 < F_{RR}(R)\,, \quad \text{when} \quad 4\Lambda < R\,. \tag{193}$$

The first condition is necessary in order to obtain the correct value of the Newton constant and avoid anti-gravitational effects, while the second condition guarantees the stability of the model with respect to the matter perturbations.

During the matter and radiation domination eras, the model we used mimics an effective cosmological constant, if the function $g(R)$ in Eq. (192) is close to unity, namely

$$c \ll \left[\left(\frac{R}{4\Lambda} \right) \log \left[\frac{R}{4\Lambda} \right] \right]^{-1} , \quad 4\Lambda \leq R \ll R_0 , \tag{194}$$

where recall that R_0 is the curvature of the Universe at the end of the inflationary era. For example, if $c = 10^{-5}$, we obtain $f_{\rm DE} \simeq 2\Lambda/\kappa^2$ up to the value $R \simeq 4\Lambda \times 10^4$. For larger values of the curvature, matter and radiation dominate strongly the evolution.

In order to investigate the behavior of our model during radiation and matter domination eras, but also during the transition to the late-time era, we need to introduce the following variable,

$$y_H \equiv \frac{\rho_{\rm DE}}{\rho_{\rm m(0)}} \equiv \frac{H(z)^2}{m^2} - (z+1)^3 - \chi(z+1)^4 , \tag{195}$$

which is known as the "scaled dark energy". This variable encompasses the ratio between the effective dark energy and the standard matter density, evaluated at the present time, with the matter density defined as follows,

$$\rho_{\rm m(0)} = \frac{6m^2}{\kappa^2} , \tag{196}$$

where m is the mass scale associated with the Planck mass. In the expression (195), the variable $z = [1/a(t) - 1]$ denotes the redshift as usual, and also χ stands for $\chi \equiv \rho_{\rm r(0)}/\rho_{\rm m(0)}$.

If one extends the expression as follows,

$$F(R) = \kappa_0^2 \left[\frac{R}{\kappa^2} + \gamma(R)R^2 + f_{\rm DE}(R) \right] , \tag{197}$$

it is possible to derive FRW eq.,

$$\frac{d^2 y_H(z)}{dz^2} + J_1 \frac{dy_H(z)}{dz} + J_2 y_H(z) + J_3 = 0 , \tag{198}$$

where the functions J_i, $i = 1, 2, 3$ stand for,

$$\begin{aligned} J_1 &= \frac{1}{(z+1)} \left[-3 - \frac{1}{y_H + (z+1)^3 + \chi(z+1)^4} \frac{1 - F_R(R)}{6m^2 F_{RR}(R)} \right] , \\ J_2 &= \frac{1}{(z+1)^2} \left[\frac{1}{y_H + (z+1)^3 + \chi(z+1)^4} \frac{2 - F_R(R)}{3m^2 F_{RR}(R)} \right] , \\ J_3 &= -3(z+1) \\ &\quad - \frac{(1 - F_R(R))((z+1)^3 + 2\chi(z+1)^4) + (R - F(R))/(3m^2)}{(z+1)^2 (y_H + (z+1)^3 + \chi(z+1)^4)} \frac{1}{6m^2 F_{RR}(R)} . \end{aligned} \tag{199}$$

At the late time regime, where $z \ll 1$, we can avoid the contribution of the matter and radiation fluids, in which case, the solution of Eq. (198) reads,

$$y_H \simeq \frac{\Lambda}{3m^2} + y_0 \text{Exp}\left[\pm i\sqrt{\frac{1}{\Lambda F_{RR}(4\Lambda)} - \frac{25}{4}}\log[z+1]\right], \tag{200}$$

with y_0 being an integration constant. Since for the exponential gravity $\Lambda F_{RR}(4\Lambda) \ll 1$, the argument of the square root is positive, in effect, dark energy oscillates around the phantom divide line $w = -1$. The frequency of the oscillation with respect to $\log[z+1]$ is given by,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{1}{\Lambda F_{RR}(4\Lambda)} - \frac{25}{4}}. \tag{201}$$

Generally speaking, since $\Lambda F_{RR}(4\Lambda) \simeq 2b^2 \exp[-4b]$, the oscillation frequency at past times may diverge. However in our model, due to the presence of the function $g(R)$ chosen as in Eq. (192), one has,

$$\nu \simeq \frac{\sqrt{2/c}}{2\pi(z+1)}. \tag{202}$$

This means that, back into the past, during the radiation and matter domination eras, the frequency of the effective dark energy oscillations, tend to decrease and the theory is protected against singularities.

Now let us investigate the dark energy oscillations issue for the model I appearing in Eqs. (326) and (157). We assume the parameters,

$$\kappa^2 = \frac{16\pi}{M_{Pl}^2}\,, \quad \gamma_0 = \frac{e^{-80}}{\gamma_1 R_0 \kappa^2}\,, \quad \gamma_1 = 10^{-4}\,, \quad R_0 = 1.8\times 10^{85}\Lambda\,, \tag{203}$$

where,

$$M_{Pl}^2 = 1.2\times 10^{28}\mathrm{eV}^2\,, \quad \Lambda = 1.1895\times 10^{-67}\mathrm{eV}^2\,. \tag{204}$$

The second condition in Eq. (203) leads to a realistic de Sitter curvature for the early-time acceleration, which is $R_{\mathrm{dS}} \simeq 10^{120}\Lambda$. Moreover, the third condition in Eq. (203) ensures that the high curvature corrections of the model I disappear after the inflation, when $R < R_0$.
The constant parameters of the function $f_{\mathrm{DE}}(R)$ in Eqs. (191)–(192) are chosen as follows,

$$b = \frac{1}{2}\,, \quad c = 10^{-5}\,. \tag{205}$$

In this way, we obtain an optimal reproduction of the ΛCDM model, and the effects of dark energy remain negligible during the early and mid stages of the matter and radiation eras.

Now we need to fix the boundary conditions of our cosmological dynamical system at large redshift $z = z_{\rm max}$. They can be inferred from the form of $\rho_{\rm DE}$ for the case of $F(R)$-modified gravity, namely,

$$\rho_{\rm DE} = \frac{1}{\kappa_0^2 F_R(R)} \left[(RF_R(R) - F(R)) - 6H\dot{F}_R(R) \right] . \tag{206}$$

When $\Lambda \ll R \ll R_0$ we obtain,

$$y_H(z) \simeq \left(\frac{\Lambda}{3m^2} \right) \left(g(R) - 6H^2 g_{RR}(R)(z+1)R \right] , \tag{207}$$

where $R \equiv R(z)$ and $H \equiv H(z)$ are functions of the redshift. At large redshift, during the matter era, we have to take $R = 3m^2(z+1)^3$ and $H = m(z+1)^{3/2}$ and the boundary conditions of the system are given by,

$$\begin{aligned} y_H(z_{\rm max}) &= \left(\frac{\Lambda}{3m^2} \right) \left[g(R_{\rm max}) - 54m^4(z_{\rm max}+1)^6 g_{RR}(R_{\rm max}) \right] , \\ \frac{dy_H}{dz}(z_{\rm max}) &= 3\Lambda(z+1)^2 \left[g_R(R_{\rm max}) - 6R_{\rm max}^2 g_{RRR}(R_{\rm max}) - 12R_{\rm max} g_{RR}(R_{\rm max}) \right] , \end{aligned} \tag{208}$$

where,

$$R_{\rm max} = 3m^2(z_{\rm max}+1)^3 . \tag{209}$$

For $z_{\rm max} = 10$, in which case $\chi(z_{\rm max}+1) \simeq 0.00341 \ll 1$, and we effectively are in a matter dominated Universe, we obtain,

$$y_H(z_{\rm max}) = 2.1818\,, \quad \frac{dy_H}{dz}(z_{\rm max}) = -2.6\times 10^{-5}\,, \quad z_{\rm max} = 10\,. \tag{210}$$

These values can be compared with the corresponding ones for the ΛCDM model, where y_H is a constant, namely $y_H = \Lambda/(3m^2) = 2.17857$. We argue that our model is extremely close to the ΛCDM model at very high redshift. Here we recall that the first observed galaxies correspond to a redshift $z \simeq 6$.
Finally, the contributions of matter and radiation are determined by the values of m^2 and χ in (195). The cosmological data indicate that,

$$m^2 \simeq 1.82\times 10^{-67}\text{eV}^2\,, \quad \chi \simeq 3.1\times 10^{-4}\,. \tag{211}$$

Numerical solution.

Despite of the fact that at high redshifts, the amplitude of the oscillations of the effective EoS parameter around the phantom divide line gradually grows, we see that their frequency decreases and thus, singularities are avoided. In order to measure the matter energy density $\rho_{\mathrm{m}}(z)$ at a given redshift, we introduce the parameter $y_m(z)$ as

$$y_m(z) = \frac{\rho_{\mathrm{m}}(z)}{\rho_{\mathrm{m}(0)}} \equiv (z+1)^3 \,. \tag{212}$$

For $-1 < z < 1$ we see that $y_H(z)$ is nearly constant and it is dominant over $y_m(z)$, for $z < 0.4$, a feature that is in full agreement with the ΛCDM description.
The $\Omega_{\mathrm{DE}}(z)$ parameter,

$$\Omega_{\mathrm{DE}}(z) \equiv \frac{\rho_{\mathrm{DE}}}{\rho_{\mathrm{eff}}} = \frac{y_H(z)}{y_H(z) + (z+1)^3 + \chi(z+1)^4} \,, \tag{213}$$

is frequently used to express the ratio between the dark energy density ρ_{DE} and the effective energy density ρ_{eff} of our FRW Universe. Thus, by extrapolating $y_H(z)$ at the current redshift $z = 0$, from Eqs. (213), we obtain,

$$\Omega_{\mathrm{DE}}(z=0) = 0.685683 \,, \quad \omega_{\mathrm{DE}}(z=0) = -0.998561 \,. \tag{214}$$

The latest cosmological data indicate that, $\Omega_{\mathrm{DE}}(z=0) = 0.685 \pm 0.013$ and $\omega_{\mathrm{DE}}(z=0) = -1.006 \pm 0.045$. Thus, our model fits the observational data at present time.

This theory makes natural unification of inflation, late-time acceleration and dark matter via unique gravitational theory. Proposal of mimetic theory:Mukhanov-Chamseddine. In the mimetic model, we parametrize the metric in the following form.

$$g_{\mu\nu} = -\hat{g}^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi\hat{g}_{\mu\nu} \, . \tag{215}$$

Instead of considering the variation of the action with respect to $g_{\mu\nu}$, we consider the variation with respect to $\hat{g}_{\mu\nu}$ and ϕ. Because the parametrization is invariant under the Weyl transformation $\hat{g}_{\mu\nu} \to \mathrm{e}^{\sigma(x)}\hat{g}_{\mu\nu}$, the variation over $\hat{g}_{\mu\nu}$ gives the traceless part of the equation. Proposal of mimetic F(R) gravity: Nojiri-Odintsov,arXiv:1408.3561. In case of $F(R)$ gravity, by using the parametrization of the metric as above,

$$S = \int d^4x\sqrt{-g\left(\hat{g}_{\mu\nu},\phi\right)}\left(F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right) + \mathcal{L}_{\mathrm{matter}}\right) \, . \tag{216}$$

Field equations have the following form:

$$\begin{aligned}0 =&\frac{1}{2}g_{\mu\nu}F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)-R\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\\&+\nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)_{\mu}\nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)_{\nu}F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\\&-g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\Box\left(\hat{g}_{\mu\nu},\phi\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)+\frac{1}{2}T_{\mu\nu}\\&+\partial_\mu\phi\partial_\nu\phi\left(2F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)-R\left(\hat{g}_{\mu\nu},\phi\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\right.\\&\left.-3\Box\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)+\frac{1}{2}T\right)\,,\end{aligned}\tag{217}$$

and

$$\begin{aligned}0 =&\nabla\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)^{\mu}\left(\partial_\mu\phi\left(2F\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)-R\left(\hat{g}_{\mu\nu},\phi\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)\right.\right.\\&\left.\left.-3\Box\left(g\left(\hat{g}_{\mu\nu},\phi\right)_{\mu\nu}\right)F'\left(R\left(\hat{g}_{\mu\nu},\phi\right)\right)+\frac{1}{2}T\right)\right)\,.\end{aligned}\tag{218}$$

We should note that any solution of the standard $F(R)$ gravity is also a solution of the mimetic $F(R)$ gravity. This is because in the standard $F(R)$ gravity, Eqs. (217)–(218) are always satisfied since we find $2F(R)-RF'(R)-3\Box F'(R)+\frac{1}{2}T=0$. The mimetic $F(R)$ gravity is ghost-free and conformally invariant theory.

FRW metric:

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} dx^{i\,2} , \tag{219}$$

with $R = 6\dot{H} + 12H^2$ and ϕ is equal to t (due to mimetic form of metric).
Field equations: Eq. (218) gives

$$\begin{aligned}\frac{C_\phi}{a^3} =& 2F(R) - RF'(R) - 3\Box F'(R) + \frac{1}{2}T \\ =& 2F(R) - 6\left(\dot{H} + 2H^2\right) F'(R) + 3\frac{d^2F'(R)}{dt^2} + 9H\frac{dF'(R)}{dt} + \frac{1}{2}\left(-\rho + 3p\right) .\end{aligned} \tag{220}$$

Here C_ϕ is a constant. Then in the second line of Eq. (217), only (t,t) component does not vanish and behaves as a^{-3} and therefore the solution of Eq. (220) with $C_\phi \neq 0$ plays a role of the mimetic dark matter. On the other hand the (t,t) and (i,j)-components in (217) give the identical equation:

$$0 = \frac{d^2F'(R)}{dt^2} + 2H\frac{dF'(R)}{dt} - \left(\dot{H} + 3H^2\right) F'(R) + \frac{1}{2}F(R) + \frac{1}{2}p . \tag{221}$$

By combining (220) and (221), we obtain

$$0 = \frac{d^2F'(R)}{dt^2} - H\frac{dF'(R)}{dt} + 2\dot{H}F'(R) + \frac{1}{2}\left(p + \rho\right) + \frac{4C_\phi}{a^3} . \tag{222}$$

When $C_\phi = 0$, the above equations reduce to those in the standard $F(R)$ gravity, or in other words, when $C_\phi \neq 0$, the equation and therefore the solutions are different from those in the standard $F(R)$ gravity. Lagrange

multiplier constraint presentation: Extended model. We may consider the following action of mimetic $F(R)$ gravity with scalar potential:

$$S = \int d^4x \sqrt{-g} \left(F\left(R\left(g_{\mu\nu}\right)\right) - V(\phi) + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) + \mathcal{L}_{\mathrm{matter}} \right) . \tag{223}$$

This action is of the sort of modified gravity with Lagrange multiplier constraint. Working with viable modified gravity one can reproduce the arbitrary evolution by changing scalar potential. This gives natural unification of inflation, dark matter and dark energy.

Singular evolution

The finite-time future singularities are classified as follows: Nojiri-Odintsov-Tsujikawa, PRD71,2005,063004.

- Type I ("Big Rip") : When $t \to t_s$, the scale factor diverges a, the effective energy density $\rho_{\rm eff}$, the effective pressure $p_{\rm eff}$ diverge, $a \to \infty$, $\rho_{\rm eff} \to \infty$, and $|p_{\rm eff}| \to \infty$. This type of singularity was presented in Caldwell-Kamionkowski-Weinberg,PRL91, 2003 where it was indicated that Rip occurs before entering singularity itself.
- Type II ("sudden") : When $t \to t_s$, the scale factor and the effective energy density is finite, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$ but the effective pressure diverges $|p_{\rm eff}| \to \infty$.
- Type III : When $t \to t_s$, the scale factor is finite, $a \to a_s$ but the effective energy density and the effective pressure diverge, $\rho_{\rm eff} \to \infty$, $|p_{\rm eff}| \to \infty$.
- Type IV : For $t \to t_s$, the scale factor, the effective energy density, and the effective pressure are finite, that is, $a \to a_s$, $\rho_{\rm eff} \to \rho_s$, $|p_{\rm eff}| \to p_s$, but the higher derivatives of the Hubble rate $H \equiv \dot{a}/a$ diverge.

There is also possibility of change to decceleration in future, or approaching dS or infinite singularity (like Little Rip). It is interesting that future singularities may occur not only dark energy epoch but also at inflationary epoch: Barrow-Graham, PRD2015;Nojiri-Odintsov-Oikonomou,PRD91 (2015)084059.

Singular evolution

We consider the following action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}R - \frac{1}{2}\omega(\phi)\partial_\mu\phi\partial^\mu\phi - V(\phi) + L_{\rm matter}\right\} . \tag{224}$$

Choice of Hubble rate.In the case of the Type II and IV singularities, the Hubble rate $H(t)$ may be chosen in the following form:

$$H(t) = f_1(t) + f_2(t)\left(t_s - t\right)^\alpha . \tag{225}$$

Here $f_1(t)$ and $f_2(t)$ are smooth (differentiable) functions of t and α is a constant. If $0 < \alpha < 1$, there appears Type II singularity and if α is larger than 1 and not integer, there appears Type IV singularity. We first consider the simple case that $f_1(t) = 0$ and $f_2(t) = f_0$ with a positive constant f_0. In the neighborhood of $t = t_s$, we find that,

$$\omega(\phi) = \frac{2\alpha f_0}{\kappa^2}\left(t_s - \phi\right)^{\alpha-1} , \quad V(\phi) \sim -\frac{\alpha f_0}{\kappa^2}\left(t_s - \phi\right)^{\alpha-1} , \tag{226}$$

and we find

$$\varphi = -\frac{2\sqrt{2\alpha f_0}}{\kappa\left(\alpha+1\right)}\left(t_s - \phi\right)^{\frac{\alpha+1}{2}} , \tag{227}$$

Consequently, the scalar potential reads,

$$V(\varphi) \sim -\frac{\alpha f_0}{\kappa^2}\left\{-\frac{\kappa\left(\alpha+1\right)}{2\sqrt{2\alpha f_0}}\varphi\right\}^{\frac{2(\alpha-1)}{\alpha+1}} . \tag{228}$$

Therefore, when the following condition holds true,

$$-2 < \frac{2\,(\alpha-1)}{\alpha+1} < 0\,, \tag{229}$$

there occurs the Type II singularity. Accordingly, the Type IV singularity occurs when the following holds true,

$$0 < \frac{2\,(\alpha-1)}{\alpha+1} < 2\,. \tag{230}$$

More examples maybe presented. Qualitatively: There could be three cases,

1. The Type IV singularity occurs during the inflationary era.
2. The inflationary era ends with the Type IV singularity.
3. The Type IV singularity occurs after the inflationary era.

Most realistically, we have second and third case, when we may get realistic inflation while universe survive transition over Type IV singularity. This scenario is also extended to F(R) gravity.Furthermore, one can get unification of singular inflation with dark energy via the same modified gravity. Singular inflation with exit thanks to singularity.

$F(R)$ Gravity Description Near the Type IV Singularity: A Singular Toy Model

SDO and V.Oikonomou, Singular Inflationary Universe from F(R)F(R) Gravity,Phys.Rev. D92 (2015) no.12, 124024 DOI: 10.1103/PhysRevD.92.124024
The main feature of the toy inflationary solution is that it produces an inflationary era, so for a long time, the toy inflationary solution should be a de Sitter solution. Also, we choose the Type IV singularity to occur at the end of the inflationary era. To state this more correctly, the Type IV singularity indicates when the inflationary era ends.
The toy inflationary solution which we shall describe, is described by the following Hubble rate,

$$H(t) = c_0 + f_0 \left(t - t_s\right)^{\alpha} \, , \tag{231}$$

with the assumption that $c_0 \gg f_0$ and also for the cosmic times near the inflationary era, it holds true that $c_0 \gg f_0 \left(t - t_s\right)^{\alpha}$, for $\alpha > 0$. So in effect, near the time instance $t \simeq t_s$, the cosmological evolution is a nearly de Sitter. Also, the Type IV singularity occurs at $t = t_s$, as it can be seen from Eq. (231). Particularly, the singularity structure of the cosmological evolution (231), is determined from the values of the parameter α, and for various values of α it is determined as follows,

- $\alpha < -1$ corresponds to the Type I singularity.
- $-1 < \alpha < 0$ corresponds to Type III singularity.
- $0 < \alpha < 1$ corresponds to Type II singularity.
- $\alpha > 1$ corresponds to Type IV singularity.

$F(R)$ Gravity Description Near the Type IV Singularity: A Singular Toy Model

So in order to have a Type IV singularity we must assume that $\alpha > 1$, and we adopt this constraint for the parameter α in the rest of this paper. For $\alpha > 1$, the cosmological evolution near the Type IV singularity is a nearly de Sitter evolution. Indeed, since $c_0 \gg f_0$, the term $\sim f_0 \left(t - t_s\right)^{\alpha}$ is negligible at early times, but it can easily be seen that it dominates the evolution at late times. The evolution is governed by c_0 at early times and for a sufficient period of time after $t = t_s$, and the evolution is governed by the term $\sim f_0 \left(t - t_s\right)^{\alpha}$ only at late times $\sim t_p$. Also it is important to note that the singularity essentially plays no particular role when one considers the Hubble rate and other observable quantities at early times. It plays a crucial role in the dynamical evolution. In the FRW background of Eq. (**??**), the Ricci scalar reads,

$$R = 6(2H^2 + \dot{H})\,, \tag{232}$$

so for the Hubble rate of Eq. (231), the Ricci scalar reads,

$$R = 12c_0^2 + 24c_0 f_0(t - t_s)^{\alpha} + 12f_0^2(t - t_s)^{2\alpha} + 6f_0(t - t_s)^{-1+\alpha}\alpha\,, \tag{233}$$

and consequently near the Type IV singularity, the Ricci scalar is $R \simeq 12c_0^2$.

$F(R)$ Gravity Description

We now investigate which vacuum $F(R)$ gravity can generate the cosmological evolution described by the Hubble rate (231).
The action of a vacuum $F(R)$ gravity is equal to,

$$\mathcal{S} = \frac{1}{2\kappa^2} \int \mathrm{d}^4 x \sqrt{-g} F(R) \,, \tag{234}$$

FRW eq.

$$-18 \left[4H(t)^2 \dot{H}(t) + H(t)\ddot{H}(t) \right] F''(R) + 3 \left[H^2(t) + \dot{H}(t) \right] F'(R) - \frac{F(R)}{2} = 0 \,. \tag{235}$$

The reconstruction method we shall adopt, makes use of an auxiliary scalar field ϕ, so the $F(R)$ gravity of Eq. (301) can be written in the following equivalent form,

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[P(\phi) R + Q(\phi) \right] \,. \tag{236}$$

Note that the auxiliary field has no kinetic form so it is a non-dynamical degree of freedom. The reconstruction method we employ is based on finding the analytic dependence of the functions $P(\phi)$ and $Q(\phi)$ on the Ricci scalar R, which can be done if we find the function $\phi(R)$. In order to find the latter, we vary the action of Eq. (303) with respect to ϕ, so we end up to the following equation,

$$P'(\phi) R + Q'(\phi) = 0 \,, \tag{237}$$

where the prime in this case indicates the derivative of the corresponding function with respect to the auxiliary scalar field ϕ.

Then by solving the algebraic equation (304) as a function of ϕ, we easily obtain the function $\phi(R)$. Correspondingly, by substituting this to Eq. (303) we can obtain the $F(R)$ gravity, which is of the following form,

$$F(\phi(R)) = P(\phi(R))R + Q(\phi(R))\,. \tag{238}$$

Essentially, finding the analytic form of the functions $P(\phi)$ and $Q(\phi)$, is the aim of the reconstruction method. These can be found by varying the action of Eq. (303) with respect to the metric tensor $g_{\mu\nu}$, and the resulting expression is,

$$\begin{aligned} & -6H^2 P(\phi(t)) - Q(\phi(t)) - 6H\frac{\mathrm{d}P\left(\phi(t)\right)}{\mathrm{d}t} = 0\,, \\ & \left(4\dot{H} + 6H^2\right) P(\phi(t)) + Q(\phi(t)) + 2\frac{\mathrm{d}^2 P(\phi(t))}{\mathrm{d}t^2} + \frac{\mathrm{d}P(\phi(t))}{\mathrm{d}t} = 0\,. \end{aligned} \tag{239}$$

By eliminating the function $Q(\phi(t))$ from Eq. (306), we obtain,

$$2\frac{\mathrm{d}^2 P(\phi(t))}{\mathrm{d}t^2} - 2H(t)\frac{\mathrm{d}P(\phi(t))}{\mathrm{d}t} + 4\dot{H}P(\phi(t)) = 0\,. \tag{240}$$

Hence, for a given cosmological evolution with Hubble rate $H(t)$, by solving the differential equation (307), we can have the analytic form of the function $P(\phi)$ at hand, and from this we can easily find $Q(t)$, by using the first relation of Eq. (306). Note that, since the action of the $F(R)$ gravity (301) with the action (303) are mathematically equivalent, the auxiliary scalar field can be identified with the cosmic time t, that is $\phi = t$.

Let us now apply it for the cosmology described by the Hubble law of Eq. (231), emphasizing to the behavior near the singularity, that is, for cosmic times $t \simeq t_s$. By substituting the Hubble rate of Eq. (231) in Eq. (307), results to the following linear second order differential equation,

$$2\frac{\mathrm{d}^2 P(t)}{\mathrm{d}t^2} - 2\left(c_0 + f_0(t-t_s)^{\alpha}\right)\frac{\mathrm{d}P(t)}{\mathrm{d}t} - 4f_0(t-t_s)^{-1+\alpha}\alpha P(t) = 0\,. \qquad (241)$$

The final form of the $F(R)$ gravity near the Type IV singularity $t = t_s$, which is,

$$F(R) \simeq R + a_2 R^2 + a_0\,, \qquad (242)$$

Note additionally that we have set $c_1 = \frac{1+c_0}{4}$, so that the coefficient of R in Eq. (315) becomes equal to one, and therefore we can have Einstein gravity plus higher curvature terms.

The inflationary evolution described by the Hubble rate of Eq. (231) provides the same physical picture that standard inflation gives. Specifically, during the inflationary era, the cosmological evolution is a nearly de Sitter evolution, so an exponential expansion occurs, and the scale factor is of the form $a(t) \sim e^{c_0 t}$. More importantly, the comoving Hubble radius $R_H = \frac{1}{a(t)H(t)}$ shrinks during inflation, and expands after inflation. Moreover, the Type IV singularity has no particular effect on the comoving quantities, like the comoving Hubble radius. This remark is very important and this is due to the presence of the parameter c_0. If this was not present, then the standard inflationary picture would not hold true anymore, since a singularity would appear in the comoving Hubble radius.

Coming back to the inflationary evolution (231), the dynamics of the $F(R)$ gravity cosmological evolution is determined by the Hubble flow parameters (also known as slow-roll parameters) given below,

$$\epsilon_1 = -\frac{\dot{H}}{H^2}, \quad \epsilon_3 = \frac{\sigma' \dot{R}}{2H\sigma}, \quad \epsilon_4 = \frac{\sigma''(\dot{R})^2 + \sigma' \ddot{R}}{H\sigma' \dot{R}}, \tag{243}$$

where $\sigma = \frac{\mathrm{d}F}{\mathrm{d}R}$ and the prime in the above equation denotes differentiation with respect to the Ricci scalar R.

It is of great importance to investigate what new qualitative features does the singularity during inflation brings along. In order to do so, we shall study the R^2 inflation model, with a singularity being included and compare our results with the ordinary R^2 inflation model. This is necessary in order to understand the new qualitative features of the singular inflation. To start with, let us present the ordinary R^2 inflation model, which we modify later on in order to include a Type IV singularity. In the following, when we mention "ordinary R^2 inflation model", we mean the non-singular version of the Starobinsky R^2 inflation model. For the R^2 inflation model, the $F(R)$ gravity is,

$$F(R) = R + \frac{1}{6M^2}R^2 \, , \tag{244}$$

with $M \gg 1$. The FRW equation corresponding to the $F(R)$ gravity (244) is given below,

$$\ddot{H} - \frac{\dot{H}^2}{2H} + \frac{M^2}{2}H = -3H\dot{H} \, , \tag{245}$$

and since during inflation, the terms $\ddot{H}$ and $\dot{H}$ can be neglected, the resulting Hubble rate that describes the R^2 inflation model of Eq. (244) is,

$$H(t) \simeq H_i - \frac{M^2}{6}(t - t_i) \, . \tag{246}$$

with t_i the time instance that inflation starts and also H_i the value of the Hubble rate at t_i. Let us calculate the Hubble flow parameters for the ordinary R^2 inflation model of Eq. (244), which we will need later in order to compare with the singular version. By substituting Eqs. (246) and (244) in Eq. (243), the Hubble flow parameters for the R^2 inflation model of Eq. (244) model become,

$$\epsilon_1 = \frac{M^2}{6\left(H_i - \frac{1}{6}M^2(t - t_i)\right)^2} \, , \tag{247}$$

$$\epsilon_3 = -\frac{2}{3\left(1 + \frac{2\left(-\frac{M^2}{6} + 2\left(H_i + \frac{1}{6}M^2(-t + t_i)\right)^2\right)}{M^2}\right)} \, ,$$

The Hubble slow-roll indices (**??**) for the ordinary R^2 inflation model, and also express these in term of the $e-$folds number N, which is defined as follows,

$$N = \int_{t_i}^{t} H(t)\mathrm{d}t\,. \tag{248}$$

The spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio in terms of the Hubble slow-roll parameters η_H and ϵ_H are equal to,

$$n_s \simeq 1 - 4\epsilon_H + 2\eta_H\,, \quad r = 48\epsilon_H^2\,, \tag{249}$$

which holds true only in the case the slow-roll expansion is valid. This is a very important observation, since if one of the Hubble slow-roll parameters is large enough so that the slow-roll expansion breaks down, then the observational indices are not given by Eq. (249).
Assuming that the Hubble slow-roll parameters are such, so that the slow-roll approximation holds true, let us calculate the Hubble slow-roll parameters and inflationary indices for the Hubble rate (246). The Hubble slow-roll indices read,

$$\epsilon_H = \frac{M^2}{6\left(H_i - \frac{1}{6}M^2(t-t_i)\right)^2}\,, \quad \eta_H = 0\,. \tag{250}$$

We can express the Hubble slow-parameter ϵ_H in term of N, and by combining Eqs. (248) and (246), we obtain,

$$t - t_i = \frac{2\left(3H_i + \sqrt{3}\sqrt{3H_i^2 - M^2 N}\right)}{M^2}\,, \tag{251}$$

so upon substitution in Eq. (250) we get,

$$\epsilon_H = \frac{M^2}{6H_i^2 - 2M^2 N}\,. \tag{252}$$

Consequently, the spectral index n_s and the scalar-to-tensor ratio r, read,

$$n_s = 1 - \frac{4M^2}{6H_i^2 - 2M^2 N}, \quad r = 48 \left(\frac{M^2}{6H_i^2 - 2M^2 N} \right)^2 . \tag{253}$$

The recent observations of the Planck collaboration have verified that the R^2 inflation model is in concordance with observations, so if we suitably choose M and H_i, concordance may be achieved. Of course our approach is based on a Jordan frame calculation, but the resulting picture with regards to the observational indices is the same in both Jordan and Einstein frame. To be more concrete, let us see for which values of H_i, M and N we can achieve concordance with observations. Assume for example that the number of e-folds is $N = 60$, so for $M \sim 10^{13}\mathrm{sec}^{-1}$, and $H_i \sim 6.29348 \times 10^{13}\mathrm{sec}^{-1}$, we obtain that the spectral index of primordial perturbations n_s and the scalar-to-tensor ratio r become approximately,

$$n_s \simeq 0.966, \quad r \simeq 0.003468 . \tag{254}$$

The latest Planck data (2015) indicate that n_s and r are approximately equal to,

$$n_s = 0.9655 \pm 0.0062 , \quad r < 0.11 , \tag{255}$$

so the values given in Eq. (254) are in concordance with the current observational data.

The ordinary R^2 inflation can also contain a Type IV singularity that we assume to occur at $t = t_s$. The Hubble rate that will describe the singular inflation evolution is the following,

$$H(t) \simeq H_i - \frac{M^2}{6}(t - t_i) + f_0 (t - t_s)^{\alpha} \ , \qquad (256)$$

and we shall assume that $\alpha > 1$, so that a Type IV singularity occurs. In addition, we assume that $H_i \gg f_0$, $M \gg f_0$ and also that $f_0 \ll 1$, and consequently the singularity term is significantly smaller in comparison to the first two terms in Eq. (256). Hence, at the Hubble rate level, the singularity term remains small during inflation and therefore it can be unnoticed. Therefore, near $t \simeq t_s$, the $F(R)$ gravity that can generate the evolution (256) is the one appearing in Eq. (244). As we demonstrated previously, the effects of the singularity will not appear at the level of observable quantities, but the singularity will strongly affect the dynamics of the system. Now we investigate in detail if this holds true in this case too. The Hubble flow indices are:

$$\epsilon_1 = \frac{M^2}{6\left(H_i - \frac{1}{6}M^2(t-t_i)\right)^2}, \qquad (257)$$

$$\epsilon_3 = \frac{f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + 4\left(H_i - \frac{1}{6}M^2(t-t_i)\right)\left(-\frac{M^2}{6}\right)}{M^2\left(1 + \frac{2\left(-\frac{M^2}{6}+2\left(H_i+\frac{1}{6}M^2(-t+t_i)\right)^2\right)}{M^2}\right)\left(H_i - \frac{1}{6}M^2(t-t_i)\right)}$$

$$\epsilon_4 = \frac{\frac{M^4}{9} + 4f_0\left(H_i - \frac{1}{6}M^2(t-t_i)\right)(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha + f_0(t-t_s)^{-3+\alpha}(-2+\alpha)(-1+\alpha)\alpha}{\left(H_i - \frac{1}{6}M^2(t-t_i)\right)\left(-\frac{2}{3}M^2\left(H_i - \frac{1}{6}M^2(t-t_i)\right) + f_0(t-t_s)^{-2+\alpha}(-1+\alpha)\alpha\right)} \, .$$

If $t_s < t_f$, and $2 < \alpha < 3$, the parameter ϵ_4 becomes singular at $t = t_s$, and the rest Hubble flow parameters are not singular. Particularly, in this case, ϵ_1 remains the same as in Eq. (247), while ϵ_3 becomes simplified and behaves as,

$$\epsilon_3 \simeq -\frac{2}{3\left(1+\frac{2\left(-\frac{M^2}{6}+2\left(H_i+\frac{1}{6}M^2(-t+t_i)\right)^2\right)}{M^2}\right)}, \tag{258}$$

which is identical to the one appearing in Eq. (247) which corresponds to the ordinary R^2 inflation model. Therefore, only the parameter ϵ_4 remains singular at $t = t_s$, and takes the following form,

$$\epsilon_4 \simeq -\frac{3\left(\frac{M^4}{9}+f_0(t-t_s)^{-3+\alpha}(-2+\alpha)(-1+\alpha)\alpha\right)}{2M^2\left(H_i-\frac{1}{6}M^2(t-t_i)\right)^2}. \tag{259}$$

The Hubble flow parameters control the slow-roll expansion, so a singularity at a higher order slow-roll parameter indicates a dynamical instability of the system. Actually, it indicates that at higher orders, the slow-roll perturbative expansion breaks down, and therefore this indicates that the solution describing the dynamical evolution of the cosmological system up to that point, ceases to be an attractor of the system. This clearly may be viewed as a mechanism for graceful exit from inflation, at least at a higher order.

It is worth calculating the spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio r in this case,

$$n_s = 1 - \frac{4M^2}{6H_i^2 - 2M^2 N}\,, \quad r = 48\left(\frac{M^2}{6H_i^2 - 2M^2 N}\right)^2 . \tag{260}$$

Obviously, concordance with the observations can be achieved, like in the ordinary R^2 inflation model. For example, if we assume that the total number of e-folds is $N = 55$, and also by choosing $M \sim 10^{13}\mathrm{sec}^{-1}$ and $H_i \sim 6.15964 \times 10^{13}\mathrm{sec}^{-1}$, the spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio become,

$$n_s \simeq 0.966\,, \quad r \simeq 0.003468\,, \tag{261}$$

as in the ordinary R^2 inflation model, so comparing with the observational data (255), it can be seen than concordance can be achieved. Note that we chose $N = 55$, since in the case at hand, inflation ends earlier than in the ordinary R^2 inflation model.

The differences of the singular inflation compared to the R^2 inflation model is that inflation ends earlier than the R^2 inflation model, and also, inflation ends abruptly, since the Hubble flow parameter ϵ_4 severely diverges. A last comment is in order: Note that, since this result we obtained for this scenario, holds for cosmic times in the vicinity of the singularity, so near $t \sim t_s$, hence it is valid only near the singularity. In principle, the singularity can be chosen arbitrarily, but then the e-folding number should be appropriately changed. In order to obtain $N \simeq 50 - 60$, we assume that t_s is near the cosmic time t_f. The most important feature of this cosmological scenario is that inflation ends abruptly, compared to the ordinary R^2 inflation model, and in fact it ends before the first Hubble slow-roll parameter becomes of order ~ 1. Recall that the first Hubble slow-roll parameter corresponds to first order in the slow-roll approximation, so in the present scenario, inflation ends at a higher order in the slow-roll expansion. We need to note that in this case, the singularity will not have any observational implications, since the indices are the same as in the R^2 inflation case, with different N, H_i and M of course. The only new feature that this scenario brings along is that inflation seems to end earlier and more abruptly.

A Preliminary toy-model: Cosmology Unifying Early and Late-time Acceleration with Matter Domination Eras

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In this section we present in some detail a preliminary cosmological model which describes in a unified way early-time acceleration compatible with observations, late-time acceleration and the matter domination era. In a later section we shall present a variant of this model which describes all the evolution eras of the Universe, but still the qualitative features of both the models are the same. However, we first study the preliminary simplified model, because it is more easy to see the qualitative behavior of the various physical quantities.
The preliminary model has two Type IV singularities as we now demonstrate, with the first occurring at the end of the inflationary era, while the second is assumed to occur at the end of the matter domination era. The chronology of the Universe will assumed to be as follows: The inflationary era is assumed to start at $t \simeq 10^{-35}$sec and is assumed to end at $t \simeq 10^{-15}$sec. After that, the matter domination era occurs, and it is assumed to end at $t \simeq 10^{17}$sec, and after that, the late-time acceleration era occurs. Note that the absence of the radiation era renders the cosmological model just a toy model, but as we mentioned earlier, later on we shall present a variant form of this model which also consistently describes the radiation domination era, in addition to all the other three eras. But the qualitative features of the two models are the same, so we first study this preliminary model for simplicity. So the transition from a decelerated expansion, to an accelerated expansion is assumed to occur nearly at $t \simeq 10^{17}$sec. The Hubble rate of the model is equal to,

$$H(t) = \mathrm{e}^{-(t-t_s)^{\gamma}} \left(\frac{H_0}{2} - H_i(t-t_i) \right) + f_0|t-t_0|^{\delta}|t-t_s|^{\gamma} + \frac{2}{3\left(\frac{4}{3H_0}+t\right)} \,, \tag{262}$$

and the values of the freely chosen parameters t_s, H_0, t_0, γ, δ, H_i, f_0 and t_i, will be determined shortly. For convenience, we shall refer to the cosmological model described by the Hubble rate of Eq. (262), as the "unification model". Before specifying the values of the parameters, it is worth discussing the finite-time singularity structure of the unification model (262), which will determine the values of the parameters γ and δ.

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Particularly, the singularity structure is the following,

- When $\gamma, \delta < -1$, then two Type I singularities occurs.
- When $-1 < \gamma, \delta < 0$, then two Type III singularities occurs.
- When $0 < \gamma, \delta < 1$, then two Type II singularities occurs.
- When $\gamma, \delta > 1$, then two Type IV singularities occurs.

Obviously, there are also more combinations that can be chosen, but we omit these for simplicity. For the purposes of this article, we assume that $\gamma, \delta > 1$, so two Type IV singularities occur. Also, if $1 < \gamma, \delta < 2$, it is possible for the slow-roll indices corresponding to the inflationary era, to develop dynamical instabilities at the singularity points. Also, the gravitational baryogenesis constraints the parameter γ to be $\gamma > 2$. For these reasons, we assume that $\gamma, \delta > 2$. Also, for consistency reasons, we assume that the parameter δ is of the following form,

$$\delta = \frac{2n+1}{2m}\,, \tag{263}$$

with n, and m, being positive integers. A convenient choice we shall make for the rest of the paper is that $\gamma = 2.1$, $\delta = 2.5$. Lets investigate the allowed values of the rest of the parameters, and specifically that of t_s, at which the first Type IV singularity occurs. The Type IV singularity at $t = t_s$, will be assumed to occur at the end of the inflationary era, so t_s is chosen to be $t_s \simeq 10^{-15}$sec. Furthermore the second Type IV singularity occurs at $t = t_0$, so at t_0 is chosen to be $t_0 \simeq 10^{17}$sec. Finally, for reasons to become clear later on, the parameters f_0, H_0 and H_i are chosen as follows, $H_0 \simeq 6.293 \times 10^{13}\mathrm{sec}^{-1}$, $H_i \simeq 0.16 \times 10^{26}\mathrm{sec}^{-1}$ and $f_0 = 10^{-95}\mathrm{sec}^{-\gamma-\delta-1}$. In conclusion, the free parameters in the theory are chosen as follows,

$$\gamma = 2.1,\quad \delta = 2.5,\quad t_0 \simeq 10^{17}\mathrm{sec},\quad t_s \simeq 10^{-15}\mathrm{sec},\quad H_0 \simeq 6.293{\times}10^{13}\mathrm{sec}^{-1},\quad H_i \simeq 6{\times}10^{26}\mathrm{sec}^{-1},\quad f_0 = 10^{-95}\mathrm{se} \tag{264}$$

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With choice of the parameters as in Eq. (264), the model has interesting phenomenology. Firstly let us investigate what happens with the first term of the Hubble rate (262). Particularly, this term describes the cosmological evolution from $t \simeq 10^{-35}$sec up to $t \simeq 10^{-15}$sec, and it is obvious that the exponential $e^{-(t-t_s)^\gamma}$ for so small values of the cosmic time, can be approximated as $e^{-(t-t_s)^\gamma} \simeq 1$. In addition, the second term is particularly small during early time, since it contains positive powers of a very small cosmic time and also f_0 is chosen to be $f_0 = 10^{-95}\mathrm{sec}^{-\gamma-\delta-1}$, so the second term can be neglected at early times. Finally, owing to the fact that $t \ll \frac{4}{3H_0}$, for $10^{-35} < t < 10^{-15}$sec, the third term at early times can be approximated as follows,

$$\frac{2}{3\left(\frac{4}{3H_0}+t\right)} \simeq \frac{2}{3\left(\frac{4}{3H_0}\right)} = \frac{H_0}{2}\,. \tag{265}$$

By combining the above facts, it can be easily seen that the Hubble rate at early times is approximately equal to,

$$H(t) \simeq H_0 - H_i\left(t-t_i\right)\,, \tag{266}$$

which is identical to the nearly R^2 quasi-de Sitter inflationary evolution. This approximate behavior for the Hubble rate at early times holds true for quite a long time after $t \simeq 10^{-15}$sec, and particularly it holds true until the exponential $e^{-(t-t_s)^\gamma}$ starts to take values smaller than one, which occurs approximately for $t \simeq 10^{-3}$sec. So for $t > 10^{-3}$sec, or more accurately, after $t > 1$sec, the exponential term takes very small values, so the first term of the Hubble rate (262) can be neglected. Then, for a large period of time, the cosmological evolution is dominated by the last term solely, which is,

$$H(t) \simeq \frac{2}{3\left(\frac{4}{3H_0}+t\right)}\,, \tag{267}$$

And since $t > 1$, and $t \gg \frac{4}{3H_0}$, for H_0 chosen as in Eq. (264), the Hubble rate is approximately equal to,

$$H(t) \simeq \frac{2}{3t}\,, \tag{268}$$

which exactly describes a matter dominated era, since the corresponding scale factor can be easily shown that it behaves as $a(t) \simeq t^{2/3}$. As we demonstrate shortly, by studying the behavior of the effective equation of state (EoS), we will arrive to the same conclusion. So after the early-time acceleration era, the unification model of Eq. (262) describes a matter dominated era. This era persists until the present time, with the second term of the Hubble rate (262) dominating over the last term, only at very late times. So at late-time, the unification model Hubble rate behaves as follows,

$$H(t) \simeq f_0|t-t_0|^{\delta}|t-t_s|^{\gamma}\,. \tag{269}$$

The same picture we just described can be verified by studying the EoS of the cosmological model of Eq. (262). Since this model will be described by $F(R)$ gravity models, the EoS reads,

$$w_{\mathrm{eff}} = -1 - \frac{2\left(e^{-(t-t_s)^\gamma} H_i - \frac{1}{2\left(\frac{1}{H_0}+t\right)^2} - e^{-(t-t_s)^\gamma}\left(\frac{H_0}{2} + H_i(t-t_i)\right)(t-t_s)^{-1+\gamma}\gamma\right)}{3\left(\frac{1}{2\left(\frac{1}{H_0}+t\right)} + e^{-(t-t_s)^\gamma}\left(\frac{H_0}{2} + H_i(t-t_i)\right) + f_0(t-t_0)^\delta(t-t_s)^\gamma\right)^2}$$
$$- \frac{2\left(f_0(t-t_0)^\delta(t-t_s)^{-1+\gamma}\gamma + f_0(t-t_0)^{-1+\delta}(t-t_s)^\gamma\delta\right)}{3\left(\frac{1}{2\left(\frac{1}{H_0}+t\right)} + e^{-(t-t_s)^\gamma}\left(\frac{H_0}{2} + H_i(t-t_i)\right) + f_0(t-t_0)^\delta(t-t_s)^\gamma\right)^2} . \tag{270}$$

Therefore, it can be easily shown that at early times, the EoS is approximately equal to,

$$w_{\mathrm{eff}} \simeq -1 - \frac{2\left(\frac{3H_0}{4} + H_i\right)}{3(H_0 + H_i(t - t_i))^2}\,, \tag{271}$$

so effectively the EoS of this form describes a nearly de Sitter acceleration, since the EoS is very close to -1, because the parameters H_0 and H_i satisfy $H_0, H_i \gg 1$. After the early times, the EoS can be approximated as follows,

$$w_{\mathrm{eff}} \simeq -1 - \frac{2\left(-\frac{2}{3t^2}\right)}{3\left(\frac{2}{3t}\right)^2} = 0\,, \tag{272}$$

which describes a matter dominated era, since $w_{\mathrm{eff}} \simeq 0$. Note that this behavior is more pronounced as the second Type IV singularity at $t = t_0$ is approached. Finally, at late times, the EoS is approximately equal to,

$$w_{\mathrm{eff}} \simeq -1 - \frac{2t^{-1-\gamma-\delta}\gamma}{3f_0} - \frac{2t^{-1-\gamma-\delta}\delta}{3f_0}\,, \tag{273}$$

which again describes a nearly de Sitter acceleration era, since f_0 satisfies $f_0 \ll 1$. Note that the EoS (273) describes a nearly de Sitter but slightly turned to phantom late-time Universe, a feature which is anticipated and partially predicted for the late-time Universe. But we need to stress that the second and third terms of the EoS in Eq. (273), are extremely small, so the difference from the exact de Sitter case can be hardly detected, as time grows.

The unimodular $F(R)$ gravity formalism was developed in **S. Nojiri, S.D. Odintsov, V.K. Oikonomou, arXiv:1512.07223, arXiv:1601.04112**.
The unimodular $F(R)$ gravity approach is based on the assumption that the metric satisfies the unimodular constraint,

$$\sqrt{-g} = 1 \,, \tag{274}$$

In addition, we assume that the metric expressed in terms of the cosmological time t is a flat Friedman-Robertson-Walker (FRW) of the form,

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \,. \tag{275}$$

The metric (275) does not satisfy the unimodular constraint (275), and in order to tackle with this problem, we redefine the cosmological time t, to a new variable τ, as follows,

$$d\tau = a(t)^3 dt \,, \tag{276}$$

in which case, the metric of Eq. (275), becomes the "unimodular metric",

$$ds^2 = -a\left(t\left(\tau\right)\right)^{-6} d\tau^2 + a\left(t\left(\tau\right)\right)^2 \sum_{i=1}^{3} \left(dx^i\right)^2 \,, \tag{277}$$

and hence the unimodular constraint is satisfied.

Assuming the unimodular metric of Eq. (277), by making use of the Lagrange multiplier method, the vacuum Jordan frame unimodular $F(R)$ gravity action is,

$$S = \int d^4x \left\{ \sqrt{-g} \left(F(R) - \lambda \right) + \lambda \right\} , \tag{278}$$

with $F(R)$ being a suitably differentiable function of the Ricci scalar R, and λ stands for the Lagrange multiplier function. Note that we assumed that no matter fluids are present and also if we vary the action (278) with respect to the function λ, we obtain the unimodular constraint (274). In the metric formalism, the action is varied with respect to the metric, so by doing the variation, we obtain the following equations of motion,

$$0 = \frac{1}{2} g_{\mu\nu} \left(F(R) - \lambda \right) - R_{\mu\nu} F'(R) + \nabla_\mu \nabla_\nu F'(R) - g_{\mu\nu} \nabla^2 F'(R) . \tag{279}$$

By using the metric of Eq. (277), the non-vanishing components of the Levi-Civita connection in terms of the scale factor $a(\tau)$ and of the generalized Hubble rate $K(\tau) = \frac{1}{a} \frac{da}{d\tau}$, are given below,

$$\Gamma^\tau_{\tau\tau} = -3K , \quad \Gamma^t_{ij} = a^8 K \delta_{ij} , \quad \Gamma^i_{jt} = \Gamma^i_{\tau j} = K \delta_j{}^i . \tag{280}$$

The non-zero components of the Ricci tensor are,

$$R_{\tau\tau} = -3\dot{K} - 12K^2 , \quad R_{ij} = a^8 \left(\dot{K} + 6K^2 \right) \delta_{ij} . \tag{281}$$

while the Ricci scalar R is the following,

$$R = a^6 \left(6\dot{K} + 30K^2 \right) . \tag{282}$$

The corresponding equations of motion become,

$$0 = -\frac{a^{-6}}{2}\left(F(R)-\lambda\right) + \left(3\dot{K}+12K^2\right)F'(R) - 3K\frac{dF'(R)}{d\tau}\,, \tag{283}$$

$$0 = \frac{a^{-6}}{2}\left(F(R)-\lambda\right) - \left(\dot{K}+6K^2\right)F'(R) + 5K\frac{dF'(R)}{d\tau} + \frac{d^2F'(R)}{d\tau^2}\,, \tag{284}$$

with the "prime" and "dot" denoting as usual differentiation with respect to the Ricci scalar and τ respectively. Equations (283) and (284) can be further combined to yield the following equation,

$$0 = \left(2\dot{K}+6K^2\right)F'(R) + 2K\frac{dF'(R)}{d\tau} + \frac{d^2F'(R)}{d\tau^2} + \frac{a^{-6}}{2}\,. \tag{285}$$

Basically, the reconstruction method for the vacuum unimodular $F(R)$ gravity is based on Eq. (285), which when it is solved it yields the function $F' = F'(\tau)$. Correspondingly, by using Eq. (282), we can obtain the function $R = R(\tau)$, when this is possible so by substituting back to $F' = F'(\tau)$ we obtain the function $F'(R) = F'\left(\tau\left(R\right)\right)$. Finally, the function $\lambda(\tau)$ can be found by using Eq. (283), and substituting the solution of the differential equation (285). Based on the reconstruction method we just presented, we demonstrate how some important bouncing cosmologies can be realized. Note that the bouncing cosmologies shall be assumed to be functions of the cosmological time t, so effectively this means that the bounce occurs in the t-dependent FRW metric of Eq. (275).

A quite convenient way of studying general $F(R)$ theories of gravity, which enables us to reveal the slow-roll inflation evolution of a specific cosmological evolution, is by treating the $F(R)$ gravity cosmological system as a perfect fluid. This approach was developed in **K. Bamba, S. Nojiri, S. D. Odintsov and D. Saez-Gomez, Phys. Rev. D 90 (2014) 124061**, and as was evinced, the slow-roll indices and the corresponding observational indices receive quite convenient form, and the study of the inflationary evolution is simplified to a great extent.

The slow-roll indices and the corresponding inflationary indices can be expressed in terms of the Hubble rate $H(N)$ as follows (N is the e-folding number, $a/a_0 = \mathrm{e}^N$),

$$\epsilon = -\frac{H(N)}{4H'(N)} \left(\frac{6\frac{H'(N)}{H(N)} + \frac{H''(N)}{H(N)} + \left(\frac{H'(N)}{H(N)}\right)^2}{3 + \frac{H'(N)}{H(N)}} \right)^2 ,$$

$$\eta = -\frac{1}{2}\frac{\left(9\frac{H'(N)}{H(N)} + 3\frac{H''(N)}{H(N)} + \frac{1}{2}\left(\frac{H'(N)}{H(N)}\right)^2 - \frac{1}{2}\left(\frac{H''(N)}{H'(N)}\right)^2 + 3\frac{H''(N)}{H'(N)} + \frac{H'''(N)}{H'(N)}\right)}{\left(3 + \frac{H'(N)}{H(N)}\right)} ,$$

$$= \frac{6\frac{H'(N)}{H(N)} + \frac{H''(N)}{H(N)} + \left(\frac{H'(N)}{H(N)}\right)^2}{4\left(3 + \frac{H'(N)}{H(N)}\right)^2} \left(3\frac{H(N)H'''(N)}{H'(N)^2} + 9\frac{H'(N)}{H(N)} - 2\frac{H(N)H''(N)H'''(N)}{H'(N)^3} + 4\frac{H''(N)}{H(N)} \right.$$

$$\left. + \frac{H(N)H''(N)^3}{H'(N)^4} + 5\frac{H'''(N)}{H'(N)} - 3\frac{H(N)H''(N)^2}{H'(N)^3} - \left(\frac{H''(N)}{H'(N)}\right)^2 + 15\frac{H''(N)}{H'(N)} + \frac{H(N)H''''(N)}{H'(N)^2} \right) . \quad (286)$$

Consider the case in which, $f_0 = \frac{1}{3}$, which corresponds to the de Sitter spacetime, because we are now interested in the slow-roll inflation regime. Then, we find,

$$H \equiv \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{d\tau}{dt}\frac{da}{d\tau} = a^2\frac{da}{d\tau} = H_0 + 3H_0\left(b(\tau) + \tau\frac{db(\tau)}{d\tau}\right) , \tag{287}$$

where the parameter H_0 satisfies $H_0 \equiv \frac{1}{3\tau_0}$. Consequently, owing to the fact that $\frac{dN}{d\tau} = K$, we find,

$$\begin{aligned}
H'(N) =& 9H_0\left(2\tau\frac{db(\tau)}{d\tau} + \tau^2\frac{d^2b(\tau)}{d\tau^2}\right) , \quad H''(N) = 27H_0\left(2\tau\frac{db(\tau)}{d\tau} + 4\tau^2\frac{d^2b(\tau)}{d\tau^2} + \tau^3\frac{d^3b(\tau)}{d\tau^3}\right) \\
H'''(N) =& 81H_0\left(2\tau\frac{db(\tau)}{d\tau} + 10\tau^2\frac{d^2b(\tau)}{d\tau^2} + 7\tau^3\frac{d^3b(\tau)}{d\tau^3} + \tau^4\frac{d^4b(\tau)}{d\tau^4}\right) , \\
H''''(N) =& 243H_0\left(2\tau\frac{db(\tau)}{d\tau} + 22\tau^2\frac{d^2b(\tau)}{d\tau^2} + 31\tau^3\frac{d^3b(\tau)}{d\tau^3} + 11\tau^4\frac{d^4b(\tau)}{d\tau^4} + \tau^5\frac{d^5b(\tau)}{d\tau^5}\right) ,
\end{aligned} \tag{288}$$

and therefore, the corresponding slow-roll indices read,

$$\begin{aligned}
\epsilon =& \frac{81\tau\left(4\frac{db(\tau)}{d\tau} + 4\tau\frac{d^2b(\tau)}{d\tau^2} + \tau^2\frac{d^3b(\tau)}{d\tau^3}\right)^2}{4\left(2\frac{db(\tau)}{d\tau} + \tau\frac{d^2b(\tau)}{d\tau^2}\right)} , \\
\eta =& \frac{3}{4}\left(\frac{2\frac{db(\tau)}{d\tau} + 4\tau\frac{d^2b(\tau)}{d\tau^2} + \tau^2\frac{d^3b(\tau)}{d\tau^3}}{2\frac{db(\tau)}{d\tau} + \tau\frac{d^2b(\tau)}{d\tau^2}}\right)^2 - \frac{3\left(4\frac{db(\tau)}{d\tau} + 14\tau\frac{d^2b(\tau)}{d\tau^2} + 8\tau^2\frac{d^3b(\tau)}{d\tau^3} + \tau^3\frac{d^4b(\tau)}{d\tau^4}\right)}{2\left(2\frac{db(\tau)}{d\tau} + \tau\frac{d^2b(\tau)}{d\tau^2}\right)} .
\end{aligned} \tag{289}$$

In the perfect fluid approach the spectral index of primordial curvature perturbations n_s and the scalar-to-tensor ratio r can be expressed in terms of the slow-roll parameters as follows,

$$n_s \simeq 1 - 6\epsilon + 2\eta \,, \quad r = 16\epsilon \,. \tag{290}$$

We need to stress that the approximations for the observational indices n_s and r, remain valid if for a wide range of values of the e-foldings number N, the slow-roll indices satisfy $\epsilon, \eta \ll 1$.
Recall that the recent Planck data indicate that the spectral index n_s and the scalar-to-tensor ratio, are constrained as follows,

$$n_s = 0.9644 \pm 0.0049 \,, \quad r < 0.10 \,, \tag{291}$$

while the most recent BICEP2/Keck-Array data further constrain r to be $r < 0.07$.
Consider the cosmological evolution with the following Hubble rate as a function of the e-folding number,

$$H(N) = \left(-\gamma\, \mathrm{e}^{\delta N} + \zeta\right)^b \,. \tag{292}$$

Substituting the Hubble rate (292) in the slow-roll parameters (286), these become,

$$\epsilon = -\frac{b\mathrm{e}^{\delta N}\gamma\delta\left(\zeta(6+\delta) - 2\mathrm{e}^{\delta N}\gamma(3+b\delta)\right)^2}{4\mathcal{G}(N)} \tag{293}$$

$$\eta = -\frac{\delta\left(8b^2\mathrm{e}^{2\delta N}\gamma^2\delta + \zeta\left(2\mathrm{e}^{\delta N}\gamma(-3+\delta) + \zeta(6+\delta)\right) + 2b\mathrm{e}^{\delta N}\gamma\left(12\mathrm{e}^{\delta N}\gamma - \zeta(12+5\delta)\right)\right)}{4\left(\mathrm{e}^{\delta N}\gamma - \zeta\right)\left(-3\zeta + \mathrm{e}^{\delta N}\gamma(3+b\delta)\right)} \,, \tag{294}$$

where we introduced the function $\mathcal{G}(N)$, which is equal to,

$$\mathcal{G}(N) = \left(\mathrm{e}^{\delta N}\gamma - \zeta\right)\left(-3\zeta + \mathrm{e}^{\delta N}\gamma(3+b\delta)\right)^2 . \tag{295}$$

Having at hand Eqs. (293) and (294), the calculation of the observational indices can easily be done, and the spectral index n_s reads,

$$\frac{2\left(\mathrm{e}^{N}\right)^{3\delta}\gamma^3(3+b\delta)^2(1+2b\delta) + 3\zeta^3\left(-6+6\delta+\delta^2\right)}{2\mathcal{G}(N)} + \frac{\mathrm{e}^{\delta N}\gamma\zeta^2\left(54+12(-3+4b)\delta+3\delta^2+2b\delta^3\right)}{2\mathcal{G}(N)}$$
$$- \frac{2\mathrm{e}^{2\delta N}\gamma^2\zeta\left(27+(-9+48b)\delta+\left(3+13b^2\right)\delta^2+b(1+b)\delta^3\right)}{2\mathcal{G}(N)} , \tag{296}$$

while the scalar-to-tensor ratio r has the following form,

$$r = -\frac{4b\mathrm{e}^{\delta N}\gamma\delta\left(\zeta(6+\delta) - 2\mathrm{e}^{\delta N}\gamma(3+b\delta)\right)^2}{\mathcal{G}(N)} . \tag{297}$$

Concordance with observations can be achieved if we appropriately choose the parameters γ, ζ, δ, and b, so by making the following choice,

$$\gamma = 0.5 , \quad \zeta = 10 , \quad \delta = \frac{1}{48} , \quad b = 1 , \tag{298}$$

the observational indices n_s and r, take the following values,

$$n_s \simeq 0.965735 , \quad r = 0.0554765 , \tag{299}$$

which are compatible with both the latest Planck data and the latest BICEP2/Keck-Array data.

The unimodular $F(R)$ gravity which generates the cosmological evolution (292) is found to be,

$$F'(R) = \frac{\left(c_2 \cos\left(\frac{1}{764}\sqrt{45887}\ln\left(\frac{R}{68832\mathcal{A}_1^8}\right)\right) + c_1 \sin\left(\frac{1}{764}\sqrt{45887}\ln\left(\frac{R}{68832\mathcal{A}_1^8}\right)\right)\right)}{\left(\frac{R}{\mathcal{A}_1^8}\right)^{95/764}} \times \left(2^{475/764} 3^{95/382} 239^{95/764}\right) . \quad (300)$$

Note that in such models of unimodular $F(R)$ gravity, graceful exit from inflation may be achieved either via the contribution of R^2 correction terms, or via a Type IV singularity, in which case singular inflation might occur.

We now investigate which vacuum $F(R)$ gravity can generate an arbitrary cosmological evolution described by a given Hubble rate.
The action of a vacuum Jordan frame $F(R)$ gravity is equal to,

$$\mathcal{S}=\frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}F(R)\,, \tag{301}$$

and by adopting the metric formalism, we vary the action of Eq. (301) with respect to the metric $g_{\mu\nu}$, so we obtain the following Friedmann equation,

$$-18\left[4H(t)^2\dot{H}(t)+H(t)\ddot{H}(t)\right]F''(R)+3\left[H^2(t)+\dot{H}(t)\right]F'(R)-\frac{F(R)}{2}=0\,. \tag{302}$$

The reconstruction method we shall adopt, makes use of an auxiliary scalar field ϕ, so the $F(R)$ gravity of Eq. (301) can be written in the following equivalent form,

$$S=\int \mathrm{d}^4x\sqrt{-g}\left[P(\phi)R+Q(\phi)\right]\,. \tag{303}$$

Note that the auxiliary field has no kinetic form so it is a non-dynamical degree of freedom. The reconstruction method we employ is based on finding the analytic dependence of the functions $P(\phi)$ and $Q(\phi)$ on the Ricci scalar R, which can be done if we find the function $\phi(R)$. In order to find the latter, we vary the action of Eq. (303) with respect to ϕ, so we end up to the following equation,

$$P'(\phi)R+Q'(\phi)=0\,, \tag{304}$$

where the prime in this case indicates the derivative of the corresponding function with respect to the auxiliary scalar field ϕ.

Then by solving the algebraic equation (304) as a function of ϕ, we easily obtain the function $\phi(R)$. Correspondingly, by substituting this to Eq. (303) we can obtain the $F(R)$ gravity, which is of the following form,

$$F(\phi(R)) = P(\phi(R))R + Q(\phi(R)) \,. \tag{305}$$

Essentially, finding the analytic form of the functions $P(\phi)$ and $Q(\phi)$, is the aim of the reconstruction method. These can be found by varying the action of Eq. (303) with respect to the metric tensor $g_{\mu\nu}$, and the resulting expression is,

$$\begin{aligned} & -6H^2P(\phi(t)) - Q(\phi(t)) - 6H\frac{\mathrm{d}P\left(\phi(t)\right)}{\mathrm{d}t} = 0\,, \\ & \left(4\dot{H} + 6H^2\right) P(\phi(t)) + Q(\phi(t)) + 2\frac{\mathrm{d}^2P(\phi(t))}{\mathrm{d}t^2} + \frac{\mathrm{d}P(\phi(t))}{\mathrm{d}t} = 0\,. \end{aligned} \tag{306}$$

By eliminating the function $Q(\phi(t))$ from Eq. (306), we obtain,

$$2\frac{\mathrm{d}^2P(\phi(t))}{\mathrm{d}t^2} - 2H(t)\frac{\mathrm{d}P(\phi(t))}{\mathrm{d}t} + 4\dot{H}P(\phi(t)) = 0\,. \tag{307}$$

Hence, for a given cosmological evolution with Hubble rate $H(t)$, by solving the differential equation (307), we can have the analytic form of the function $P(\phi)$ at hand, and from this we can easily find $Q(t)$, by using the first relation of Eq. (306). Note that, since the action of the $F(R)$ gravity (301) with the action (303) are mathematically equivalent, the auxiliary scalar field can be identified with the cosmic time t, that is $\phi = t$.

A bounce cosmology is described by two eras of evolution, the contraction and expansion eras, and in between is the bouncing point, at which the Universe bounces off. During the contraction era, the scale factor of the Universe decreases, so the scale factor satisfies $\dot{a} < 0$. The Universe continues to contract until it reaches a minimal radius, at a time instance $t = t_s$, where it bounces off and the scale factor satisfies $\dot{a} = 0$. This minimal radius point is the bouncing point, and it is exactly due to this minimal size that the Universe avoids the initial singularity. After the bouncing point, the Universe starts to expand, and hence the scale factor satisfies $\dot{a} > 0$. During the contraction era, that is, when $t < t_s$, the Hubble rate satisfies $H(t) < 0$, until the bouncing point, at which $H(t_s) = 0$, and after the bouncing point and during the expansion era, the Hubble rate satisfies, $H(t) > 0$. Hence the bounce cosmology conditions are the following,

$$\begin{aligned} &\text{Before the bouncing point } t < t_s : \quad \dot{a}(t) < 0, \;\; H(t) < 0\,, \\ &\text{At the bouncing point } \quad t = t_s : \quad \dot{a}(t) = 0, \;\; H(t) = 0\,, \\ &\text{After the bouncing point } t > t_s : \quad \dot{a}(t) > 0, \;\; H(t) > 0\,, \end{aligned} \tag{308}$$

where we assumed that the bouncing point is at $t = t_s$.

Consider the following bounce cosmology, studied in **Odintsov and Oikonomou**, Phys.Rev. D91 (2015) 6, 064036, **Oikonomou** Astrophys.Space Sci. 359 (2015) 1, 30.
The scale factor and the Hubble rate for the superbounce are given below,

$$a(t) = (-t + t_s)^{\frac{2}{c^2}}\,, \quad H(t) = -\frac{2}{c^2(-t + t_s)}\,, \tag{309}$$

with c being an arbitrary parameter of the theory while the bounce in this case occurs at $t = t_s$.

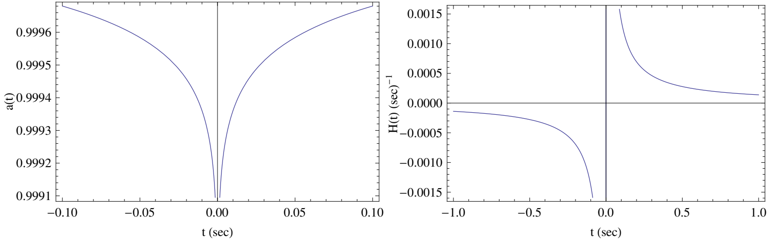

Figure: The scale factor $a(t)$ (left plot) and the Hubble rate (right plot) as a function of the cosmological time t, for the superbounce scenario $a(t) = (-t + t_s)^{\frac{2}{c^2}}$.

In the figure, we have plotted the time dependence of the scale factor and of the Hubble rate for the superbounce case.

It can be seen that in this case too, the bounce cosmology conditions (308) are satisfied, and in addition, the scale factor decreases for $t < 0$ and increases for $t > 0$, as in every bounce cosmology, so contraction and expansion occurs. In addition, the physics of the cosmological perturbations are the same to the matter bounce case, since the Hubble radius decreases for $t < 0$ and increases for $t > 0$, so the correct description for the superbounce is the following: Initially, the Universe starts with an infinite Hubble radius, at $t \to -\infty$, so the primordial modes are at subhorizon scales at that time. Gradually, the Hubble horizon decreases and consequently the modes exit the horizon and possibly freeze. Eventually, after the bouncing point, the Hubble horizon increases again, so it is possible for the primordial modes to reenter the horizon. Hence this model can harbor a conceptually complete phenomenology. The behavior of the Hubble horizon as a function of the cosmological time can be found in Fig. 3

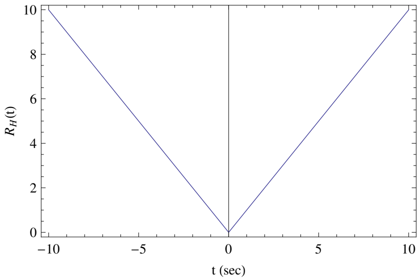

Figure: The Hubble radius $R_H(t)$ as a function of the cosmological time t, for the superbounce scenario $a(t) = (-t + t_s)^{\frac{2}{c^2}}$.

The $F(R)$ Gravity that generates this cosmology is found to be **Odintsov and Oikonomou**, Phys.Rev. D91 (2015) 6, 064036, **Oikonomou** Astrophys.Space Sci. 359 (2015) 1, 30,

$$F(R) = c_1 R^{\rho_1} + c_2 R^{\rho_2}, \tag{310}$$

where c_1, c_2 are arbitrary parameters, and ρ_1 and ρ_2 are equal to,

$$\rho_1 = \frac{-(a_2 - a_1) + \sqrt{(a_2 - a_1)^2 + 2a_1}}{2a_1}$$
$$\rho_2 = \frac{-(a_2 - a_1) - \sqrt{(a_2 - a_1)^2 + 2a_1}}{2a_1}. \tag{311}$$

and also

$$a_1 = \frac{c^2}{4 - c^2}$$
$$a_2 = \frac{2 - c^2}{2(4 - c^2)}. \tag{312}$$

In the Singular Bounce (**Odintsov and Oikonomou** Phys.Rev. D92 (2015) 2, 024016 and arXiv:1512.04
Oikonomou Phys.Rev. D92 (2015) 12, 124027),

$$a(t) = \mathrm{e}^{\frac{f_0}{\alpha+1}(t-t_s)^{\alpha+1}}\,, \quad H(t) = f_0\,(t-t_s)^{\alpha}\,, \tag{313}$$

with f_0 an arbitrary positive real number, and t_s is the time instance at which the bounce occurs and also coincides with the time that the singularity occurs. In order for a Type IV singularity to occur, the parameter α has to satisfy $\alpha > 1$. In addition, in order for the singular bounce to obey the bounce cosmology conditions, the parameter α has to be chosen in the following way,

$$\alpha = \frac{2n+1}{2m+1}\,, \tag{314}$$

with n and m integers chosen so that $\alpha > 1$. For example, for $\alpha = \frac{5}{3}$, the time dependence of the scale factor and of the Hubble rate are given in Fig. 4, and as it can be seen, the bounce conditions are satisfied, and in this case, contraction and expansion occurs.

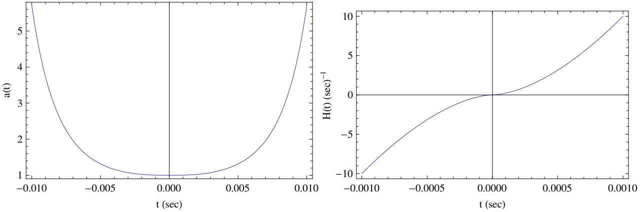

Figure: The scale factor $a(t)$ (left plot) and the Hubble rate (right plot) as a function of the cosmological time t, for the singular bounce scenario $a(t) = \mathrm{e}^{\frac{f_0}{\alpha+1}(t-t_s)^{\alpha+1}}$.

The $F(R)$ gravity that generates the bounce, near the Type IV singularity is **Odintsov and Oikonomou** Phys.Rev. D92 (2015) 2, 024016, **Oikonomou** Phys.Rev. D92 (2015) 12, 124027, **Odintsov and Oikonomou**, arXiv:1512.04787,

$$F(R) \simeq -\frac{\mathcal{A}^2}{\mathcal{C}}R^2 - 2\frac{\mathcal{B}\mathcal{A}}{\mathcal{C}}R - \frac{\mathcal{B}^2}{\mathcal{C}} + \mathcal{C}\,. \tag{315}$$

and the parameters $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ depend on the free parameters of the theory, see **Odintsov and Oikonomou** Phys.Rev. D92 (2015) 2, 024016, **Oikonomou** Phys.Rev. D92 (2015) 12, 124027, **Odintsov and Oikonomou**, arXiv:1512.04787.

Trace anomaly reads (Duff 1994,Buchbinder-Odintsov-Shapiro 1992))

$$\langle T^{\mu}_{\mu} \rangle = \alpha \left(W + \frac{2}{3}\Box R \right) - \beta \mathcal{G} + \xi \Box R \,, \tag{316}$$

where $W = C^{\xi\sigma\mu\nu} C_{\xi\sigma\mu\nu}$ is the "square" of the Weyl tensor $C_{\xi\sigma\mu\nu}$ and $\mathcal{G}$ the Gauss-Bonnet topological invariant, given by

$$W = R^{\xi\sigma\mu\nu} R_{\xi\sigma\mu\nu} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3}R^2 \,, \quad \mathcal{G} = R^{\xi\sigma\mu\nu} R_{\xi\sigma\mu\nu} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \,, \tag{317}$$

The dimensionfull coefficients α, β, and ξ of the above expression are related to the number of conformal fields present in the theory. We introduce real scalar fields N_{S}, the Dirac (fermion) fields N_{F}, vector fields N_{V}, gravitons $N_2(=0\,,1)$, and higher-derivative conformal scalars N_{HD}. Then

$$\alpha = \frac{N_{\mathrm{S}} + 6N_{\mathrm{F}} + 12N_{\mathrm{V}} + 611N_2 - 8N_{\mathrm{HD}}}{120(4\pi)^2} \,, \quad \beta = \frac{N_{\mathrm{S}} + 11N_{\mathrm{F}} + 62N_{\mathrm{V}} + 1411N_2 - 28N_{\mathrm{HD}}}{360(4\pi)^2} \,, \tag{318}$$

If we exclude the contribution of gravitons and higher-derivative conformal scalars, we get

$$\alpha = \frac{1}{120(4\pi)^2}(N_{\mathrm{S}} + 6N_{\mathrm{F}} + 12N_{\mathrm{V}}) \,, \quad \beta = \frac{1}{360(4\pi)^2}(N_{\mathrm{S}} + 11N_{\mathrm{F}} + 62N_{\mathrm{V}}) \,, \quad \xi = -\frac{N_{\mathrm{V}}}{6(4\pi)^2} \,, \tag{319}$$

For $\mathcal{N}_{\mathrm{super}} = 4$ SU(N) super Yang-Mills (SYM) theory, we have $N_{\mathrm{S}} = 6N^2$, $N_{\mathrm{F}} = 2N^2$, and $N_{\mathrm{V}} = N^2$, where N is a very large number. Therefore, we obtain a relation among the numbers of scalars, spinors and vector fields.

$$\alpha = \beta = \frac{N^2}{64\pi^2} \,, \quad \xi = -\frac{N^2}{96\pi^2} \,. \tag{320}$$

Note that

$$\frac{2}{3}\alpha + \xi = 0\,, \tag{321}$$

and in principle the contribution of the $\Box R$ term to the conformal anomaly vanishes, but it could be reintroduced via a higher curvature term in the action (see below). Owing to the conformal anomaly, the classical Einstein equation is corrected as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = \kappa^2 \langle T_{\mu\nu} \rangle\,. \tag{322}$$

By taking the trace of the last equation (322), we derive

$$R = -\kappa^2 \langle T^{\mu}_{\mu} \rangle \equiv -\kappa^2 \left[\alpha \left(W + \frac{2}{3}\Box R \right) - \beta \mathcal{G} + \xi \Box R \right]\,. \tag{323}$$

Despite the fact that in Eq. (321), the coefficient of the $\Box R$ term is equal to zero, we can set it to any desired value by adding the finite R^2 counter term in the action. In the classical Einstein gravity, this additional term is necessary to exit from inflation (Starobinsky 1980). Concretely, by adding the following action

$$I = \frac{\gamma N^2}{192\pi^2} \int_{\mathcal{M}} d^4x \sqrt{-g}\, R^2\,, \quad \gamma > 0\,, \tag{324}$$

Eq. (322) becomes (Dowker-Critchley 1976,Fishetti-Hartle-Hu 1979,Mamaev-Mostepanenko 1980, Starobinsky 1980)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\, R = -\frac{\gamma N^2 \kappa^2}{48\pi^2} R R_{\mu\nu} + \frac{\gamma N^2 \kappa^2}{192\pi^2} R^2 g_{\mu\nu} + \frac{\gamma N^2 \kappa^2}{48\pi^2} \nabla_\mu \nabla_\nu R - \frac{\gamma N^2 \kappa^2}{48\pi^2} g_{\mu\nu} \Box R^2 + \kappa^2 \langle T_{\mu\nu} \rangle\,. \tag{325}$$

Account of F(R) gravity

The action is given by

$$I = \frac{1}{2\kappa^2} \int_{\mathcal{M}} d^4 x \sqrt{-g} \left[R + 2\kappa^2 \tilde{\gamma} R^2 + f(R) + 2\kappa^2 \mathcal{L}_{\rm QC} \right] , \quad \tilde{\gamma} \equiv \frac{\gamma N^2}{192\pi^2} , \tag{326}$$

where we have considered the R^2 term in the action with $\tilde{\gamma}$ as in (324) and we have added a correction given by a function $f(R)$ of the Ricci scalar. The field equations are

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \langle T_{\mu\nu} \rangle - 4\tilde{\gamma}\kappa^2 R R_{\mu\nu} + \tilde{\gamma} R^2 \kappa^2 g_{\mu\nu} + 4\tilde{\gamma}\kappa^2 \nabla_\mu \nabla_\nu R - 4\tilde{\gamma}\kappa^2 g_{\mu\nu} \Box R^2$$
$$-f_R(R) \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{2} g_{\mu\nu} [f(R) - R f_R(R)] + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_R(R) , \tag{327}$$

The trace is described as

$$R = -\kappa^2 (\alpha W - \beta G + \delta \Box R) - 2f(R) + R f_R(R) + 3\Box f_R(R) , \tag{328}$$

where we have imposed the condition in Eq. (321) and introduced δ defined as

$$\delta \equiv -12\tilde{\gamma} = -\frac{\gamma N^2}{16\pi^2} , \quad \delta < 0 . \tag{329}$$

Here, $\gamma(>0)$ is a free parameter. The flat FLRW space-time

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2 \right) , \tag{330}$$

The energy density ρ and pressure p of quantum corrections are represented as

$$\langle T_{00} \rangle = \rho , \quad \langle T_{ij} \rangle = p\, a(t)^2 \delta_{ij} , \quad (i, j = 1, 2, 3) . \tag{331}$$

In the FLRW background, it follows from $(\mu, \nu) = (0, 0)$ component and the trace part of $(\mu, \nu) = (i, j)$ of Eq. (327), we obtain the equations of motion

$$\frac{3}{\kappa^2} H^2 = \rho + \frac{1}{2\kappa^2} \left[R f_R(R) - f(R) - 6H^2 f_R(R) - 6H \dot{f}_R(R) \right] \equiv \rho_{\rm eff} \,, \tag{332}$$

$$-\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2 \right) = p + \frac{1}{2\kappa^2} \left[-R f_R(R) + f(R) + (4\dot{H} + 6H^2) f_R(R) + 4H \dot{f}_R(R) + 2\ddot{f}_R(R) \right] \equiv p_{\rm eff} \,. \tag{333}$$

In these equations, $\rho_{\rm eff}$ and $p_{\rm eff}$ are the effective energy density and pressure of the universe. The effective conservation law

$$\dot{\rho}_{\rm eff} + 3H \left(\rho_{\rm eff} + p_{\rm eff} \right) = 0 \,. \tag{334}$$

The effective energy density is

$$\rho_{\rm eff} = \frac{\rho_0}{a^4} + 6\beta H^4 + \delta \left(18 H^2 \dot{H} + 6 \ddot{H} H - 3 \dot{H}^2 \right) + \frac{1}{2\kappa^2} \left(R f_R(R) - f(R) - 6H^2 f_R(R) - 6H \dot{f}_R(R) \right) \,, \tag{335}$$

where ρ_0 is the constant of integration. The effective pressure is

$$p_{\rm eff} = \frac{\rho_0}{3a^4} - \beta \left(6H^4 + 8H^2 \dot{H} \right) - \delta \left(9\dot{H}^2 + 12 H \ddot{H} + 2 \dddot{H} + 18 H^2 \dot{H} \right) + \frac{1}{2\kappa^2} \left[-R f_R(R) + f(R) + (4\dot{H} + 6H^2) f_R(R) + 4H \dot{f}_R(R) + 2\ddot{f}_R(R) \right] . \tag{336}$$

In the expressions of $\rho_{\rm eff}$ in Eq. (335) and $p_{\rm eff}$ in Eq. (336), we can recognize the contributions from not only modified gravity but also quantum corrections. .

Trace-anomaly driven inflation in exponential gravity

Exponential $f(R)$ (Cognola-Elizalde-Nojiri-Odintsov-Zerbini 2007)

$$f(R) = -2\Lambda_{\rm eff}\left[1 - \exp\left(-\frac{R}{R_0}\right)\right] . \tag{337}$$

Indistinguishable from LCDM.
de Sitter solutions:

$$H^2_{\rm dS\pm} = \frac{1}{4\beta\kappa^2}\left(1 \pm \sqrt{1 - \frac{8\zeta}{3}}\right) = \frac{2\pi M_{\rm Pl}^2}{N^2}\left(1 \pm \sqrt{1 - \frac{8\zeta}{3}}\right) ,$$
$$\Lambda_{\rm eff} = \frac{\zeta}{\beta\kappa^2} = \zeta\left[\frac{8\pi M_{\rm Pl}^2}{N^2}\right] , \quad 0 < \zeta < \frac{3}{8} . \tag{338}$$

There are two special solutions

$$H^2_{\rm dS} = \frac{1}{2\beta\kappa^2} = \frac{4\pi M_{\rm Pl}^2}{N^2} , \quad \Lambda_{\rm eff} = 0 , \tag{339}$$

$$H^2_{\rm dS} = \frac{1}{4\beta\kappa^2} = \frac{2\pi M_{\rm Pl}^2}{N^2} , \quad \Lambda_{\rm eff} = \frac{3}{8\beta\kappa^2} = \frac{3}{8}\left(\frac{8\pi M_{\rm Pl}^2}{N^2}\right) . \tag{340}$$

Stability of the de Sitter solutions We define the perturbations $\Delta H(t)$ as

$$H = H_{\rm dS\pm} + \Delta H(t) , \quad |\Delta H(t)| \ll 1 . \tag{341}$$

The solution is given by

$$\Delta H(t) = A_0 {\rm e}^{\lambda_{1,2} t} , \quad \lambda_{1,2} = \frac{-3H_{\rm dS\pm} \pm \sqrt{9H^2_{\rm dS\pm} + \frac{4}{\delta}\left(\frac{1}{\kappa^2} - 4H^2_{\rm dS\pm}\beta\right)}}{2} , \tag{342}$$

where A_0 is a constant.

The de Sitter solutions of the model (337) are unstable (and adopted to describe the inflation) only if λ_1 (the eigenvalue with the positive sign in front of the square root) is real and positive, i.e.,

$$4\beta - \frac{1}{\kappa^2 H_{\mathrm{dS}\pm}^2} > 0\,, \quad 9H_{\mathrm{dS}\pm}^2 + \frac{4}{\delta}\left(\frac{1}{\kappa^2} - 4H_{\mathrm{dS}\pm}^2\beta\right) > 0\,. \tag{343}$$

Here, we have taken into account the fact that $\beta > 0$ and $\delta < 0$.
Dynamics of inflation
Given the unstable de Sitter solution $H_{\mathrm{dS}\pm}^2$ in , to analyze inflation occurring in the model in Eq. (337), we have to calculate the amplitude of the perturbations in Eq. (342).
At the time $t = 0$ when inflation starts, we have to set $\Delta H(t = 0) = 0$. The complete solution of this equation is given by the homogeneous part in Eq. (342) plus the contribute of modified gravity as follows

$$\Delta H(t) = A_0 e^{\lambda_{1,2} t} - \frac{e^{-R_{\mathrm{dS}}/R_0}\Lambda_{\mathrm{eff}}}{12 H_{\mathrm{dS}}\kappa^2}\left(\frac{R_{\mathrm{dS}}}{R_0} + 2\right)\left(\frac{1}{\kappa^2} - 4H_{\mathrm{dS}}^2\beta\right)^{-1}\,. \tag{344}$$

Thus, at $t = 0$, by putting $\Delta H(t = 0) = 0$, we can estimate the amplitude A_0 as

$$A_0 = -\frac{e^{-R_{\mathrm{dS}}/R_0}\zeta}{12 H_{\mathrm{dS}}(\beta\kappa^2)}\left(\frac{R_{\mathrm{dS}}}{R_0} + 2\right)\left(1 - \frac{8}{3}\zeta\right)^{-1/2} < 0\,. \tag{345}$$

Here, we have considered only the unstable solution $H_{\mathrm{dS}} \equiv H_{\mathrm{dS}+}$ in Eq. (338).
The time at the end of inflation

$$t_{\mathrm{f}} \simeq \frac{R_{\mathrm{dS}}}{R_0\,\lambda_1}\,. \tag{346}$$

The number of e-folds $\mathcal{N}$ is

$$\mathcal{N} = \ln\left(\frac{a_{\mathrm{f}}}{a_{\mathrm{i}}}\right)\,, \tag{347}$$

and inflation is viable if $\mathcal{N} > 76$.

For the model (337), by taking account of the fact that we have chosen $t_{\mathrm{i}} = 0$ and using Eq. (346), we acquire

$$\mathcal{N} \equiv H_{\mathrm{dS}} t_f = \frac{2R_{\mathrm{dS}}}{3R_0} \left[-1 + \sqrt{1 - \frac{16\beta}{9\delta} \left(\frac{\sqrt{1 - \frac{8}{3}\zeta}}{1 + \sqrt{1 - \frac{8}{3}\zeta}} \right)} \right]^{-1} . \qquad (348)$$

By combining this relation, the expressions for β in Eq. (320) and δ in Eq. (329), and Eq. (348), we have

$$\mathcal{N} = \frac{2b}{3} \left[-1 + \sqrt{1 + \frac{4}{9\gamma} \left(\frac{\sqrt{1 - \frac{8}{3}\zeta}}{1 + \sqrt{1 - \frac{8}{3}\zeta}} \right)} \right]^{-1} . \qquad (349)$$

Spectral index

The second time derivative of $a(t)$ is

$$\frac{\ddot{a}}{a} = H^2 + \dot{H} = H^2 (1 - \epsilon) , \qquad (350)$$

with the parameter ϵ. When the approximate de Sitter solution is realized, it has to be very small as

$$\epsilon = -\frac{\dot{H}}{H^2} \ll 1 . \qquad (351)$$

Moreover, ϵ has to change very slowly. There is another parameter η, which has to also be very small as

$$|\eta| = \left| -\frac{\ddot{H}}{2H\dot{H}} \right| \equiv \left| \epsilon - \frac{1}{2\epsilon H} \dot{\epsilon} \right| \ll 1 . \qquad (352)$$

These two parameters are the so-called slow-roll parameters.

The amplitude of scalar-mode power spectrum of the primordial curvature perturbations at $k = 0.002\,\mathrm{Mpc}^{-1}$ is described as

$$\Delta_{\mathcal{R}}^2 = \frac{\kappa^2 H^2}{8\pi^2 \epsilon}\,, \tag{353}$$

and the last cosmological data constrain the spectral index n_{s} and the tensor-to-scalar ratio r are given by (Mukhanov:1981),

$$n_{\mathrm{s}} = 1 - 6\epsilon + 2\eta\,, \quad r = 16\epsilon\,. \tag{354}$$

In the model (337), we find

$$\Delta_{\mathcal{R}}^2 = \frac{1}{32\pi^2 \beta \epsilon}\left(1 + \sqrt{1 - \frac{8}{3}\zeta}\right) = \frac{2}{N^2 \epsilon}\left(1 + \sqrt{1 - \frac{8}{3}\zeta}\right)\,, \tag{355}$$

The parameters ϵ and η read

$$\begin{aligned} \epsilon &\simeq -\frac{\Delta \dot{H}(t)}{H_{\mathrm{dS}}^2} = \frac{b^2}{\mathcal{N}^2}\left(-\frac{\delta}{4\beta}\right)\frac{\mathrm{e}^{(\lambda_1 t - b)}\zeta\,(b+2)}{\left(1 - \frac{8}{3}\zeta\right)}\left(\frac{b}{3\mathcal{N}} + 1\right) = \frac{b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{(\lambda_1 t - b)}\zeta\,(b+2)}{\left(1 - \frac{8}{3}\zeta\right)}\left(\frac{b}{3\mathcal{N}} + 1\right)\,, \\ \eta &= \epsilon - \frac{\dot{\epsilon}}{2\epsilon H_{\mathrm{dS}}} = \epsilon - \frac{\lambda_1}{2H_{\mathrm{dS}}} = \epsilon - \frac{b}{2\mathcal{N}}\,. \end{aligned} \tag{356}$$

During inflation, when $t \ll t_{\mathrm{f}}$, since $\mathcal{N} \gg 1$, we have

$$\epsilon \simeq \frac{b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1 - \frac{8}{3}\zeta\right)} \ll 1\,, \quad |\eta| \simeq \left|-\frac{b}{2\mathcal{N}}\right| \ll 1\,. \tag{357}$$

Thus, the spectral index and the tensor-to-scalar ratio in Eq. (354) for the model (337) are derived as

$$n_{\mathrm{s}} = 1 - \frac{b}{\mathcal{N}} - \frac{6b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1 - \frac{8}{3}\zeta\right)}\,, \quad r = \frac{16b^2}{\mathcal{N}^2}\frac{\mathrm{e}^{-b}\zeta\,(b+2)}{\left(1 - \frac{8}{3}\zeta\right)}\,. \tag{358}$$

We mention the recent observations of the spectral index n_s as well as the tensor-to-scalar ratio r. The results observed by the Planck satellite are $n_s = 0.9603 \pm 0.0073\,(68\%\,\mathrm{CL})$ and $r < 0.11\,(95\%\,\mathrm{CL})$. Since $b/\mathcal{N} \ll 1$ and $1 \ll b$, the constraints from the Planck satellite described above can be satisfied. For instance, for $b = 3$, $\zeta = 1/8$, and $\mathcal{N} = 76$, we have $n_s \simeq 0.9601$ and $r = 1.20 \times 10^{-3}$.
On the other hand, the BICEP2 experiment has detected the B-mode polarization of the cosmic microwave background (CMB) radiation with the tensor to scalar ratio $r = 0.20^{+0.07}_{-0.05}\,(68\%\,\mathrm{CL})$, and also the case that r vanishes has been rejected at 7.0σ level.
For our model, even if the dependence of the tensor-to-scalar ratio on $\mathcal{N}^2$ makes it very small, we can play with a value of ζ close to 3/8 in order to increase its value. For instance, with the choice $\zeta = 0.37125$, we can still describe the unstable de Sitter solution for $b > 1$, since $R_{\mathrm{dS}} \gg R_0$ and $f(R_{\mathrm{dS}}) \simeq -2\Lambda_{\mathrm{eff}}$. Thus, the number of e-folds $\mathcal{N}$ depends on γ only as in Eq. (349). Indeed, when we take the combination of the values of b and γ, e.g., $(b = 2, \gamma > 1.14)$, $(b = 3, \gamma > 0.76)$, and $(b = 4, \gamma > 0.57)$, and so on, we obtain $\mathcal{N} > 76$.
For example, if $\mathcal{N} = 76$, for $b = 2$, 3 and 4, we acquire $r = 0.22$, 0.23, and 0.18, respectively.

Thus, unification of realistic inflation with viable dark energy era occurs in exponential F(R) gravity with account of quantum effects (trace anomaly). This is in full accord with first discovery of such unification proposed in Nojiri-Odintsov2003.

Nariai metric in the cosmological patch with $R_0 = 4\Lambda$ and cosmological time t given by $\tau = \arccos\left[\cosh t\right]^{-1}$ reads

$$ds^2 = -\frac{1}{\Lambda\cos^2\tau}\left(-d\tau^2 + dx^2\right) + \frac{1}{\Lambda}d\Omega^2\,, \tag{359}$$

$-\pi/2 < \tau < \pi/2$. $F(R)$-gravity admits such a metric as the limiting case of the Schwarzshild-de Sitter solution under the condition

$$2F(R_0) = R_0 F_R(R_0)\,. \tag{360}$$

Perturbations around the Nariai space-time are described by

$$ds^2 = \mathrm{e}^{2\rho(x,\tau)}\left(-d\tau^2 + dx^2\right) + \mathrm{e}^{-2\varphi(x,\tau)}d\Omega^2\,, \quad \rho = -\ln\left[\sqrt{\Lambda}\cos\tau\right] + \delta\rho\,, \quad \varphi = \ln\sqrt{\Lambda} + \delta\varphi\,. \tag{361}$$

From the field equations of $F(R)$-gravity one finds

$$\frac{1}{\alpha\cos^2\tau}\left[2(2\alpha-1)\delta\varphi\right] - 3\delta\ddot{\varphi} + 3\delta\varphi'' = 0\,, \quad \alpha = \frac{2\Lambda F_{RR}(R_0)}{F'(R_0)}\,, \tag{362}$$

and

$$\delta R \equiv 4\Lambda\left(-\delta\rho + \delta\varphi\right) + \Lambda\cos^2\tau\left(2\delta\ddot{\rho} - 2\delta\rho'' - 4\delta\ddot{\varphi} + 4\delta\varphi''\right) = 2\frac{F_R(R_0)}{F_{RR}(R_0)}\delta\varphi\,. \tag{363}$$

Equation (362) can be used to study the evolution of $\varphi(\tau, x)$. In principle, one may insert the result in (363) in order to obtain $\rho(\tau, x)$. However, the radius of the Nariai black hole depends on $\varphi(\tau, x)$ only, so that we will limit our analysis to Eq. (362).

Horizon perturbations.
The position of the horizon moves on the one-sphere S_1 and it is located in the correspondence of $\nabla\delta\varphi \cdot \nabla\delta\varphi = 0$. For a black hole located at $x = x_0$, the horizon is defined as

$$r_0(\tau)^{-2} = \mathrm{e}^{2\varphi(\tau, x_0)} = \frac{1 + \delta\varphi(x_0, \tau)}{\Lambda} \,. \tag{364}$$

Therefore, evaporation/anti-evaporation correspond to increasing/decreasing values of $\delta\varphi(\tau)$ on the horizon.
Following [J. C. Niemeyer and R. Bousso, Phys. Rev. D **62** (2000) 023503 [gr-qc/0004004]] we can decompose the two-sphere radius of Nariai solution into Fourier modes on the S_1 sphere, namely

$$\delta\varphi(x, t) = \epsilon \sum_{n=1}^{+\infty} \left(A_n(\tau)\cos[nx] + B_n(\tau)\sin[nx]\right) \,, \quad 1 \gg \epsilon > 0 \,. \tag{365}$$

Here, ϵ is assumed to be positive and small. From Eq. (362) we get

$$\delta\varphi(x, t) = \epsilon \sum_{n=1}^{\infty} P_\nu^\mu(\xi) \Big[a_n \cos(nx) + b_n \sin(nx)\Big] \,, \quad \xi = \sin\tau \,, \tag{366}$$

with

$$\mu = \sqrt{\frac{2(2\alpha - 1)}{3\alpha}} \,, \quad \nu = -\frac{1}{2} \pm \sqrt{n^2 + \frac{1}{4}} \,, \quad \alpha = \frac{2\Lambda F_{RR}(R_0)}{F'(R_0)} \,. \tag{367}$$

Above, $P_\nu^\mu(\xi)$ are the Legendre polynomials regular on the boundary $\xi = 0$ (i.e. $t = 0$) and the unknown coefficients $\{a_n, b_n\}$ can in principle be obtained by using the initial boundary conditions at $\xi = 0$.

By using this formalism, we can study the stability/unstability of Nariai solutions in $F(R)$-gravity for different modes of $\delta\varphi(x,t)$. For $n=1$ one has near to $\xi=1$ (i.e. $t\to+\infty$):

- When μ is real

$$P_\nu^\mu(\xi)\simeq(1-\xi)^{-\frac{\mu}{2}}\left[\frac{2^{\mu/2}}{\Gamma(1-\mu)}-\frac{2^{\mu/2}(\mu-\mu^2+2\nu(1+\nu))}{4\Gamma(2-\mu)}(1-\xi)+\mathcal{O}(1-\xi)^2\right]. \quad (368)$$

  This is the case of α real and $1/2<\alpha$ or $\alpha<0$, for example models like $F(R)=R+\gamma R^m$. The Legendre polynomial and therefore the Nariai horizon diverge. We have anti-evaporation (or evaporation if $\epsilon<0$ from the beginning).

- When μ is complex number

$$P_\nu^{i|\mu|}(\xi)\simeq(1-\xi)^{-\frac{i|\mu|}{2}}\left[\frac{2^{\frac{i|\mu|}{2}}}{\Gamma(1-i|\mu|)}-\frac{2^{\frac{i|\mu|}{2}}(1-\xi)}{4\Gamma(2-i|\mu|)}(|\mu|(i+|\mu|)+2\nu(\nu+1))+\mathcal{O}(1-\xi)^2\right]. \quad (369)$$

  This is the case of $0<\alpha<1/2$, for example models like $F(R)=R-2\Lambda(1-e^{R/R^*})$. The Legendre polynomial and therefore the Nariai horizon do not diverge. Solution is stable, we can have only transient evaporation/antievaporation.

It is convenient to write function $f(R)$ as

$$f(R) = R + \alpha h(R), \tag{370}$$

The field equations are

$$(1 + \alpha h_R)G_{\mu\nu} - \frac{1}{2}\alpha(h - h_R R)g_{\mu\nu} - \alpha(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)h_R = \frac{8\pi G}{c^4} T_{\mu\nu}\,. \tag{371}$$

Spherically symmetric metric with two independent functions of radial coordinate:

$$ds^2 = -e^{2\phi}c^2 dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{372}$$

The energy–momentum tensor $T_{\mu\nu} = \mathrm{diag}(e^{2\phi}\rho c^2, e^{2\lambda}P, r^2 P, r^2\sin^2\theta P)$, where ρ is the matter density and P is the pressure. The components of the field equations are

$$\begin{aligned} \frac{-8\pi G}{c^2}\rho \quad &= \quad -r^{-2} + e^{-2\lambda}(1 - 2r\lambda')r^{-2} + \alpha h_R(-r^{-2} + e^{-2\lambda}(1 - 2r\lambda')r^{-2}) \\ &\quad -\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha[h'_R r^{-1}(2 - r\lambda') + h''_R], \end{aligned} \tag{373}$$

$$\begin{aligned} \frac{8\pi G}{c^4}P \quad &= \quad -r^{-2} + e^{-2\lambda}(1 + 2r\phi')r^{-2} + \alpha h_R(-r^{-2} + e^{-2\lambda}(1 + 2r\phi')r^{-2}) \\ &\quad -\frac{1}{2}\alpha(h - h_R R) + e^{-2\lambda}\alpha h'_R r^{-1}(2 + r\phi'), \end{aligned} \tag{374}$$

where prime denotes derivative with respect to radial distance, r.

For the exterior solution, we assume a Schwarzschild solution. For this reason, it is convenient to define the change of variable

$$e^{-2\lambda} = 1 - \frac{2GM}{c^2 r} \,. \tag{375}$$

The value of parameter M on the surface of a neutron star can be considered as a gravitational star mass. Useful relation

$$\frac{GdM}{c^2 dr} = \frac{1}{2}\left[1 - e^{-2\lambda}(1 - 2r\lambda'] \right) \,, \tag{376}$$

.

The hydrostatic condition of equilibrium can be obtained from the Bianchi identities

$$\frac{dP}{dr} = -(\rho + P/c^2)\frac{d\phi}{dr} \,, . \tag{377}$$

The second TOV equation can be obtained by substitution of the derivative $d\phi/dr$ from (377) in Eq.(374). The dimensionless variables

$$M = mM_\odot, \quad r \to r_g r, \quad \rho \to \rho M_\odot/r_g^3, \quad P \to pM_\odot c^2/r_g^3, \quad R \to R/r_g^2.$$

Here $M_\odot$ is the Sun mass and $r_g = GM_\odot/c^2 = 1.47473$ km. Eqs. (373), (374) can be rewritten as

$$\left(1 + \alpha r_g^2 h_R + \frac{1}{2}\alpha r_g^2 h_R' r\right)\frac{dm}{dr} = 4\pi\rho r^2 - \frac{1}{4}\alpha r^2 r_g^2\left(h - h_R R - 2\left(1 - \frac{2m}{r}\right)\left(\frac{2h_R'}{r} + h_R''\right)\right) \,, \tag{378}$$

$$8\pi p = -2\left(1 + \alpha r_g^2 h_R\right)\frac{m}{r^3} - \left(1 - \frac{2m}{r}\right)\left(\frac{2}{r}(1 + \alpha r_g^2 h_R) + \alpha r_g^2 h_R'\right)(\rho + p)^{-1}\frac{dp}{dr} - \tag{379}$$

$$-\frac{1}{2}\alpha r_g^2\left(h - h_R R - 4\left(1 - \frac{2m}{r}\right)\frac{h_R'}{r}\right) \,,$$

where $' = d/dr$.

For $\alpha = 0$, Eqs. (378), (379) reduce to

$$\frac{dm}{dr} = 4\pi\tilde{\rho}r^2 \tag{380}$$

$$\frac{dp}{dr} = -\frac{4\pi pr^3 + m}{r(r-2m)}\left(\tilde{\rho} + p\right), \tag{381}$$

i.e. to ordinary dimensionless TOV equations. These equations can be solved numerically for a given EoS $p = f(\rho)$ and initial conditions $m(0) = 0$ and $\rho(0) = \rho_c$.

For non-zero α, one needs the third equation for the Ricci curvature scalar. The trace of field Eqs. (371) gives the relation

$$3\alpha\Box h_R + \alpha h_R R - 2\alpha h - R = -\frac{8\pi G}{c^4}(-3P + \rho c^2). \tag{382}$$

In dimensionless variables, we have

$$3\alpha r_g^2\left(\left(\frac{2}{r} - \frac{3m}{r^2} - \frac{dm}{rdr} - \left(1-\frac{2m}{r}\right)\frac{dp}{(\rho+p)dr}\right)\frac{d}{dr} + \left(1-\frac{2m}{r}\right)\frac{d^2}{dr^2}\right)h_R$$

$$+\alpha r_g^2 h_R R - 2\alpha r_g^2 h - R = -8\pi(\rho - 3p)\,. \tag{383}$$

We need to add the EoS for matter inside star to the Eqs. (378), (379), (383). Standard polytropic EoS $p \sim \rho^\gamma$ works, although a more realistic EoS has to take into account different physical states for different regions of the star and it is more complicated.

Perturbative solution. For a perturbative solution the density, pressure, mass and curvature can be expanded as

$$p = p^{(0)} + \alpha p^{(1)} + \ldots, \quad \rho = \rho^{(0)} + \alpha\rho^{(1)} + \ldots, \tag{384}$$

$$m = m^{(0)} + \alpha m^{(1)} + \ldots, \quad R = R^{(0)} + \alpha R^{(1)} + \ldots,$$

where functions $\rho^{(0)}$, $p^{(0)}$, $m^{(0)}$ and $R^{(0)}$ satisfy to standard TOV equations assumed at zeroth order. Terms containing h_R are assumed to be of first order in the small parameter α, so all such terms should be evaluated at $\mathcal{O}(\alpha)$ order.

For $m = m^{(0)} + \alpha m^{(1)}$, the following equation

$$\frac{dm}{dr} = 4\pi\rho r^2 - \alpha r^2 \left(4\pi\rho^{(0)} h_R + \frac{1}{4}(h - h_R R)\right) + \frac{1}{2}\alpha \left(\left(2r - 3m^{(0)} - 4\pi\rho^{(0)} r^3\right) \frac{d}{dr} + r(r - 2m^{(0)}) \frac{d^2}{dr^2}\right) h_R \tag{385}$$

for pressure $p = p^{(0)} + \alpha p^{(1)}$

$$\frac{r - 2m}{\rho + p} \frac{dp}{dr} = 4\pi r^2 p + \frac{m}{r} - \alpha r^2 \left(4\pi p^{(0)} h_R + \frac{1}{4}(h - h_R R)\right) - \alpha \left(r - 3m^{(0)} + 2\pi p^{(0)} r^3\right) \frac{dh_R}{dr}. \tag{386}$$

The Ricci curvature scalar, in terms containing h_R and h, has to be evaluated at $\mathcal{O}(1)$ order, i.e.

$$R \approx R^{(0)} = 8\pi(\rho^{(0)} - 3p^{(0)}) . \tag{387}$$

We can consider various EoS for the description of the behavior of nuclear matter at high densities. For example the SLy and FPS equation have the same analytical representation:

$$\zeta = \frac{a_1 + a_2\xi + a_3\xi^3}{1 + a_4\xi} f(a_5(\xi - a_6)) + (a_7 + a_8\xi) f(a_9(a_{10} - \xi)) + \tag{388}$$

$$+(a_{11} + a_{12}\xi) f(a_{13}(a_{14} - \xi)) + (a_{15} + a_{16}\xi) f(a_{17}(a_{18} - \xi)),$$

where

$$\zeta = \log(P/\text{dyncm}^{-2}) , \qquad \xi = \log(\rho/\text{gcm}^{-3}) , \qquad f(x) = \frac{1}{\exp(x) + 1} .$$

The coefficients a_i for SLy and FPS EoS are different.
Neutron star with a quark core. The quark matter can be described by the very simple EoS:

$$p_Q = a(\rho - 4B), \tag{389}$$

where a is a constant and the parameter B can vary from $\sim$ 60 to 90 Mev/fm^3.

For quark matter with massless strange quark, it is $a = 1/3$. We consider $a = 0.28$ corresponding to $m_s = 250$ Mev. For numerical calculations, Eq. (389) is used for $\rho \geq \rho_{tr}$, where ρ_{tr} is the transition density for which the pressure of quark matter coincides with the pressure of ordinary dense matter. For example for FPS equation, the transition density is $\rho_{tr} = 1.069 \times 10^{15}$ g/cm^3 ($B = 80$ Mev/fm^3), for SLy equation $\rho_{tr} = 1.029 \times 10^{15}$ g/cm^3 ($B = 60$ Mev/fm^3).

Model 1.

$$f(R) = R + \beta R(\exp(-R/R_0) - 1), \tag{390}$$

We can assume, for example, $R = 0.5 r_g^{-2}$. For $R << R_0$ this model coincides with quadratic model of $f(R)$ gravity.

For neutron stars models with quark core, there is no significant differences with respect to General Relativity. For a given central density, the star mass grows with α. The dependence is close to linear for $\rho \sim 10^{15}$g/cm^3. For the piecewise equation of state (FPS case for $\rho < \rho_{tr}$) the maximal mass grows with increasing α. For $\beta = -0.25$, the maximal mass is $1.53 M_\odot$, for $\beta = 0.25$, $M_{max} = 1.59 M_\odot$ (in General Relativity, it is $M_{max} = 1.55 M_\odot$). With an increasing β, the maximal mass is reached at lower central densities. Furthermore, for $dM/d\rho_c < 0$, there are no stable star configurations. A similar situation is observed in the SLy case but mass grows with β more slowly.

For the simplified EoS (388), other interesting effects can occur. For $\beta \sim -0.15$ at high central densities ($\rho_c \sim 3.0 - 3.5 \times 10^{15}$g/cm^3), we have the dependence of the neutron star mass from radius and from central density. For $\beta < 0$ for high central densities we have the stable star configurations ($dM/d\rho_c > 0$).

For example the measurement of mass of the neutron star PSR J1614-2230 with $1.97 \pm 0.04\ M_\odot$ provides a stringent constraint on any $M - R$ relation. The model with SLy equation is more interesting: in the context of model (390), the upper limit of neutron star mass is around $2M_\odot$ and there is second branch of stability star configurations at high central densities. This branch describes observational data better than the model with SLy EoS in GR.
Possibility of a stabilization mechanism in $f(R)$ gravity which leads to the existence of stable neutron stars which are more compact objects than in General Relativity. Cubic model.

$$f(R) = R + \alpha R^2(1 + \gamma R) \,. \tag{391}$$

Let $|\gamma R| \sim \mathcal{O}(1)$ for large R and $\alpha R^2(1 + \gamma R) << R$. For small masses, the results coincide with R^2 model. For $\gamma = -10$ (in units r_g^2) the maximal mass of neutron star at high densities $\rho > 3.7 \times 10^{15}$ g/cm^3 is nearly $1.88M_\odot$ and radius is about $\sim$ 9 km (SLy equation). For $\gamma = -20$ the maximal mass is $1.94M_\odot$ and radius is about $\sim$ 9.2 km . In the GR, for SLy equation, the minimal radius of neutron stars is nearly 10 km. Therefore such a model of $f(R)$ gravity can give rise to neutron stars with smaller radii than in GR. Therefore such theory can describe (assuming only the SLy equation), the existence of peculiar neutron stars with mass $\sim 2M_\odot$ (the measured mass of PSR J1614-2230) and compact stars ($R \sim 9$ km) with masses $M \sim 1.6 - 1.7M_\odot$.
For smaller values of γ the minimal neutron star mass (and minimal central density at which stable stars exist) on second branch of stability decreases.

It is interesting to note that for negative and sufficiently large values of ϵ, the maximal limit of neutron star mass can exceed the limit in General Relativity for given EoS (the stable stars exist for higher central densities). Therefore some EoS which ruled out by observational constraints in GR can describe real star configurations in frames of such model of gravity. One has to note that the upper limit in this model of gravity is achieved for smaller radii than in GR for acceptable EoS.

Topological Gauss-Bonnet invariant:

$$\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma} \,, \tag{392}$$

is added to the action of the Einstein gravity. One starts with the following action:

$$S = \int d^4x\sqrt{-g}\left(\frac{1}{2\kappa^2}R + f(\mathcal{G}) + \mathcal{L}_{\mathrm{matter}}\right) \,. \tag{393}$$

Here, $\mathcal{L}_{\mathrm{matter}}$ is the Lagrangian density of matter. The variation of the metric $g_{\mu\nu}$:

$$\begin{aligned} 0 =& \frac{1}{2\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + T^{\mu\nu}_{\mathrm{matter}} + \frac{1}{2}g^{\mu\nu}f(\mathcal{G}) - 2f'(\mathcal{G})RR^{\mu\nu} \\ &+ 4f'(\mathcal{G})R^{\mu}_{\ \rho}R^{\nu\rho} - 2f'(\mathcal{G})R^{\mu\rho\sigma\tau}R^{\nu}_{\ \rho\sigma\tau} - 4f'(\mathcal{G})R^{\mu\rho\sigma\nu}R_{\rho\sigma} + 2\left(\nabla^\mu\nabla^\nu f'(\mathcal{G})\right)R \\ &- 2g^{\mu\nu}\left(\nabla^2 f'(\mathcal{G})\right)R - 4\left(\nabla_\rho\nabla^\mu f'(\mathcal{G})\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu f'(\mathcal{G})\right)R^{\mu\rho} \\ &+ 4\left(\nabla^2 f'(\mathcal{G})\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\rho\sigma} - 4\left(\nabla_\rho\nabla_\sigma f'(\mathcal{G})\right)R^{\mu\rho\nu\sigma} \,. \end{aligned} \tag{394}$$

The first FRW equation:

$$0 = -\frac{3}{\kappa^2}H^2 - f(\mathcal{G}) + \mathcal{G}f'(\mathcal{G}) - 24\dot{\mathcal{G}}f''(\mathcal{G})H^3 + \rho_{\mathrm{matter}} \,. \tag{395}$$

Here $\mathcal{G}$ has the following form:

$$\mathcal{G} = 24\left(H^2\dot{H} + H^4\right) \,. \tag{396}$$

the FRW-like equations (fluid description):

$$\rho^{\mathcal{G}}_{\mathrm{eff}} = \frac{3}{\kappa^2}H^2 \,, \quad p^{\mathcal{G}}_{\mathrm{eff}} = -\frac{1}{\kappa^2}\left(3H^2 + 2\dot{H}\right) \,. \tag{397}$$

Here,

$$\rho_{\rm eff}^{\mathcal{G}} \equiv - f(\mathcal{G}) + \mathcal{G} f'(\mathcal{G}) - 24\dot{\mathcal{G}} f''(\mathcal{G}) H^3 + \rho_{\rm matter} \,,$$
$$p_{\rm eff}^{\mathcal{G}} \equiv f(\mathcal{G}) - \mathcal{G} f'(\mathcal{G}) + \frac{2\mathcal{G}\dot{\mathcal{G}}}{3H} f''(\mathcal{G}) + 8H^2 \ddot{\mathcal{G}} f''(\mathcal{G}) + 8H^2 \dot{\mathcal{G}}^2 f'''(\mathcal{G}) + p_{\rm matter} \,. \tag{398}$$

When $\rho_{\rm matter} = 0$, Eq. (395) has a de Sitter universe solution where H, and therefore $\mathcal{G}$, are constant. For $H = H_0$, with a constant H_0, Eq. (395) turns into

$$0 = -\frac{3}{\kappa^2} H_0^2 + 24 H_0^4 f'\left(24 H_0^4\right) - f\left(24 H_0^4\right) \,. \tag{399}$$

As an example, we consider the model

$$f(\mathcal{G}) = f_0 \left|\mathcal{G}\right|^{\beta} \,, \tag{400}$$

with constants f_0 and β. Then, the solution of Eq. (399) is given by

$$H_0^4 = \frac{1}{24\left(8\left(n-1\right)\kappa^2 f_0\right)^{\frac{1}{\beta-1}}} \,. \tag{401}$$

No matter and GR. Eq. (395) reduces to

$$0 = \mathcal{G} f'(\mathcal{G}) - f(\mathcal{G}) - 24\dot{\mathcal{G}} f''(\mathcal{G}) H^3 \,. \tag{402}$$

If $f(\mathcal{G})$ behaves as (400), assuming

$$a = \left\{ \begin{array}{ll} a_0 t^{h_0} & \text{when } h_0 > 0 \text{ (quintessence)} \\ a_0 \left(t_s - t\right)^{h_0} & \text{when } h_0 < 0 \text{ (phantom)} \end{array} \right. \,, \tag{403}$$

one obtains

$$0 = \left(\beta - 1\right) h_0^6 \left(h_0 - 1\right)\left(h_0 - 1 + 4\beta\right) \,. \tag{404}$$

As $h_0 = 1$ implies $\mathcal{G} = 0$, one may choose

$$h_0 = 1 - 4\beta\,, \tag{405}$$

and Eq. (**??**) gives

$$w_{\rm eff} = -1 + \frac{2}{3(1-4\beta)}\,. \tag{406}$$

Therefore, if $\beta > 0$, the universe is accelerating ($w_{\rm eff} < -1/3$), and if $\beta > 1/4$, the universe is in a phantom phase ($w_{\rm eff} < -1$). Thus, we are led to consider the following model:

$$f(\mathcal{G}) = f_i \left|\mathcal{G}\right|^{\beta_i} + f_l \left|\mathcal{G}\right|^{\beta_l}\,, \tag{407}$$

where it is assumed that

$$\beta_i > \frac{1}{2}\,, \quad \frac{1}{2} > \beta_l > \frac{1}{4}\,. \tag{408}$$

Then, when the curvature is large, as in the primordial universe, the first term dominates, compared with the second term and the Einstein term, and it gives

$$-1 > w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_i)} > -\frac{5}{3}\,. \tag{409}$$

On the other hand, when the curvature is small, as is the case in the present universe, the second term in (407) dominates compared with the first term and the Einstein term and yields

$$w_{\rm eff} = -1 + \frac{2}{3(1-4\beta_l)} < -\frac{5}{3}\,. \tag{410}$$

Therefore, theory (407) can produce a model that is able to describe inflation and the late-time acceleration of the universe in a unified manner.

The action (393) can be rewritten by introducing the auxiliary scalar field ϕ as,

$$S = \int d^4 x \sqrt{-g} \left[\frac{R}{2\kappa^2} - V(\phi) - \xi(\phi)\mathcal{G} \right] . \tag{411}$$

By variation over ϕ, one obtains

$$0 = V'(\phi) + \xi'(\phi)\mathcal{G} , \tag{412}$$

which could be solved with respect to ϕ as

$$\phi = \phi(\mathcal{G}) . \tag{413}$$

By substituting the expression (413) into the action (411), we obtain the action of $f(\mathcal{G})$ gravity, with

$$f(\mathcal{G}) = -V\left(\phi(\mathcal{G})\right) + \xi\left(\phi(\mathcal{G})\right)\mathcal{G} . \tag{414}$$

Assuming a spatially-flat FRW universe and the scalar field ϕ to depend only on t, we obtain the field equations:

$$0 = -\frac{3}{\kappa^2}H^2 + V(\phi) + 24H^3 \frac{d\xi(\phi(t))}{dt} , \tag{415}$$

$$\begin{aligned} 0 =& \frac{1}{\kappa^2}\left(2\dot{H} + 3H^2\right) - V(\phi) - 8H^2 \frac{d^2\xi(\phi(t))}{dt^2} \\ & - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} - 16H^3 \frac{d\xi(\phi(t))}{dt} . \end{aligned} \tag{416}$$

Combining the above equations, we obtain

$$\begin{aligned} 0 =& \frac{2}{\kappa^2}\dot{H} - 8H^2 \frac{d^2\xi(\phi(t))}{dt^2} - 16H\dot{H}\frac{d\xi(\phi(t))}{dt} + 8H^3 \frac{d\xi(\phi(t))}{dt} \\ =& \frac{2}{\kappa^2}\dot{H} - 8a\frac{d}{dt}\left(\frac{H^2}{a}\frac{d\xi(\phi(t))}{dt}\right) , \end{aligned} \tag{417}$$

which can be solved with respect to $\xi(\phi(t))$ as

$$\xi(\phi(t)) = \frac{1}{8}\int^t dt_1 \frac{a(t_1)}{H(t_1)^2} W(t_1)\,, \quad W(t) \equiv \frac{2}{\kappa^2}\int^t \frac{dt_1}{a(t_1)} \dot{H}(t_1)\,. \tag{418}$$

Combining (415) and (418), the expression for $V(\phi(t))$ follows:

$$V(\phi(t)) = \frac{3}{\kappa^2} H(t)^2 - 3a(t)H(t)W(t)\,. \tag{419}$$

As there is a freedom of redefinition of the scalar field ϕ, we may identify t with ϕ. Hence, we consider the model where $V(\phi)$ and $\xi(\phi)$ can be expressed in terms of a single function g as

$$\begin{aligned} V(\phi) =& \frac{3}{\kappa^2} g'(\phi)^2 - 3g'(\phi)\,\mathrm{e}^{g(\phi)} U(\phi)\,, \\ \xi(\phi) =& \frac{1}{8}\int^\phi d\phi_1 \frac{\mathrm{e}^{g(\phi_1)}}{g'(\phi_1)^2} U(\phi_1)\,, \\ U(\phi) \equiv& \frac{2}{\kappa^2}\int^\phi d\phi_1 \mathrm{e}^{-g(\phi_1)} g''(\phi_1)\,. \end{aligned} \tag{420}$$

By choosing $V(\phi)$ and $\xi(\phi)$ as (420), one can easily find the following solution for Eqs.(415) and (416):

$$a = a_0 \mathrm{e}^{g(t)} \quad \left(H = g'(t)\right)\,. \tag{421}$$

Therefore one can reconstruct $F(G)$ gravity to generate arbitrary expansion history of the universe.
Thus, we reviewed the modified Gauss-Bonnet gravity and demonstrated that it may naturally lead to the unified cosmic history, including the inflation and dark energy era.

Stringy gravity:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2} + \mathcal{L}_\phi + \mathcal{L}_c + \ldots\right] , \tag{422}$$

where ϕ is the dilaton, $\mathcal{L}_\phi$ is the Lagrangian of ϕ, and $\mathcal{L}_c$ expresses the string curvature correction terms,

$$\mathcal{L}_\phi = -\partial_\mu\phi\partial^\mu\phi - V(\phi) , \quad \mathcal{L}_c = c_1\alpha' \mathrm{e}^{2\frac{\phi}{\phi_0}}\mathcal{L}_c^{(1)} + c_2{\alpha'}^2\mathrm{e}^{4\frac{\phi}{\phi_0}}\mathcal{L}_c^{(2)} + c_3{\alpha'}^3\mathrm{e}^{6\frac{\phi}{\phi_0}}\mathcal{L}_c^{(3)} , \tag{423}$$

where $1/\alpha'$ is the string tension, $\mathcal{L}_c^{(1)}$, $\mathcal{L}_c^{(2)}$, and $\mathcal{L}_c^{(3)}$ express the leading-order (Gauss-Bonnet term $\mathcal{G}$ in (392)), the second-order, and the third-order curvature corrections, respectively:

$$\mathcal{L}_c^{(1)} = \Omega_2 , \quad \mathcal{L}_c^{(2)} = 2\Omega_3 + R^{\mu\nu}_{\alpha\beta}R^{\alpha\beta}_{\lambda\rho}R^{\lambda\rho}_{\mu\nu} , \quad \mathcal{L}_c^{(3)} = \mathcal{L}_{31} - \delta_H\mathcal{L}_{32} - \frac{\delta_B}{2}\mathcal{L}_{33} . \tag{424}$$

Here, δ_B and δ_H take the value of 0 or 1 and

$$\Omega_2 = \mathcal{G} ,$$

$$\Omega_3 \propto \epsilon^{\mu\nu\rho\sigma\tau\eta}\epsilon_{\mu'\nu'\rho'\sigma'\tau'\eta'}R_{\mu\nu}^{\ \ \mu'\nu'}R_{\rho\sigma}^{\ \ \rho'\sigma'}R_{\tau\eta}^{\ \ \tau'\eta'} ,$$

$$\mathcal{L}_{31} = \zeta(3)R_{\mu\nu\rho\sigma}R^{\alpha\nu\rho\beta}\left(R^{\mu\gamma}_{\ \ \delta\beta}R_{\alpha\gamma}^{\ \ \delta\sigma} - 2R^{\mu\gamma}_{\ \ \delta\alpha}R_{\beta\gamma}^{\ \ \delta\sigma}\right) ,$$

$$\mathcal{L}_{32} = \frac{1}{8}\left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 + \frac{1}{4}R_{\mu\nu}^{\ \ \gamma\delta}R_{\gamma\delta}^{\ \ \rho\sigma}R_{\rho\sigma}^{\ \ \alpha\beta}R_{\alpha\beta}^{\ \ \mu\nu} - \frac{1}{2}R_{\mu\nu}^{\ \ \alpha\beta}R_{\alpha\beta}^{\ \ \rho\sigma}R^{\mu}_{\ \sigma\gamma\delta}R_{\rho}^{\ \nu\gamma\delta} - R_{\mu\nu}^{\ \ \alpha\beta}R_{\alpha\beta}^{\ \ \rho\nu}R_{\rho\sigma}^{\ \ \gamma\delta}R_{\gamma\delta}^{\ \ \mu\sigma} ,$$

$$\mathcal{L}_{33} = \left(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right)^2 - 10R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\sigma}R_{\sigma\gamma\delta\rho}R^{\beta\gamma\delta\rho} - R_{\mu\nu\alpha\beta}R^{\mu\nu\rho}_{\ \ \ \ \sigma}R^{\beta\sigma\gamma\delta}R_{\delta\gamma\rho}^{\ \ \ \ \alpha} . \tag{425}$$

The correction terms are different depending on the type of string theory; the dependence is encoded in the curvature invariants and in the coefficients (c_1, c_2, c_3) and δ_H, δ_B, as follows,

- For the Type II superstring theory: $(c_1, c_2, c_3) = (0, 0, 1/8)$ and $\delta_H = \delta_B = 0$.
- For the heterotic superstring theory: $(c_1, c_2, c_3) = (1/8, 0, 1/8)$ and $\delta_H = 1$, $\delta_B = 0$.
- For the bosonic superstring theory: $(c_1, c_2, c_3) = (1/4, 1/48, 1/8)$ and $\delta_H = 0$, $\delta_B = 1$.

The starting action is:

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \xi(\phi)\mathcal{G}\right] . \tag{426}$$

Field equations:

$$\begin{aligned}
0 = &\frac{1}{\kappa^2}\left(-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R\right) + \frac{1}{2}\partial^\mu\phi\partial^\nu\phi - \frac{1}{4}g^{\mu\nu}\partial_\rho\phi\partial^\rho\phi + \frac{1}{2}g^{\mu\nu}\left(-V(\phi) + \xi(\phi)\mathcal{G}\right)\\
&- 2\xi(\phi)RR^{\mu\nu} - 4\xi(\phi)R^\mu{}_\rho R^{\nu\rho} - 2\xi(\phi)R^{\mu\rho\sigma\tau}R^\nu{}_{\rho\sigma\tau} + 4\xi(\phi)R^{\mu\rho\nu\sigma}R_{\rho\sigma}\\
&+ 2\left(\nabla^\mu\nabla^\nu\xi(\phi)\right)R - 2g^{\mu\nu}\left(\nabla^2\xi(\phi)\right)R - 4\left(\nabla_\rho\nabla^\mu\xi(\phi)\right)R^{\nu\rho} - 4\left(\nabla_\rho\nabla^\nu\xi(\phi)\right)R^{\mu\rho}\\
&+ 4\left(\nabla^2\xi(\phi)\right)R^{\mu\nu} + 4g^{\mu\nu}\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\rho\sigma} + 4\left(\nabla_\rho\nabla_\sigma\xi(\phi)\right)R^{\mu\rho\nu\sigma} .
\end{aligned} \tag{427}$$

FRW eq.:

$$0 = -\frac{3}{\kappa^2}H^2 + \frac{1}{2}\dot\phi^2 + V(\phi) + 24H^3\frac{d\xi(\phi(t))}{dt} , \tag{428}$$

$$\begin{aligned}
0 = &\frac{1}{\kappa^2}\left(2\dot H + 3H^2\right) + \frac{1}{2}\dot\phi^2 - V(\phi) - 8H^2\frac{d^2\xi(\phi(t))}{dt^2}\\
&- 16H\dot H\frac{d\xi(\phi(t))}{dt} - 16H^3\frac{d\xi(\phi(t))}{dt} .
\end{aligned} \tag{429}$$

Scalar equation

$$0 = \ddot\phi + 3H\dot\phi + V'(\phi) + \xi'(\phi)\mathcal{G} . \tag{430}$$

In particular when we consider the following string-inspired model,

$$V = V_0 \mathrm{e}^{-\frac{2\phi}{\phi_0}} , \quad \xi(\phi) = \xi_0 \mathrm{e}^{\frac{2\phi}{\phi_0}} , \tag{431}$$

the de Sitter space solution follows:

$$H^2 = H_0^2 \equiv - \frac{\mathrm{e}^{-\frac{2\varphi_0}{\phi_0}}}{8\xi_0 \kappa^2} , \quad \phi = \varphi_0 . \tag{432}$$

Here, φ_0 is an arbitrary constant. If φ_0 is chosen to be larger, the Hubble rate $H = H_0$ becomes smaller. Then, if $\xi_0 \sim \mathcal{O}(1)$, by choosing $\varphi_0/\phi_0 \sim 140$, the value of the Hubble rate $H = H_0$ is consistent with the observations. The model (431) also has another solution:

$$\begin{aligned} H &= \frac{h_0}{t} , \quad \phi = \phi_0 \ln \frac{t}{t_1} \qquad \text{when } h_0 > 0 , \\ H &= - \frac{h_0}{t_s - t} , \quad \phi = \phi_0 \ln \frac{t_s - t}{t_1} \qquad \text{when } h_0 < 0 . \end{aligned} \tag{433}$$

Here, h_0 is obtained by solving the following algebraic equations:

$$0 = - \frac{3h_0^2}{\kappa^2} + \frac{\phi_0^2}{2} + V_0 t_1^2 - \frac{48\xi_0 h_0^3}{t_1^2} , \quad 0 = (1 - 3h_0)\, \phi_0^2 + 2V_0 t_1^2 + \frac{48\xi_0 h_0^3}{t_1^2} (h_0 - 1) . \tag{434}$$

Eqs. (434) can be rewritten as

$$V_0 t_1^2 = - \frac{1}{\kappa^2 (1 + h_0)} \left\{ 3h_0^2 (1 - h_0) + \frac{\phi_0^2 \kappa^2 (1 - 5h_0)}{2} \right\} , \tag{435}$$

$$\frac{48\xi_0 h_0^2}{t_1^2} = - \frac{6}{\kappa^2 (1 + h_0)} \left(h_0 - \frac{\phi_0^2 \kappa^2}{2} \right) . \tag{436}$$

The arbitrary value of h_0 can be realized by properly choosing V_0 and ξ_0. With the appropriate choice of V_0 and ξ_0, we can obtain a negative h_0 and, therefore, the effective EoS parameter (**??**) is less than -1, $w_{\mathrm{eff}} < -1$, which corresponds to the effective phantom.

$F(R)$ bigravity

Non-linear massive gravity (with non-dynamical background metric) was extended to the ghost-free construction with the dynamical metric (Hassan et al).
The convenient description of the theory gives bigravity or bimetric gravity which contains two metrics (symmetric tensor fields). One of two metrics is called physical metric while second metric is called reference metric.
Next is $F(R)$ bigravity which is also ghost-free theory. We introduce four kinds of metrics, $g_{\mu\nu}$, $g^{\mathrm{J}}_{\mu\nu}$, $f_{\mu\nu}$, and $f^{\mathrm{J}}_{\mu\nu}$. The physical observable metric $g^{\mathrm{J}}_{\mu\nu}$ is the metric in the Jordan frame. The metric $g_{\mu\nu}$ corresponds to the metric in the Einstein frame in the standard $F(R)$ gravity and therefore the metric $g_{\mu\nu}$ is not physical metric. In the bigravity theories, we have to introduce another reference metrics or symmetric tensor $f_{\mu\nu}$ and $f^{\mathrm{J}}_{\mu\nu}$. The metric $f_{\mu\nu}$ is the metric corresponding to the Einstein frame with respect to the curvature given by the metric $f_{\mu\nu}$. On the other hand, the metric $f^{\mathrm{J}}_{\mu\nu}$ is the metric corresponding to the Jordan frame.
The starting action is given by

$$\begin{aligned} S_{\mathrm{bi}} =& M_g^2 \int d^4 x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4 x \sqrt{-\det f}\, R^{(f)} \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4 x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n\, e_n \left(\sqrt{g^{-1} f}\right) . \end{aligned} \tag{438}$$

Here $R^{(g)}$ is the scalar curvature for $g_{\mu\nu}$ and $R^{(f)}$ is the scalar curvature for $f_{\mu\nu}$. M_{eff} is defined by

$$\frac{1}{M_{\mathrm{eff}}^2} = \frac{1}{M_g^2} + \frac{1}{M_f^2} . \tag{439}$$

Furthermore, tensor $\sqrt{g^{-1}f}$ is defined by the square root of $g^{\mu\rho} f_{\rho\nu}$, that is, $\left(\sqrt{g^{-1}f}\right)^{\mu}_{\ \rho} \left(\sqrt{g^{-1}f}\right)^{\rho}_{\ \nu} = g^{\mu\rho} f_{\rho\nu}$.

For general tensor $X^\mu_{\ \nu}$, $e_n(X)$'s are defined by

$$\begin{aligned} & e_0(X) = 1\,, \quad e_1(X) = [X]\,, \quad e_2(X) = \tfrac{1}{2}([X]^2 - [X^2])\,, \\ & e_3(X) = \tfrac{1}{6}([X]^3 - 3[X][X^2] + 2[X^3])\,, \\ & e_4(X) = \tfrac{1}{24}([X]^4 - 6[X]^2[X^2] + 3[X^2]^2 + 8[X][X^3] - 6[X^4])\,, \\ & e_k(X) = 0 \ \text{ for } k > 4\,. \end{aligned} \tag{440}$$

Here $[X]$ expresses the trace of arbitrary tensor $X^\mu_{\ \nu}$: $[X] = X^\mu_{\ \mu}$. In order to construct the consistent $F(R)$ bigravity, we add the following terms to the action (438):

$$S_\varphi = -M_g^2 \int d^4x \sqrt{-\det g}\left\{\frac{3}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + V(\varphi)\right\} + \int d^4x \mathcal{L}_{\mathrm{matter}}\left(\mathrm{e}^{\varphi} g_{\mu\nu}, \Phi_i\right)\,, \tag{441}$$

$$S_\xi = -M_f^2 \int d^4x \sqrt{-\det f}\left\{\frac{3}{2} f^{\mu\nu}\partial_\mu\xi\partial_\nu\xi + U(\xi)\right\}\,. \tag{442}$$

By the conformal transformations $g_{\mu\nu} \to \mathrm{e}^{-\varphi} g^{\mathrm{J}}_{\mu\nu}$ and $f_{\mu\nu} \to \mathrm{e}^{-\xi} f^{\mathrm{J}}_{\mu\nu}$, the total action $S_F = S_{\mathrm{bi}} + S_\varphi + S_\xi$ is transformed as

$$\begin{aligned} S_F =& M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}}\left\{\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} - \mathrm{e}^{-2\xi} U(\xi)\right\} \\ & + 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) \\ & + M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}}\left\{\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} - \mathrm{e}^{-2\varphi} V(\varphi)\right\} \\ & + \int d^4x \mathcal{L}_{\mathrm{matter}}\left(g^{\mathrm{J}}_{\mu\nu}, \Phi_i\right)\,. \end{aligned} \tag{443}$$

The kinetic terms for φ and ξ vanish. By the variations with respect to φ and ξ as in the case of convenient $F(R)$ gravity, we obtain

$$0 = 2m^2 M_{\text{eff}}^2 \sum_{n=0}^{4} \beta_n \left(\frac{n}{2} - 2\right) \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) + M_g^2 \left\{-\mathrm{e}^{-\varphi} R^{\mathrm{J}(g)} + 2\mathrm{e}^{-2\varphi} V(\varphi) + \mathrm{e}^{-2\varphi} V'(\varphi)\right\} , \tag{444}$$

$$0 = -\, 2m^2 M_{\text{eff}}^2 \sum_{n=0}^{4} \frac{\beta_n n}{2} \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi - \frac{n}{2}\xi} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) + M_f^2 \left\{-\mathrm{e}^{-\xi} R^{\mathrm{J}(f)} + 2\mathrm{e}^{-2\xi} U(\xi) + \mathrm{e}^{-2\xi} U'(\xi)\right\} . \tag{445}$$

The Eqs. (444) and (445) can be solved algebraically with respect to φ and ξ as

$$\varphi = \varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)$$

and

$$\xi = \xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)$$

. Substituting above φ and ξ into (443), one gets $F(R)$ bigravity:

$$S_F = M_f^2 \int d^4x \sqrt{-\det f^{\mathrm{J}}}\, F^{(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right) + 2m^2 M_{\text{eff}}^2 \int d^4x \sqrt{-\det g} \sum_{n=0}^{4} \beta_n \mathrm{e}^{\left(\frac{n}{2}-2\right)\varphi\left(R^{\mathrm{J}(g)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right)} e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right) + M_g^2 \int d^4x \sqrt{-\det g^{\mathrm{J}}}\, F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1} f^{\mathrm{J}}}\right)\right) + \int d^4x \mathcal{L}_{\text{matter}}\left(g_{\mu\nu}^{\mathrm{J}}, \Phi_i\right) , \tag{446}$$

$$F^{\mathrm{J}(g)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(g)} \right.$$
$$\left. -\mathrm{e}^{-2\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} V\left(\varphi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right)\right\}, \qquad (447)$$

$$F^{(f)}\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right) \equiv \left\{ \mathrm{e}^{-\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} R^{\mathrm{J}(f)} \right.$$
$$\left. -\mathrm{e}^{-2\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)} U\left(\xi\left(R^{\mathrm{J}(g)}, R^{\mathrm{J}(f)}, e_n\left(\sqrt{g^{\mathrm{J}\,-1}f^{\mathrm{J}}}\right)\right)\right)\right\}. \qquad (448)$$

Note that it is difficult to solve Eqs. (444) and (445) with respect to φ and ξ explicitly. Therefore, it might be easier to define the model in terms of the auxiliary scalars φ and ξ as in (443).

Let us consider the cosmological reconstruction program. For simplicity, we start from the minimal case

$$
\begin{aligned}
S_{\mathrm{bi}} =& M_g^2 \int d^4x \sqrt{-\det g}\, R^{(g)} + M_f^2 \int d^4x \sqrt{-\det f}\, R^{(f)} \\
&+ 2m^2 M_{\mathrm{eff}}^2 \int d^4x \sqrt{-\det g} \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} + \det \sqrt{g^{-1}f} \right) . \qquad (449)
\end{aligned}
$$

In order to evaluate $\delta\sqrt{g^{-1}f}$, two matrices M and N, which satisfy the relation $M^2 = N$ are taken. Since $\delta M M + M \delta M = \delta N$, one finds

$$
\mathrm{tr}\, \delta M = \frac{1}{2} \mathrm{tr}\, \left(M^{-1} \delta N \right) . \qquad (450)
$$

For a while, we consider the Einstein frame action (449) with (441) and (442) but matter contribution is neglected. Then by the variation over $g_{\mu\nu}$, we obtain

$$
\begin{aligned}
0 =& M_g^2 \left(\frac{1}{2} g_{\mu\nu} R^{(g)} - R^{(g)}_{\mu\nu} \right) + m^2 M_{\mathrm{eff}}^2 \left\{ g_{\mu\nu} \left(3 - \mathrm{tr}\, \sqrt{g^{-1}f} \right) \right. \\
&\left. + \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1}f} \right)^{-1\,\rho}_{\quad\ \nu} + \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1}f} \right)^{-1\,\rho}_{\quad\ \mu} \right\} \\
&+ M_g^2 \left[\frac{1}{2} \left(\frac{3}{2} g^{\rho\sigma} \partial_\rho \varphi \partial_\sigma \varphi + V(\varphi) \right) g_{\mu\nu} - \frac{3}{2} \partial_\mu \varphi \partial_\nu \varphi \right] . \qquad (451)
\end{aligned}
$$

On the other hand, by the variation over $f_{\mu\nu}$, we get

$$
\begin{aligned}
0 =& M_f^2 \left(\frac{1}{2} f_{\mu\nu} R^{(f)} - R^{(f)}_{\mu\nu} \right) + m^2 M_{\mathrm{eff}}^2 \sqrt{\det \left(f^{-1} g \right)} \left\{ - \frac{1}{2} f_{\mu\rho} \left(\sqrt{g^{-1}f} \right)^{\rho}_{\ \nu} \right. \\
&\left. - \frac{1}{2} f_{\nu\rho} \left(\sqrt{g^{-1}f} \right)^{\rho}_{\ \mu} + \det \left(\sqrt{g^{-1}f} \right) f_{\mu\nu} \right\} + M_f^2 \left[\frac{1}{2} \left(\frac{3}{2} f^{\rho\sigma} \partial_\rho \xi \partial_\sigma \xi + U(\xi) \right) f_{\mu\nu} - \frac{3}{2} \partial_\mu \xi \partial_\nu \xi \right] . \qquad (452)
\end{aligned}
$$

We should note that $\det\sqrt{g}\det\sqrt{g^{-1}f} \neq \sqrt{\det f}$ in general. The variations of the scalar fields φ and ξ are given by

$$0 = -3\Box_g\varphi + V'(\varphi)\,, \quad 0 = -3\Box_f\xi + U'(\xi)\,. \tag{453}$$

Here $\Box_g$ ($\Box_f$) is the d'Alembertian with respect to the metric g (f). By multiplying the covariant derivative ∇_g^μ with respect to the metric g with Eq. (451) and using the Bianchi identity $0 = \nabla_g^\mu\left(\frac{1}{2}g_{\mu\nu}R^{(g)} - R^{(g)}_{\mu\nu}\right)$ and Eq. (453), we obtain

$$0 = -\, g_{\mu\nu}\nabla_g^\mu\left(\mathrm{tr}\sqrt{g^{-1}f}\right) + \frac{1}{2}\nabla_g^\mu\left\{f_{\mu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\nu} + f_{\nu\rho}\left(\sqrt{g^{-1}f}\right)^{-1\,\rho}{}_{\mu}\right\}\,. \tag{454}$$

Similarly by using the covariant derivative ∇_f^μ with respect to the metric f, from (452), we obtain

$$0 = \nabla_f^\mu\left[\sqrt{\det\left(f^{-1}g\right)}\left\{-\frac{1}{2}\left(\sqrt{g^{-1}f}\right)^{-1\nu}{}_{\sigma}g^{\sigma\mu} - \frac{1}{2}\left(\sqrt{g^{-1}f}\right)^{-1\mu}{}_{\sigma}g^{\sigma\nu} + \det\left(\sqrt{g^{-1}f}\right)f^{\mu\nu}\right\}\right]\,. \tag{455}$$

In case of the Einstein gravity, the conservation law of the energy-momentum tensor depends from the Einstein equation. It can be derived from the Bianchi identity. In case of bigravity, however, the conservation laws of the energy-momentum tensor of the scalar fields are derived from the scalar field equations. These conservation laws are independent of the Einstein equation. The Bianchi identities give equations (454) and (455) independent of the Einstein equation.

We now assume the FRW universes for the metrics $g_{\mu\nu}$ and $f_{\mu\nu}$ and use the conformal time t for the universe with metric $g_{\mu\nu}$:

$$ds_g^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu = a(t)^2\left(-dt^2 + \sum_{i=1}^{3}\left(dx^i\right)^2\right)\,,$$

$$ds_f^2 = \sum_{\mu,\nu=0}^{3} f_{\mu\nu}dx^\mu dx^\nu = -c(t)^2dt^2 + b(t)^2\sum_{i=1}^{3}\left(dx^i\right)^2\,. \tag{456}$$

Then (t, t) component of (451) gives

$$0 = -3M_g^2 H^2 - 3m^2 M_{\text{eff}}^2 \left(a^2 - ab\right) + \left(\frac{3}{4}\dot{\varphi}^2 + \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 \,, \tag{457}$$

and (i, j) components give

$$\begin{aligned} 0 =& M_g^2 \left(2\dot{H} + H^2\right) + m^2 M_{\text{eff}}^2 \left(3a^2 - 2ab - ac\right) \\ &+ \left(\frac{3}{4}\dot{\varphi}^2 - \frac{1}{2}V(\varphi)a(t)^2\right) M_g^2 \,. \end{aligned} \tag{458}$$

Here $H = \dot{a}/a$. On the other hand, (t, t) component of (452) gives

$$0 = -3M_f^2 K^2 + m^2 M_{\text{eff}}^2 c^2 \left(1 - \frac{a^3}{b^3}\right) + \left(\frac{3}{4}\dot{\xi}^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,, \tag{459}$$

and (i, j) components give

$$\begin{aligned} 0 =& M_f^2 \left(2\dot{K} + 3K^2 - 2LK\right) + m^2 M_{\text{eff}}^2 \left(\frac{a^3 c}{b^2} - c^2\right) \\ &+ \left(\frac{3}{4}\dot{\xi}^2 - \frac{1}{2}U(\xi)c(t)^2\right) M_f^2 \,. \end{aligned} \tag{460}$$

Here $K = \dot{b}/b$ and $L = \dot{c}/c$. Both of Eq. (454) and Eq. (455) give the identical equation:

$$cH = bK \text{ or } \frac{c\dot{a}}{a} = \dot{b} \,. \tag{461}$$

If $\dot{a} \neq 0$, we obtain $c = a\dot{b}/\dot{a}$. On the other hand, if $\dot{a} = 0$, we find $\dot{b} = 0$, that is, a and b are constant and c can be arbitrary.

We now redefine scalars as $\varphi = \varphi(\eta)$ and $\xi = \xi(\zeta)$ and identify η and ζ with the conformal time t, $\eta = \zeta = t$. Hence, one gets

$$\omega(t)M_g^2 = -4M_g^2\left(\dot{H} - H^2\right) - 2m^2 M_{\rm eff}^2(ab - ac)\,, \tag{462}$$

$$\tilde{V}(t)a(t)^2 M_g^2 = M_g^2\left(2\dot{H} + 4H^2\right) + m^2 M_{\rm eff}^2(6a^2 - 5ab - ac)\,, \tag{463}$$

$$\sigma(t)M_f^2 = -4M_f^2\left(\dot{K} - LK\right) - 2m^2 M_{\rm eff}^2\left(-\frac{c}{b} + 1\right)\frac{a^3 c}{b^2}\,, \tag{464}$$

$$\tilde{U}(t)c(t)^2 M_f^2 = M_f^2\left(2\dot{K} + 6K^2 - 2LK\right) + m^2 M_{\rm eff}^2\left(\frac{a^3 c}{b^2} - 2c^2 + \frac{a^3 c^2}{b^3}\right)\,. \tag{465}$$

Here

$$\omega(\eta) = 3\varphi'(\eta)^2\,, \quad \tilde{V}(\eta) = V\left(\varphi\left(\eta\right)\right)\,, \quad \sigma(\zeta) = 3\xi'(\zeta)^2\,, \quad \tilde{U}(\zeta) = U\left(\xi\left(\zeta\right)\right)\,. \tag{466}$$

Therefore for arbitrary $a(t)$, $b(t)$, and $c(t)$ if we choose $\omega(t)$, $\tilde{V}(t)$, $\sigma(t)$, and $\tilde{U}(t)$ to satisfy Eqs. (462-465), the cosmological model with given $a(t)$, $b(t)$ and $c(t)$ evolution can be reconstructed. Following this technique we presented number of inflationary and/or dark energy models as well as unified inflation-dark energy cosmologies. The method is general and maybe applied to more exotic and more complicated cosmological solutions.

What is the next?

What is the next? So far F(R) gravity which also admits extensions as HL or massive gravity is considered to be the best: simplest formulation, ghost-free, easy emergence of unified description for the universe evolution, friendly passing of cosmological bounds and local tests, absence of singularities in some versions(Bamba-Nojiri-Odintsov 2007), possibility of easy further modifications. More deep cosmological tests are necessary to understand if this is final phenomenological theory of universe and how it is related with yet to be constructed QG!