Dirac analysis of cosmological perturbation theory

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Motivation

Cosmological perturbation theory, applicable to early universe,

- deals with the dynamics of linear perturbations to the FLRW universe
- admits the gauge group of linear space-time diffeomorphisms (the issue of coordinate dependence)
- within the canonical formalism we have a standard set of concepts to handle gauge symmetries such as first-class constraints, gauge fixing conditions, gauge invariant quantities, Dirac brackets, canonical isomorphisms between gauge-fixed surfaces. . . (Dirac procedure)
- the knowledge of the canonical gauge-invariant quantities and their Hamiltonian dynamics are necessary for canonical quantization
- method for handling gauge-fixing conditions is necessary to obtain physical interpretation of the formalism

Previous results

The approach by *D. Langlois, CQG 11 389-407 (1994)*:

- starts with the ADM canonical formalism
- employs a method by *Goldberg, Newman and Rovelli, J Math Phys 32(10) (1991)* inspired by the Hamilton-Jacobi theory
- makes a canonical transformation that separates physical DOFs from gauge DOFs in the kinematical phase space
- provides an efficient computation of the physical Hamiltonian (in terms of physical DOFs) as e.g. the curvature term vanishes (a gauge quantity)

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In this talk, I will present a "new" approach:

- based on the Dirac procedure for constrained systems
- it not only identifies the gauge-invariant variables but also uses them to establish the relation between different gauge-fixing conditions (which give them physical meaning)
- gives a quick derivation of the physical Hamiltonian and this has a simple explanation

ADM + Schutz formalism

Gravity + perfect fluid ($p = w\rho$),

$$\mathbf{H} = \int_\Sigma (N\mathcal{H}_0 + N^a\mathcal{H}_a)\,\mathrm{d}^3x, \quad (q_{ab}, \pi^{ab}, \phi, p^\phi),\ \text{where}$$

$$\mathcal{H}_0 = \underbrace{\sqrt{q}\left(-{}^3R + q^{-1}(\pi_a{}^b\pi_b{}^a - \frac{1}{2}\pi^2)\right)}_{\mathcal{H}_{g,0}} + \underbrace{\frac{(p^\phi)^2}{\sqrt{q}K\alpha\mu^{\alpha-2}} - \sqrt{q}K\mu^\alpha}_{\mathcal{H}_{f,0}},$$

$$\mathcal{H}_a = \underbrace{-2D_b\pi_a{}^b}_{\mathcal{H}_{g,a}} + \underbrace{p^\phi\phi_{,a}}_{\mathcal{H}_{f,a}}.$$

The fluid pressure: $p(\mu) = K\mu^\alpha, \quad \alpha = \frac{w+1}{w}$.

The fluid's flow: $U_\nu = \mu^{-1}\phi_{,\nu}$, where $\frac{1}{N}\phi_{,0} = \frac{p^\phi}{\sqrt{q}K\alpha\mu^{\alpha-2}} + \frac{N^a}{N}\phi_{,a}$.

The Hamiltonian: $\mathcal{H}_{f,0} = \sqrt{q}\cdot\rho\big|_{t=const}, \quad \mathcal{H}_{f,a} = -N\sqrt{q}(\rho_0 + p)U^0U_a$.

Expansion of ADM + Schutz formalism

FLRW + perturbations split:

$$\delta q_{ab} = q_{ab} - a^2\delta_{ab}, \quad \delta\pi^{ab} = \pi^{ab} - \frac{1}{3}p\delta^{ab}, \quad \delta\phi = \phi - \bar{\phi}, \quad \delta p^{\phi} = p^{\phi} - \bar{p}^{\phi},$$

$$a^2 = \int \frac{q_{ab}\delta^{ab}}{3}\,\mathrm{d}^3x, \quad p = \int \pi^{ab}\delta_{ab}\,\mathrm{d}^3x, \quad \bar{\phi} = \int \phi\,\mathrm{d}^3x, \quad \bar{p}^{\phi} = \int p^{\phi}\,\mathrm{d}^3x,$$

The Poisson brackets:

$$\{\delta\phi(x), \delta p^{\phi}(x')\} = \delta^3(x - x'), \quad \{\delta q_{ab}(x), \delta\pi^{cd}(x')\} = \delta_{(a}^{\ c}\delta_{b)}^{\ d}\delta^3(x - x'),$$

$$\{\bar{\phi}, \bar{p}^{\phi}\} = 1, \quad \{a^2, p\} = 1.$$

Canonical cosmological perturbation theory

The total Hamiltonian is a combination of the zero-order constraint, first-order constraints and a nonvanishing second-order term:

$$\mathbf{H} = N\mathcal{H}_0^{(0)} + \int_\Sigma N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^a \delta\mathcal{H}_a$$

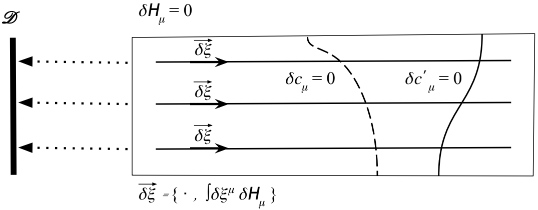
Not an exact gauge system. Time problem at zero order. No time problem at first order.

One explicit formula,

$$\mathcal{H}_{g,0}\big|^{(2)} = a\delta\pi_{ab}\delta\pi^{ab} - \frac{1}{2}a(\delta\pi)^2 + \frac{a^{-1}p}{3}\delta\pi^{ab}\delta q_{ab} - \frac{a^{-1}p}{6}\delta\pi\delta q + \frac{5a^{-3}p^2}{72}\delta q_{ab}\delta q^{ab}$$
$$+ \frac{7a^{-3}p^2}{48}(\delta q)^2 + \frac{a^{-3}}{2}\left(\delta q_{ab,ab}\delta q + \frac{3}{2}\delta q_{aa,c}\delta q_{bb,c} - \delta q_{ab,b}\delta q_{ac,c} + \frac{1}{2}\delta q_{ab,c}\delta q_{ab,c}\right).$$

Dirac procedure

$$\mathbf{H}^{(2)} = \int N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^a \delta\mathcal{H}_a$$



Dirac observables D:

$$\delta\vec{\xi}(D) \approx 0, \quad \delta\vec{\xi}(\cdot) = \{\cdot\ , \int \delta\xi^\mu \delta\mathcal{H}_\mu\}$$

Gauge-fixing conditions "$\delta c_\mu = 0$":

$$det|\{\delta\phi_\mu, \delta\phi_\nu\}| \neq 0, \quad \delta\phi_\mu \in \{\delta\mathcal{H}_\mu, \delta c_\mu\}$$

Determination of δN, δN^a:

$$\{\delta c_\mu, \mathbf{H}^{(2)}\} \approx \int N\{\delta c_\mu, \mathcal{H}_0^{(2)}\} + \delta N\{\delta c_\mu, \delta\mathcal{H}_0\} + \delta N^a\{\delta c_\mu, \delta\mathcal{H}_a\} \approx 0$$

Dirac procedure

Dirac brackets:

$$\{\cdot,\cdot\}_D = \{\cdot,\cdot\} - \{\cdot,\delta\phi_\mu\}\{\delta\phi_\mu,\delta\phi_\nu\}^{-1}\{\delta\phi_\nu,\cdot\}$$

Properties:

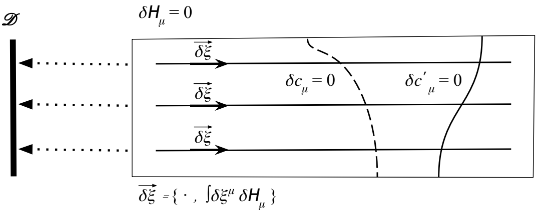
1. $\{\delta\phi_\mu,\cdot\,\}_D = 0,$
2. $\{D_i, D_j\}_D \approx D_k \approx \{D_i, D_j\},$
3. $\{\cdot,\cdot\}_D = \{\cdot, D_i\}\{D_i, D_j\}^{-1}\{D_j,\cdot\},\ \text{where } \{D_j,\delta\phi_\mu\} = 0.$

Reduced Hamiltonian to be used with $\{\cdot,\cdot\}_D$:

$$\mathbf{H}^{(2)} = \int (N\mathcal{H}_0^{(2)} + \delta N\delta\mathcal{H}_0 + \delta N^a\delta\mathcal{H}_a)\Big|_{\delta\phi_\mu=0,\ \mu=1,\dots,8}$$

$$= N\int \mathcal{H}_0^{(2)}\Big|_{\delta\phi_\mu=0,\ \mu=1,\dots,8} = N\int \mathcal{H}\left(D\Big|_{\delta\phi_\mu=0,\ \mu=1,\dots,8}\right)$$

Relation between gauges



Suppose there are two sets of gauge fixing conditions,

$$\delta c_\mu = 0 \quad \text{and} \quad \delta c'_\mu = 0,$$

then, there exists the canonical isomorphism

$$D_i|_{\delta c_\mu = 0} \mapsto D_i|_{\delta c'_\mu = 0},$$

$$\left\{ D_i|_{\delta c_\mu = 0}, D_j|_{\delta c_\mu = 0} \right\}_D \approx \left\{ D_i|_{\delta c'_\mu = 0}, D_j|_{\delta c'_\mu = 0} \right\}_{D'} \approx \left\{ D_i, D_j \right\}.$$

SVT Decomposition

We Fourier-transform the metric perturbations:

$$\delta\check{q}_{ab} = \underbrace{\delta q_1 A^1_{ab} + \delta q_2 A^2_{ab}}_{scalar\ \ perts} + \underbrace{\delta q_3 A^3_{ab} + \delta q_4 A^4_{ab}}_{vector\ \ perts} + \underbrace{\delta q_5 A^5_{ab} + \delta q_6 A^6_{ab}}_{tensor\ \ perts}$$

SVT decomposition $(\vec{k}, \vec{v}, \vec{w})$:

$$\mathbf{H}^{(2)} = \underbrace{\int_\Sigma N\mathcal{H}_0^{(2S)} + \delta N \delta\mathcal{H}_0 + \delta N^{\vec{k}} \delta\mathcal{H}_{\vec{k}}}_{scalar\ \ part}$$
$$+ \underbrace{\int_\Sigma N\mathcal{H}_0^{(2V)} + \delta N^{\vec{v}} \delta\mathcal{H}_{\vec{v}} + \delta N^{\vec{w}} \delta\mathcal{H}_{\vec{w}}}_{vector\ \ part} + \underbrace{\int_\Sigma N\mathcal{H}_0^{(2T)}}_{tensor\ \ part}$$

phase space $= \mathbb{R}^6$ (scalars) $\times$ $\mathbb{R}^4$ (vectors) $\times$ $\mathbb{R}^4$ (tensors)

Reduced phase space for vector perturbations

The reduced Hamiltonian:

$$\mathbf{H}_V^{(2)} = \int_\Sigma N\mathcal{H}_0^{(2V)} + \delta N^{\vec{v}}\delta\mathcal{H}_{\vec{v}} + \delta N^{\vec{w}}\delta\mathcal{H}_{\vec{w}}$$

The phase space:

$$(\delta q_3, \delta q_4, \delta\pi_3, \delta\pi_4)$$

The gauge-fixing conditions:

$$\delta c_{\vec{v}} = 0,\ \delta c_{\vec{w}} = 0 \implies \delta q_3 = \delta q_4 = \delta\pi_3 = \delta\pi_4 = 0.$$

The consistency condition:

$$\{\delta c_{\vec{v}}, \mathbf{H}_V^{(2)}\} = 0,\ \{\delta c_{\vec{w}}, \mathbf{H}_V^{(2)}\} = 0 \ \Rightarrow\ \delta N^{\vec{v}} = 0,\ \delta N^{\vec{w}} = 0.$$

Reduced phase space for scalar perturbations

The reduced Hamiltonian:

$$\mathbf{H}_S^{(2)} = \int N\mathcal{H}_0^{(2S)} + \delta N \delta\mathcal{H}_0 + \delta N^{\vec{k}} \delta\mathcal{H}_{\vec{k}}$$

Assume $\delta D = a\delta q_1 + b\delta q_2 + c\delta\pi_1 + d\delta\pi_2 + e\delta\phi + f\delta p^\phi$ and insert into

$$\forall_{\delta\xi^0, \delta\xi^{\vec{k}}} \quad \int \delta\xi^0 \{\delta D, \delta\mathcal{H}_0\} + \int \delta\xi^{\vec{k}} \{\delta D, \delta\mathcal{H}_{\vec{k}}\} \approx 0,$$

Two basic Dirac observables:

$$\Phi := p^\phi \delta\phi + \frac{\alpha}{\alpha - 1} \left(\frac{p}{6}\right) (3\delta q_1 - \delta q_2),$$

$$\Pi := \frac{\delta p^\phi}{p^\phi} - \frac{3(\alpha - 1)}{2} a^{-2} p^{-1} p^\phi \delta\phi - \frac{3\alpha}{4} \frac{\delta q_1}{a^2} + \frac{\alpha - 2}{4} \frac{\delta q_2}{a^2},$$

where $\{\Phi_{\vec{k}}, \Pi_{-\vec{l}}\} = \delta_{\vec{k},\vec{l}}$ (up to first order).

Physical dynamics

In the *spatially flat slicing* one kills the geometry perturbation:

$$\delta c_1 := \delta q_1, \quad \delta c_2 := \delta q_2,$$

so that

$$\mathbf{H}^{(2)}\Big|_{\delta c_1=\delta c_2=\delta\mathcal{H}_0=\delta\mathcal{H}_{\vec{k}}=0} = N\int \mathcal{H}_0^{(2S)}\Big|_{\delta c_1=\delta c_2=\delta\mathcal{H}_0=\delta\mathcal{H}_{\vec{k}}=0}(\delta\phi, \delta p^{\phi}),$$

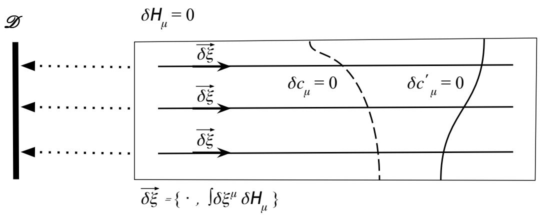
where $\{\delta\phi, \delta p^{\phi}\}_D = \{\delta\phi, \delta p^{\phi}\} = 1$ and the curvature term vanishes. Finally,

$$\mathbf{H}_{phys} = N\int\left(\frac{\alpha a p^2}{12(\alpha-1)^2}\Pi|^2_{\delta c_1=0=\delta c_2} + \frac{3(\alpha-1)}{\alpha}\frac{a^{-3}k^2}{p^2}\Phi|^2_{\delta c_1=0=\delta c_2}\right),$$

where

$$\Pi|_{\delta c_1=0=\delta c_2} = \frac{\delta p^{\phi}}{p^{\phi}} - \frac{3(\alpha-1)}{2}a^{-2}p^{-1}p^{\phi}\delta\phi, \quad \Phi|_{\delta c_1=0=\delta c_2} = p^{\phi}\delta\phi.$$

Relation between gauges



Suppose there are two sets of gauge fixing conditions,

$$\delta c_\mu = 0 \ \text{ and } \ \delta c'_\mu = 0,$$

then, there exists the canonical isomorphism

$$\Phi|_{\delta c_\mu = 0} \mapsto \Phi|_{\delta c'_\mu = 0}, \quad \Pi|_{\delta c_\mu = 0} \mapsto \Pi|_{\delta c'_\mu = 0},$$

$$\left\{ \Phi|_{\delta c_\mu = 0}, \Pi|_{\delta c_\mu = 0} \right\}_D \approx \left\{ \Phi|_{\delta c'_\mu = 0}, \Pi|_{\delta c'_\mu = 0} \right\}_{D'} \approx \left\{ \Phi, \Pi \right\}.$$

Gauge-fixing conditions

1. The lapse & shifts are on equal footing with the three-metric perturbations in the configuration space approach.

$$\int N\{\delta c_i, \mathcal{H}_0^{(2S)}\} + \delta N\{\delta c_i, \delta\mathcal{H}_0\} + \delta N^{\vec{k}}\{\delta c_i, \delta\mathcal{H}_{\vec{k}}\} \approx 0,$$

   is a condition for δN and $\delta N^{\vec{k}}$ or for δc_i.
2. Invertible map:

$$(\delta q_1, \delta q_2, \delta\pi_1, \delta\pi_2, \delta p^{\phi}, \delta\phi) \quad \longleftrightarrow \quad (\delta q, \delta R, \delta\theta, \delta\sigma, \delta\rho, \delta\phi).$$

3. Examples of gauges

Uniform density gauge: $\delta\rho = 0$ and $\delta q = 0$,

$$\Phi|_{\delta c_\mu = 0} = \frac{4}{3} a\delta\sigma + a^3 \left(\frac{\alpha}{6(\alpha-1)}\frac{p^2}{k^2} - \frac{4}{3}\right)\delta\theta,$$

$$\Pi|_{\delta c_\mu = 0} = -2(\alpha-1)a^{-1}p^{-1}\delta\sigma + ap^{-1}\left(2(\alpha-1) - \frac{1}{4}(\alpha-2)\frac{p^2}{k^2}\right)\delta\theta$$

$$\frac{\delta N}{N} = \frac{8ak^2}{3p}(\delta\sigma - a^2\delta\theta) + \frac{2a}{p}\delta\theta, \quad \frac{\delta N^{\vec{k}}}{N} = -\frac{8(\alpha-1)}{a^2p^2\alpha}(\delta\sigma - a^2\delta\theta),$$

Gauge-fixing conditions II

Comoving orthogonal: $\delta\phi = 0$ and $\delta\rho + \frac{\alpha}{12(\alpha-1)}\frac{p^2}{a^8}\delta q = 0$,

$$\Phi|_{\delta c_\mu = 0} = \frac{\alpha}{\alpha - 1}\frac{p}{4}\frac{a^4}{k^2}\delta R,$$

$$\Pi|_{\delta c_\mu = 0} = -\frac{\delta q}{2a^6} - \frac{3(\alpha-2)}{8}\frac{a^2}{k^2}\delta R.$$

$$\frac{\delta N^{\vec{k}}}{N} = 0, \frac{\delta N}{N} = \frac{\delta q}{2a^6(\alpha - 1)}.$$

Longitudinal gauge: $\delta R - \frac{2}{3}a^{-8}k^2\delta q = 0$ and $\frac{\delta N^{\vec{k}}}{N} = 0 \Rightarrow \delta\sigma = 0$, $\frac{\delta N}{N} = \frac{1}{6}a^{-6}\delta q$,

$$\Phi|_{\delta c_\mu = 0} = -\frac{4}{3}a^3\delta\theta + \frac{\alpha}{\alpha - 1}\frac{p}{4}\frac{a^4}{k^2}\delta R,$$

$$\Pi|_{\delta c_\mu = 0} = \frac{2(\alpha-1)(\alpha-2)}{\alpha}ap^{-1}\delta\theta + a^2\left(\frac{6(\alpha-1)}{\alpha p^2} - \frac{3(\alpha-2)}{8k^2}\right)\delta R.$$

Extension to the multifluid case

Replace $\mathcal{H}_f$ with $\sum_i \mathcal{H}_{f_i}$. There are $2n$ Dirac observables for n fluids,

$$\Phi_i := \delta D_{2i} = \bar{p}^{\phi_i}\delta\phi_i + \frac{a^{\frac{-3}{\alpha_i-1}} p^{T_i}}{ap}\left(3\delta q_1 - \delta q_2\right),$$

$$\Pi_i := \delta D_{1i} = \frac{\delta p^{\phi_i}}{\bar{p}^{\phi_i}} - \frac{\delta q_2}{2a^2},$$

where $\{\Phi_i(\vec{k}), \Pi_j(-\vec{l})\} = \delta_{ij}\delta_{\vec{k},\vec{l}}$ (up to first order).
The reduced Hamiltonian:

$$\mathbf{H} = N\mathcal{H}_0\Big|^{(0)} + \sum_i \frac{a^{\frac{-3}{\alpha_i-1}} p^{T_i}}{2(\alpha_i-1)}\Pi_i^2 + \sum_i \frac{k^2 a^{-2}}{2a^{\frac{-3}{\alpha_i-1}} p^{T_i}}\Phi_i^2$$
$$-\frac{3a^{-2}p^{-1}}{2}\left(\sum_k \Phi_k\right)\left(\sum_i a^{\frac{-3}{\alpha_i-1}} p^{T_i}\Pi_i\right) + \frac{3a^{-3}}{8}\left(\sum_k \Phi_k\right)^2.$$

Summary

- Dirac procedure works nicely for cosmological perturbation theory for *any* number of matter components
- It provides a convenient way to define and relate various gauge-fixing conditions
- It provides a *simple calculation* of the physical dynamics
- It establishes a starting point for quantization of the entire formalism (next step)
- It should be more or less straightforward to implement this approach in *anisotropic* cosmological perturbation theories (next step)
- It could be of great use for developing higher order cosmological perturbation theory (next step)