Dirac analysis of cosmological perturbation theory

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Motivation

Cosmological perturbation theory, applicable to early universe,

- deals with the dynamics of linear perturbations to the FLRW universe
- admits the gauge group of linear space-time diffeomorphisms (the issue of coordinate dependence)
- within the canonical formalism we have a standard set of concepts to handle gauge symmetries such as first-class constraints, gauge fixing conditions, gauge invariant quantities, Dirac brackets, canonical isomorphisms between gauge-fixed surfaces... (Dirac procedure)
- the knowledge of the canonical gauge-invariant quantities and their Hamiltonian dynamics are necessary for canonical quantization
- method for handling gauge-fixing conditions is necessary to obtain physical interpretation of the formalism
Previous results

The approach by D. Langlois, CQG 11 389-407 (1994):

- starts with the ADM canonical formalism
- makes a canonical transformation that separates physical DOFs from gauge DOFs in the kinematical phase space
- provides an efficient computation of the physical Hamiltonian (in terms of physical DOFs) as e.g. the curvature term vanishes (a gauge quantity)
Previous results

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In this talk, I will present a “new” approach:

- based on the Dirac procedure for constrained systems
- it not only identifies the gauge-invariant variables but also uses them to establish the relation between different gauge-fixing conditions (which give them physical meaning)
- gives a quick derivation of the physical Hamiltonian and this has a simple explanation
Gravity + perfect fluid \((p = w \rho)\),

\[
H = \int_{\Sigma} (N\mathcal{H}_0 + N^a\mathcal{H}_a) \, d^3x, \quad (q_{ab}, \pi^{ab}, \phi, p^\phi), \quad \text{where}
\]

\[
\mathcal{H}_0 = \sqrt{q} \left( -3R + q^{-1}(\pi_a^b \pi_b^a - \frac{1}{2} \pi^2) \right) + \frac{(p^\phi)^2}{\sqrt{q}K\mu^{\alpha-2}} - \sqrt{q}K\mu^\alpha,
\]

\[
\mathcal{H}_a = -2D_b\pi^b_a + p^\phi \phi_{,a}.
\]

The fluid pressure: \(p(\mu) = K\mu^\alpha, \quad \alpha = \frac{w+1}{w}\).

The fluid's flow: \(U_\nu = \mu^{-1}\phi_{,\nu}, \quad \text{where} \quad \frac{1}{N}\phi_{,0} = \frac{p^\phi}{\sqrt{q}K\mu^{\alpha-2}} + \frac{N^a}{N}\phi_{,a}\).

The Hamiltonian: \(H_{f,0} = \sqrt{q} \cdot \rho \bigg|_{t=\text{const}}, \quad H_{f,a} = -N \sqrt{q}(\rho_0 + p)U^0 U_a.\)
Expansion of ADM + Schutz formalism

FLRW + perturbations split:

\[ \delta q_{ab} = q_{ab} - a^2 \delta_{ab}, \quad \delta \pi^{ab} = \pi^{ab} - \frac{1}{3} p \delta^{ab}, \quad \delta \phi = \phi - \bar{\phi}, \quad \delta p^{\phi} = p^{\phi} - \bar{p}^{\phi}, \]

\[ a^2 = \int \frac{q_{ab} \delta^{ab}}{3} \, d^3 x, \quad p = \int \pi^{ab} \delta_{ab} \, d^3 x, \quad \bar{\phi} = \int \phi \, d^3 x, \quad \bar{p}^{\phi} = \int p^{\phi} \, d^3 x, \]

The Poisson brackets:

\[ \{ \delta \phi(x), \delta p^{\phi}(x') \} = \delta^3(x - x'), \quad \{ \delta q_{ab}(x), \delta \pi^{cd}(x') \} = \delta^3_{(a} \delta^d_{b)}(x - x'), \]

\[ \{ \bar{\phi}, \bar{p}^{\phi} \} = 1, \quad \{ a^2, p \} = 1. \]
 Canonical cosmological perturbation theory

The total Hamiltonian is a combination of the zero-order constraint, first-order constraints and a nonvanishing second-order term:

\[ H = N\mathcal{H}_0^{(0)} + \int \Sigma N\mathcal{H}_0^{(2)} + \delta N\delta\mathcal{H}_0 + \delta N^a\delta\mathcal{H}_a \]

Not an exact gauge system. Time problem at zero order. No time problem at first order.

One explicit formula,

\[
\mathcal{H}_{g,0}^{(2)} = a\delta\pi_{ab}\delta\pi^{ab} - \frac{1}{2}a(\delta\pi)^2 + \frac{a^{-1}p}{3}\delta\pi^{ab}\delta q_{ab} - \frac{a^{-1}p}{6}\delta\pi\delta q + \frac{5a^{-3}p^2}{72}\delta q_{ab}\delta q^{ab} + \frac{7a^{-3}p^2}{48}(\delta q)^2 + \frac{a^{-3}}{2} \left(\delta q_{ab,ab}\delta q + \frac{3}{2}\delta q_{aa,c}\delta q_{bb,c} - \delta q_{ab,b}\delta q_{ac,c} + \frac{1}{2}\delta q_{ab,c}\delta q_{ab,c}\right). 
\]
Dirac procedure

\[ \mathbf{H}^{(2)} = \int N \mathcal{H}_0^{(2)} + \delta N \delta \mathcal{H}_0 + \delta N^a \delta \mathcal{H}_a \]

Dirac observables \( D \):

\[ \delta \tilde{\xi}(D) \approx 0, \quad \delta \tilde{\xi}(\cdot) = \{ \cdot, \int \delta \xi^\mu \delta \mathcal{H}_\mu \} \]

Gauge-fixing conditions “\( \delta c_\mu = 0 \)”: \[ \det |\{ \delta \phi_\mu, \delta \phi_\nu \}| \neq 0, \quad \delta \phi_\mu \in \{ \delta \mathcal{H}_\mu, \delta c_\mu \} \]

Determination of \( \delta N, \delta N^a \):

\[ \{ \delta c_\mu, \mathbf{H}^{(2)} \} \approx \int N \{ \delta c_\mu, \mathcal{H}_0^{(2)} \} + \delta N \{ \delta c_\mu, \delta \mathcal{H}_0 \} + \delta N^a \{ \delta c_\mu, \delta \mathcal{H}_a \} \approx 0 \]
Dirac procedure

Dirac brackets:

\[ \{ \cdot, \cdot \}_D = \{ \cdot, \cdot \} - \{ \cdot, \delta \phi_\mu \} \{ \delta \phi_\mu, \delta \phi_\nu \}^{-1} \{ \delta \phi_\nu, \cdot \} \]

Properties:

1. \( \{ \delta \phi_\mu, \cdot \}_D = 0 \),
2. \( \{ D_i, D_j \}_D \approx D_k \approx \{ D_i, D_j \} \),
3. \( \{ \cdot, \cdot \}_D = \{ \cdot, D_i \} \{ D_i, D_j \}^{-1} \{ D_j, \cdot \} \), where \( \{ D_j, \delta \phi_\mu \} = 0 \).

Reduced Hamiltonian to be used with \( \{ \cdot, \cdot \}_D \):

\[
H^{(2)} = \int \left( N H_0^{(2)} + \delta N \delta H_0 + \delta N^a \delta H_a \right) \bigg|_{\delta \phi_\mu = 0, \mu = 1, \ldots, 8}
\]

\[
= N \int H_0^{(2)} \bigg|_{\delta \phi_\mu = 0, \mu = 1, \ldots, 8}
= N \int H \left( D \bigg|_{\delta \phi_\mu = 0, \mu = 1, \ldots, 8} \right)
\]
Suppose there are two sets of gauge fixing conditions, 

\[ \delta c_\mu = 0 \text{ and } \delta c'_\mu = 0, \]

then, there exists the canonical isomorphism

\[ D_i|_{\delta c_\mu = 0} \mapsto D_i|_{\delta c'_\mu = 0}, \]

\[ \left\{ D_i|_{\delta c_\mu = 0}, D_j|_{\delta c_\mu = 0} \right\}_D \approx \left\{ D_i|_{\delta c'_\mu = 0}, D_j|_{\delta c'_\mu = 0} \right\}_{D'} \approx \left\{ D_i, D_j \right\}. \]
SVT Decomposition

We Fourier-transform the metric perturbations:

\[
\delta q_{ab} = \delta q_1 A^1_{ab} + \delta q_2 A^2_{ab} + \delta q_3 A^3_{ab} + \delta q_4 A^4_{ab} + \delta q_5 A^5_{ab} + \delta q_6 A^6_{ab}
\]

\(\text{scalar perts}\) \(\text{vector perts}\) \(\text{tensor perts}\)

SVT decomposition \((\vec{k}, \vec{v}, \vec{w})\):

\[
H^{(2)} = \int \sum \left( N H_0^{(2S)} + \delta N \delta H_0 + \delta N \vec{k} \delta H_{\vec{k}} \right) + \int \sum \left( N H_0^{(2V)} + \delta N \vec{v} \delta H_{\vec{v}} + \delta N \vec{w} \delta H_{\vec{w}} \right) + \int \sum N H_0^{(2T)}
\]

\(\text{scalar part}\) \(\text{vector part}\) \(\text{tensor part}\)

phase space = \(\mathbb{R}^6\) (scalars) \(\times\) \(\mathbb{R}^4\) (vectors) \(\times\) \(\mathbb{R}^4\) (tensors)
Reduced phase space for vector perturbations

The reduced Hamiltonian:

\[
H^{(2)}_V = \int_{\Sigma} N H^{(2)}_0 + \delta N^\vec{v} \delta H_{\vec{v}} + \delta N^{\vec{w}} \delta H_{\vec{w}}
\]

The phase space:

\((\delta q_3, \delta q_4, \delta \pi_3, \delta \pi_4)\)

The gauge-fixing conditions:

\[\delta c_{\vec{v}} = 0, \delta c_{\vec{w}} = 0 \implies \delta q_3 = \delta q_4 = \delta \pi_3 = \delta \pi_4 = 0.\]

The consistency condition:

\[\{\delta c_{\vec{v}}, H^{(2)}_V\} = 0, \{\delta c_{\vec{w}}, H^{(2)}_V\} = 0 \implies \delta N^\vec{v} = 0, \delta N^{\vec{w}} = 0.\]
Reduced phase space for scalar perturbations

The reduced Hamiltonian:

$$H^{(2)}_S = \int N H_0^{(2S)} + \delta N \delta H_0 + \delta N \vec{k} \delta \mathcal{H}_k$$

Assume $\delta D = a \delta q_1 + b \delta q_2 + c \delta \pi_1 + d \delta \pi_2 + e \delta \phi + f \delta p^\phi$ and insert into

$$\forall \delta \xi^0, \delta \vec{k} \int \delta \xi^0 \{ \delta D, \delta H_0 \} + \int \delta \vec{k} \{ \delta D, \delta \mathcal{H}_k \} \approx 0,$$

Two basic Dirac observables:

$$\Phi := p^\phi \delta \phi + \frac{\alpha}{\alpha - 1} \left( \frac{p^\phi}{6} \right) (3 \delta q_1 - \delta q_2),$$

$$\Pi := \frac{\delta p^\phi}{p^\phi} - \frac{3(\alpha - 1)}{2} a^{-2} p^{-1} p^\phi \delta \phi - \frac{3\alpha}{4} \frac{\delta q_1}{a^2} + \frac{\alpha - 2}{4} \frac{\delta q_2}{a^2},$$

where $\{ \Phi_{\vec{k}}, \Pi_{\vec{l}} \} = \delta_{\vec{k}, \vec{l}}$ (up to first order).
Physical dynamics

In the *spatially flat slicing* one kills the geometry perturbation:

\[ \delta c_1 := \delta q_1, \quad \delta c_2 := \delta q_2, \]

so that

\[
\left. H^{(2)} \right|_{\delta c_1=\delta c_2=\delta H_0=\delta H_k=0} = N \int \left. H_0^{(2S)} \right|_{\delta c_1=\delta c_2=\delta H_0=\delta H_k=0} (\delta \phi, \delta p^\phi),
\]

where \( \{\delta \phi, \delta p^\phi\}_D = \{\delta \phi, \delta p^\phi\} = 1 \) and the curvature term vanishes. Finally,

\[
H_{phys} = N \int \left( \frac{\alpha a p^2}{12(\alpha - 1)^2} \Pi|_{\delta c_1=0=\delta c_2}^2 + \frac{3(\alpha - 1)}{\alpha} \frac{a^{-3} k^2}{p^2} \Phi|_{\delta c_1=0=\delta c_2}^2 \right),
\]

where

\[
\Pi|_{\delta c_1=0=\delta c_2} = \frac{\delta p^\phi}{p^\phi} - \frac{3(\alpha - 1)}{2} a^{-2} p^{-1} p^\phi \delta \phi, \quad \Phi|_{\delta c_1=0=\delta c_2} = p^\phi \delta \phi.
\]
Relation between gauges

\[ \delta H_\mu = 0 \]

\[ \delta \chi = \{ , \int \delta \chi' \delta H_\mu \} \]

Suppose there are two sets of gauge fixing conditions,

\[ \delta c_\mu = 0 \text{ and } \delta c'_\mu = 0, \]

then, there exists the canonical isomorphism

\[ \Phi |_{\delta c_\mu=0} \mapsto \Phi |_{\delta c'_\mu=0}, \quad \Pi |_{\delta c_\mu=0} \mapsto \Pi |_{\delta c'_\mu=0}, \]

\[ \{ \Phi |_{\delta c_\mu=0}, \Pi |_{\delta c_\mu=0} \}_D \approx \{ \Phi |_{\delta c'_\mu=0}, \Pi |_{\delta c'_\mu=0} \}_{D'}, \approx \{ \Phi, \Pi \}. \]
Gauge-fixing conditions

1. The lapse & shifts are on equal footing with the three-metric perturbations in the configuration space approach.

\[
\int N\{\delta c_i, \mathcal{H}_0(2S)\} + \delta N\{\delta c_i, \delta \mathcal{H}_0\} + \delta N\tilde{k}\{\delta c_i, \delta \mathcal{H}_{k}\} \approx 0,
\]

is a condition for \(\delta N\) and \(\delta N\tilde{k}\) or for \(\delta c_i\).

2. Invertible map:

\[
(\delta q_1, \delta q_2, \delta \pi_1, \delta \pi_2, \delta p^\phi, \delta \phi) \leftrightarrow (\delta q, \delta R, \delta \theta, \delta \sigma, \delta \rho, \delta \phi).
\]

3. Examples of gauges

**Uniform density gauge:** \(\delta \rho = 0\) and \(\delta q = 0\),

\[
\Phi|_{\delta c_\mu=0} = \frac{4}{3}a\delta \sigma + a^3 \left(\frac{\alpha}{6(\alpha - 1)} \frac{p^2}{k^2} - \frac{4}{3}\right) \delta \theta,
\]

\[
\Pi|_{\delta c_\mu=0} = -2(\alpha - 1)a^{-1}p^{-1}\delta \sigma + ap^{-1}\left(2(\alpha - 1) - \frac{1}{4}(\alpha - 2)\frac{p^2}{k^2}\right) \delta \theta
\]

\[
\frac{\delta N}{N} = \frac{8ak^2}{3p}(\delta \sigma - a^2\delta \theta) + \frac{2a}{p}\delta \theta, \quad \frac{\delta N\tilde{k}}{N} = -\frac{8(\alpha - 1)}{a^2p^2\alpha}(\delta \sigma - a^2\delta \theta),
\]
Gauge-fixing conditions II

**Comoving orthogonal:** $\delta \phi = 0$ and $\delta \rho + \frac{\alpha}{12(\alpha-1)} \frac{p^2}{a^8} \delta q = 0,$

$$\Phi|_{\delta c_\mu = 0} = \frac{\alpha}{\alpha - 1} \frac{p}{4} \frac{a^4}{k^2} \delta R,$$

$$\Pi|_{\delta c_\mu = 0} = -\frac{\delta q}{2a^6} - \frac{3(\alpha - 2)}{8} \frac{a^2}{k^2} \delta R.$$ 

$$\frac{\delta N^k}{N} = 0, \frac{\delta N}{N} = \frac{\delta q}{2a^6(\alpha - 1)}.$$ 

**Longitudinal gauge:** $\delta R - \frac{2}{3} a^{-8} k^2 \delta q = 0$ and $\frac{\delta N^k}{N} = 0 \Rightarrow \delta \sigma = 0,$

$$\frac{\delta N}{N} = \frac{1}{6} a^{-6} \delta q,$$

$$\Phi|_{\delta c_\mu = 0} = -\frac{4}{3} a^3 \delta \theta + \frac{\alpha}{\alpha - 1} \frac{p}{4} \frac{a^4}{k^2} \delta R,$$

$$\Pi|_{\delta c_\mu = 0} = \frac{2(\alpha - 1)(\alpha - 2)}{\alpha} a p^{-1} \delta \theta + a^2 \left( \frac{6(\alpha - 1)}{\alpha p^2} - \frac{3(\alpha - 2)}{8k^2} \right) \delta R.$$
Extension to the multifluid case

Replace $\mathcal{H}_f$ with $\sum_i \mathcal{H}_{f_i}$. There are $2n$ Dirac observables for $n$ fluids,

$$\Phi_i := \delta D_{2i} = \bar{p}^{\Phi_i} \delta \phi_i + \frac{a^{-3} \alpha_i^{-1} p^{T_i}}{2a p} (3 \delta q_1 - \delta q_2),$$

$$\Pi_i := \delta D_{1i} = \frac{\delta p^{\phi_i}}{\bar{p}^{\phi_i}} - \frac{\delta q_2}{2a^2},$$

where $\{\Phi_i(k), \Pi_j(-\bar{l})\} = \delta_{ij} \delta_{k,\bar{l}}$ (up to first order).

The reduced Hamiltonian:

$$H = N\mathcal{H}_0\big|^{(0)} + \sum_i a^{-3} \alpha_i^{-1} p^{T_i} \Pi_i^2 + \sum_i \frac{k^2 a^{-2}}{2a^{-3} \alpha_i^{-1} p^{T_i}} \Phi_i^2$$

$$- \frac{3a^{-2} p^{-1}}{2} \left( \sum_k \Phi_k \right) \left( \sum_i a^{-3} \alpha_i^{-1} p^{T_i} \Pi_i \right) + \frac{3a^{-3}}{8} \left( \sum_k \Phi_k \right)^2.$$
Summary

- Dirac procedure works nicely for cosmological perturbation theory for *any* number of matter components.
- It provides a convenient way to define and relate various gauge-fixing conditions.
- It provides a *simple calculation* of the physical dynamics.
- It establishes a starting point for quantization of the entire formalism (next step).
- It should be more or less straightforward to implement this approach in *anisotropic* cosmological perturbation theories (next step).
- It could be of great use for developing higher order cosmological perturbation theory (next step).