Phenomenology with the Bethe-Salpeter equation in Minkowski space

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Outline

1. General concepts
   - Bethe-Salpeter equation
   - BS amplitude: Three-body bound state
   - Nakanishi integral representation and LF projection
2. Two-body bound state within the BSE
   - Bosonic BSE in Minkowski space
   - The interaction kernel
   - Fermion-boson bound state
   - Fermion-antifermion BSE in Minkowski space
   - The mock pion
3. Conclusions
4. Outlook

General goals

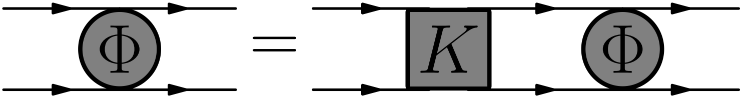
- Fully covariant relativistic description in Minkowski space;
- Bethe-Salpeter equation (BSE) to study non-perturbative systems;
- How bad is to ignore the crosses in the BSE kernel?
- Make the numerics feasible;
- No Fock space truncation;
- Phenomenological studies within the approach;

Bethe-Salpeter equation

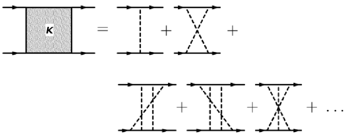
- The BSE for the bound state with total four momentum $p^2 = M^2$, composed of two scalar particles of mass m reads

$$\Phi(k,p) = S(p/2+k)S(p/2-k)\int \frac{d^4k'}{(2\pi)^4} iK(k,k',p)\Phi(k',p),$$

$$S(k) = \frac{i}{k^2 - m^2 + i\epsilon} \quad \text{: Feynman propagator}$$



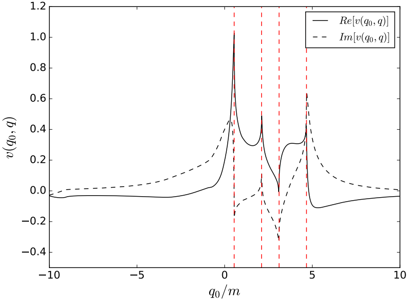
- The kernel K is given as a sum of irreducible Feynman diagrams (ladder, cross-ladder, etc).

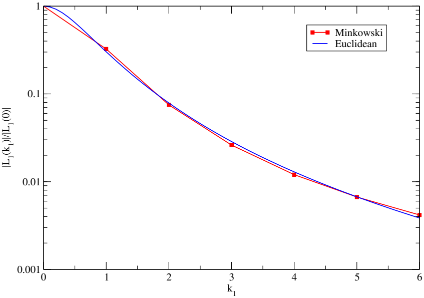


E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951)
N. Nakanishi, Graph Theory and Feynman Integrals (Gordon and Breach, New York, 1971)

Three-body problem in Minkowski space

- By direct integration of the poles: **singular equation** to be handled numerically;





- (left) Real and imaginary parts of $v(q_0, q_v)$ ($q_v/m = 0.5$, $B_3/m = 0.395$);
- (right) Transverse amplitude compared with Euclidean space solution;

E. Ydrefors, JHAN et al., Solving the three-body bound-state Bethe-Salpeter equation in Minkowski space, in preparation.

Nakanishi integral representation and LF projection

- General spectral representation for N-leg transition amplitudes;
- For the 3-leg amplitude (Bound state):

$$\Phi(k,p) = \int_{-1}^{1} dz' \int_{0}^{\infty} d\gamma' \frac{g(\gamma',z';\kappa^2)}{(\gamma' + \kappa^2 - k^2 - (p\cdot k)z' - i\epsilon)^3}, \quad \kappa^2 = m^2 - M^2/4$$

where M is the bound state mass and m is the constituent mass.

- Much easier to treat Minkowski space poles properly;
- Valence light-front wave function (LFWF) from the BS amplitude:

$$\psi(x,\mathbf{k}_\perp) = \frac{p^+}{\sqrt{2}}\, x\,(1-x) \int_{-\infty}^{\infty} \frac{dk^-}{2\pi}\, \Phi(k,p), \quad k^\pm = k_0 \pm k_z$$

- Corresponding to eliminate the relative LF time $t+z=0$;

T. Frederico, G. Salme and M. Viviani, Phys. Rev. D 85, 036009 (2012)

BSE in Minkowski space

- Applying the NIR on both sides of the BSE and integrating over k^- leads to the **non singular** integral equation:

$$\int_0^\infty d\gamma' \frac{g(\gamma', z; \kappa^2)}{[\gamma + \gamma' + z^2 m^2 + (1 - z^2)\kappa^2]^2} = \int_0^\infty d\gamma' \int_{-1}^{1} dz' V(\alpha; \gamma, z, \gamma', z') g(\gamma', z'; \kappa^2)$$

where $\gamma \equiv |\mathbf{k}_\perp|^2$ and $z \equiv 2\,x - 1$; V is expressed in terms of the BS interaction kernel.

  - Cross-ladder impact - huge effect in different observables [1];
  - Suppression with color dof [2];
  - Check of NIR as a solution - agreement with Euclidean BSE;

[1] V. Gigante, J.H. Alvarenga Nogueira et al., Phys. Rev. D 95 (2017) 056012;

[2] J.H. Alvarenga Nogueira et al., Phys.Lett. B777 (2018) 207-211;

Interaction kernel truncation

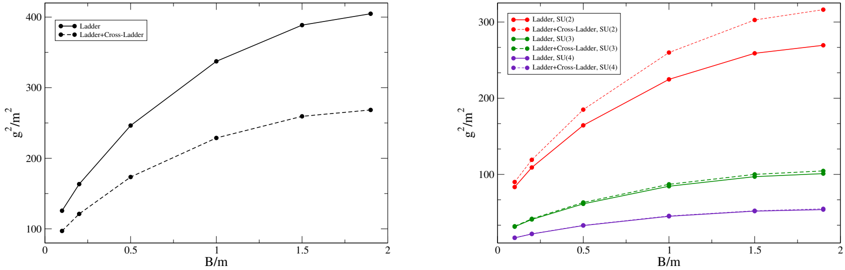


Figure: Coupling constant for various values of the binding energy B obtained by using the Bethe-Salpeter ladder (L) and ladder plus cross-ladder (CL) kernels, for an exchanged mass of $\mu = 0.5m$ computed with no color factors (left) and compared with the results for $N = 2, 3$ and 4 colors (right).

- Suppression already sizable for $N = 3$;
  - Essential for the framework viability;
  - BSE is hardly applicable to QED, but practicable for QCD.

V. Gigante, JHAN et al., Phys. Rev. D 95 (2017) 056012 JHAN et al., Phys.Lett. B777 (2018) 207-211

Suppression on the light-front wave function

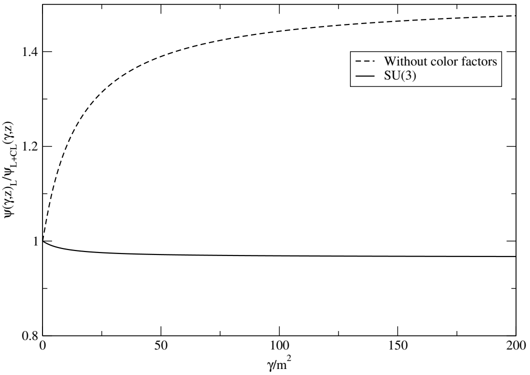


Figure: Ratio between valence LFWFs, as a function of $\gamma = k_\perp^2$, solved with a L and a L+CL kernels. Results for $N = 3$ colors (solid line) are compared with the ones where no color factors (dashed line) are included. Used values of the input parameters: $B = 1.0m$, $\mu = 0.5m$ and $z = 0.0$.

- Suppression also seen in the electromagnetic form factor.

V. Gigante, JHAN et al., Phys. Rev. D 95 (2017) 056012 JHAN et al., Phys.Lett. B777 (2018) 207-211

Asymmetric system: Fermion-scalar bound state

- Asymmetry brings new numerical challenges;
- Quark-diquark system: a fist step towards the three-body bound state;
- Scalar coupling $\alpha^s = \lambda_F^s \lambda_S^s/(8\pi m_S)$, for $m_F = m_S$ and $\mu/\bar{m} = 0.15,\ 0.50$

$B/\bar{m}$	$\alpha^s_M(0.15)$	$\alpha^s_{WR}(0.15)$	$\alpha^s_M(0.50)$	$\alpha^s_{WR}(0.50)$
0.10	1.5057	1.5057	2.6558	2.6558
0.20	2.2969	2.2969	3.2644	3.6244
0.30	3.0467	3.0467	4.5354	4.5354
0.40	3.7963	3.7963	5.4505	5.4506
0.50	4.5680	4.5681	6.4042	6.4043
0.80	7.2385	7.2387	9.8789	9.8794
1.00	9.7779	9.7783	13.7379	13.7380

$WR \equiv$ Wick-rotated (Euclidean), $M \equiv$ Minkowski and $\bar{m} = (m_S + m_F)/2$;

Fermion-antifermion BSE in Minkowski space

- Introducing spin dof

$$\Phi(k,p) = S(p/2+k) \int d^4k' \, F^2(k-k') \, iK(k,k') \, \Gamma_1 \, \Phi(k',p) \, \hat{\Gamma}_2 \, S(k-p/2)$$

where $\Gamma_1 = \Gamma_2 = 1 \; (scalar), \; \gamma_5 \; (pseudo), \; \gamma^\mu \; (vector)$

$$iK_V^{\mu\nu}(k,k') = -i\, g^2 \frac{g^{\mu\nu}}{(k-k')^2 - \mu^2 + i\epsilon}, \quad F(k-k') = \frac{(\mu^2 - \Lambda^2)}{[(k-k')^2 - \Lambda^2 + i\epsilon]}$$

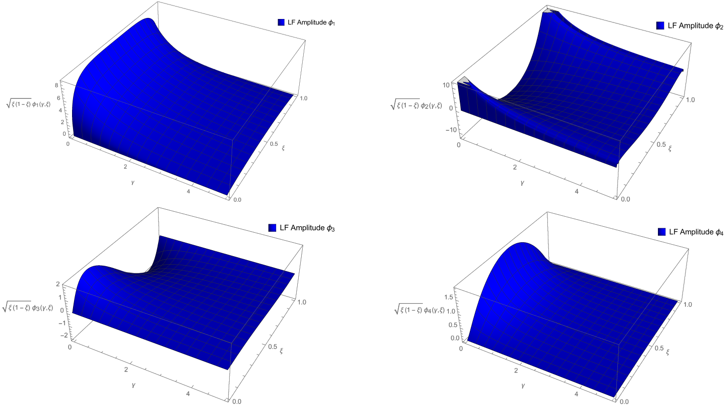
- Taking benefit from orthogonality properties for the decomposition

$$\Phi(k,p) = \sum_{i=1}^{4} S_i(k,p) \phi_i(k,p)$$

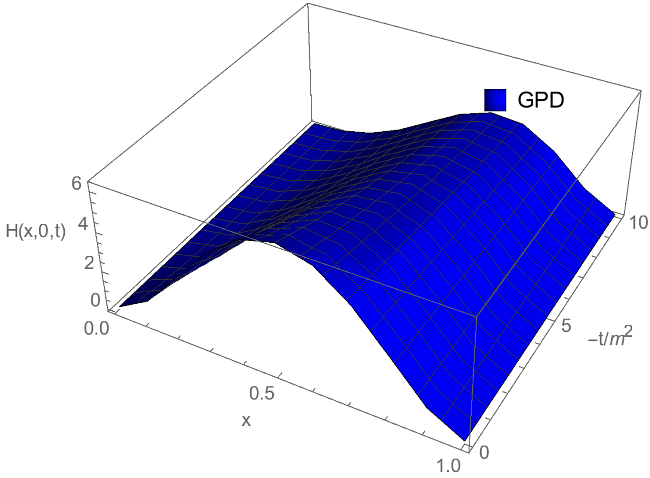
$$S_1 = \gamma_5, S_2 = \frac{\not{p}}{M}\gamma_5, S_3 = \frac{k\cdot p}{M^3}\not{p}\,\gamma_5 - \frac{1}{M}\not{k}\gamma_5 \text{ and } S_4 = \frac{i}{M^2}\sigma^{\mu\nu}p_\mu k_\nu\,\gamma_5$$

- The scalar amplitudes ϕ_i are represented by the NIR;

- Parameters for a mock pion:
  - $B/m = 1.35$ ($M_\pi = 140$ MeV fixed), $\mu/m = 2.0$ (≈ 500 MeV), $\Lambda/m = 1.0$, m_q=215 MeV $\rightarrow f_\pi = 96$ MeV, $P_{val} = 68\%$;

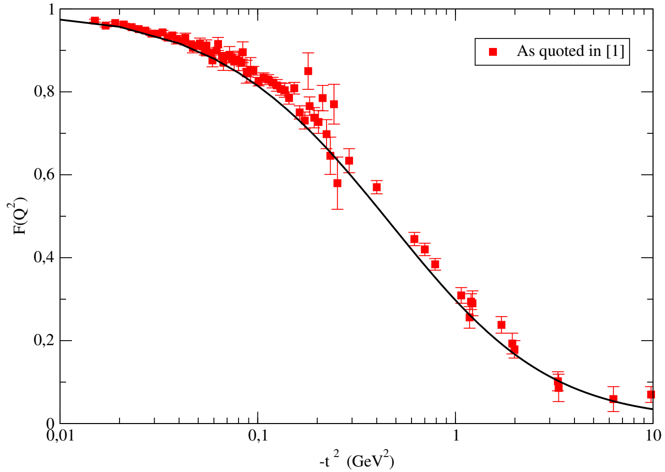


Preliminary: GPD in the DGLAP region



- GPD given by overlap of LFWFs;

Preliminary: Pion's electromagnetic form factor



[1] Collection of R. Baldini, et al., Eur. Phys. J. C 11, 709 (1999); Nucl. Phys. A 666 & 667, 3 (2000);

Conclusions

- NIR + LF projection $\rightarrow$ Showing to be essential tools to deal with the Bethe-Salpeter equation in Minkowski space;
  - Many systems already treated and several subtle points under control;
  - Approach is even more robust - now able to deal properly with all singularities within fermionic systems;
  - Valence seems to be far from enough;
  - Approach gives information beyond the valence;
  - Ladder approx. supported by the color suppression of the non-planar (crossed) diagrams;
- Simple and direct connection with physical observables;
- Still the most stable method in Minkowski space;

Outlook

- Pion GPD: in collaboration with H. Moutarde (CEA, Saclay);
- Confining kernel consistent with spectral representation;
- Dressed propagators - Schwinger-Dyson equation in Minkowski space;
- Calculations in Landau gauge;

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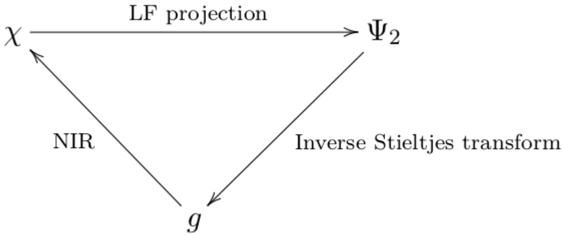
Thank you!

BACKUP

Relations: LF, NIR and BS amplitude

- The Nakanishi integral representation (NIR) gives the Bethe-Salpeter amplitude χ (BSA) through the weight function g;
- The Light-Front projection of the BSA gives the valence light-front wave function (LFWF) Ψ_2;
- The inverse Stieltjes transform gives g from the valence LFWF;



Carbonell, Frederico, Karmanov Phys.Lett. B769 (2017) 418-423

Extra singular contribution of the fermionic system

- The coupled integral equation system is given by

$$\psi_i(\gamma, z) = g^2 \sum_j \int_{-1}^{1} dz' \int_0^\infty d\gamma' \, g_j(\gamma', z'; \kappa^2) \, \mathcal{L}_{ij}(\gamma, z, \gamma', z'; p)$$

  - S_i operators + fermionic propagators: $(k^-)^n$ extra singularities;
  - Singularities have generic form:

$$\mathcal{C}_n = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} (k^-)^n \, \mathcal{S}(k^-, v, z, z', \gamma, \gamma') \qquad n = 0, 1, 2, 3$$

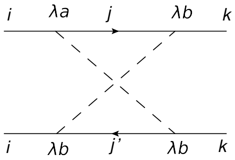
  - End-point singularities can be analytically treated by

$$\mathcal{I}(\beta, y) = \int_{-\infty}^{\infty} \frac{dx}{[\beta x - y \mp i\epsilon]^2} = \pm \frac{2\pi i \delta(\beta)}{[-y \mp i\epsilon]}$$

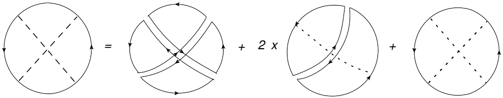
de Paula, Frederico, Salmè, Viviani PRD 94 (2016) 071901; EPJC 77 (2017) 764
Yan et al PRD 7 (1973) 1780
Pole-dislocation method: de Melo et al. NPA631 (1998) 574C, PLB708 (2012) 87

Color flow

Cross graph with color factors:



Diagrams ilustrating the color flow in the cross graph:



Each loop brings a factor of N and each "phantom" (dotted line) gluon a factor $-1/N$.

Color factor

- For the ladder kernel this computation is straightforward:

$$
\begin{aligned}
tr[(\lambda^a)_{ji}(\lambda^a)_{ij}] &= \sum_a (\lambda^a)_{ji}(\lambda^a)_{ij} = \frac{1}{2}\sum_{i,j=1}^{3}\left(\delta_{jj}\delta_{ii} - \frac{1}{N}\delta_{ji}\delta_{ij}\right) \\
&= \frac{1}{2}\left(N^2 - \frac{1}{N}\sum_{i=1}^{3}\delta_{ii}\right) = \frac{N^2-1}{2}, \qquad (1)
\end{aligned}
$$

- Cross-box:

$$
\begin{aligned}
tr[\lambda^a\lambda^b\lambda^a\lambda^b] &= \frac{1}{2}\left(\delta_{jk}\delta_{ij'} - \frac{1}{N}\delta_{ij}\delta_{j'k}\right)\frac{1}{2}\left(\delta_{kj'}\delta_{ji} - \frac{1}{N}\delta_{kj}\delta_{ij'}\right) \\
&= \frac{1}{4}\left[\delta_{jj'}\delta_{j'j} - \frac{1}{N}\delta_{jj}\delta_{ii} - \frac{1}{N}\delta_{ii}\delta_{kk} + \frac{1}{N^2}\delta_{ki}\delta_{ik}\right] \\
&= \frac{1}{4}\left[\sum_{j=1}^{N}\delta_{jj} - \frac{1}{N}\sum_{i,j=1}^{N}\delta_{jj}\delta_{ii} - \sum_{i,k=1}^{N}\frac{1}{N}\delta_{ii}\delta_{kk} + \frac{1}{N^2}\sum_{k=1}^{N}\delta_{kk}\right] \\
&= \frac{1}{4}\left[N - 2\frac{N^2}{N} + \frac{N}{N^2}\right] == \frac{1}{4}\left[-N + \frac{1}{N}\right] = -\frac{(N^2-1)}{4N}. \qquad (2)
\end{aligned}
$$

where we use the $SU(N)$ projection operators.

P. Cvitanović, Group theory for Feynman diagrams in non-Abelian gauge theories, Phys. Rev. D **14** (1976) 1536

Three-body problem with zero-range interaction

- The vertex function within the three-body BS equation reads (Faddeev decomposition)

$$v(q,p) = 2\mathrm{i}F(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}\epsilon} \frac{\mathrm{i}}{(p-q-k)^2 - m^2 + i\epsilon} v(k,p)$$

- The equation can be solved in Minkowski space, by Wick-rotation $k_0 \to ik_0$ or integrating over k^-:

$$\Gamma(k_\perp, x) = F(M_{12}) \frac{1}{(2\pi)^3} \int_0^{1-x} \frac{dx'}{x'(1-x-x')} \int_0^\infty \frac{d^2k'_\perp}{M_0^2 - M_3^2} \, \Gamma\left(k'_\perp, x'\right),$$

$$M_0^2 = \frac{\vec{k'}_\perp^2 + m^2}{x'} + \frac{\vec{k}_\perp^2 + m^2}{x} + \frac{(\vec{k'}_\perp + \vec{k}_\perp)^2 + m^2}{1-x-x'}.$$

- $F(M_{12})$: two-body scattering amplitude characterized by scattering length a (or bound state mass M_2) and $M_{12}^2 = (p-q)^2$.

T. Frederico, PLB 282 (1992) 409

E. Ydrefors, J.H. Alvarenga Nogueira et al., Phys. Lett. B 770 (2017) 131-137

Three-body within covariant BS equation

$$v(q,p) = 2iF(M_{12}) \int \frac{d^4k}{(2\pi)^4} \frac{i}{[k^2 - m^2 + i\epsilon]} \frac{i}{[(p-q-k)^2 - m^2 + i\epsilon]} v(k,p)$$

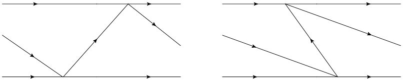


Figure: The three-body LF graphs obtained by time-ordering of the Feynman graph.

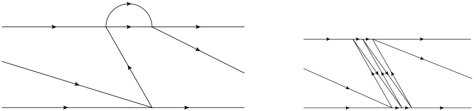


Figure: Examples of many-body intermediate state contributions to the LF three-body forces.

Direct method

- Direct integration of the BS equation, treating explicitly the singularities.
- The equation for $v(q_0, q_v)$ can be written in the "non-singular" form

$$v(q_0,q_v) = \frac{\mathcal{F}(M_{12})}{(2\pi)^4}\int_0^\infty k_v^2 dk_v \Big\{ i\,\frac{[\Pi(q_0,q_v;\varepsilon_k,k_v)v(\varepsilon_k,k_v)+\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)]}{2\varepsilon_k}$$
$$-2\int_{-\infty}^0 dk_0\left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q_v;-\varepsilon_k,k_v)v(-\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right]$$
$$-2\int_0^\infty dk_0\left[\frac{\Pi(q_0,q_v;k_0,k_v)v(k_0,k_v)-\Pi(q_0,q;\varepsilon_k,k_v)v(\varepsilon_k,k_v)}{k_0^2-\varepsilon_k^2}\right]\Big\},$$

  - $[k_0^2 - k_v^2 - m^2 + i\epsilon]^{-1} = PV[k_0^2 - \varepsilon_k^2]^{-1} - i\pi/(2\varepsilon_k)[\delta(k_0-\varepsilon_k)+\delta(k_0+\varepsilon_k)]$.
  - $\varepsilon_k = \sqrt{k_v^2+m^2}$, $k_v = |\vec{k}|$;
  - Kernel Π only has weak, logaritmic, singularities;
  - For $a < 0$ (considered here) $F(M_{12})$ has no pole;
  - The singularities at $k_0 = \pm\varepsilon_k$ were subtracted;
- We have solved the above equation by using a spline expansion for v, i.e. $v(q_0,q_v) = \sum_{ij} C_{ij} S_i(q_0) S_j(q_v)$.

Carbonell and Karmanov PRD 90 (2014) 056002

$$\Pi(q_0, q; k_0, k) = \frac{i\pi}{qk} \left[\log \left| \frac{(\eta+1)}{(\eta-1)} \right| - i\pi I(\eta) \right]$$

with

$$I(\eta) = \begin{cases} 1 & \text{if} \quad |\ \eta\ | \quad \leq \quad 1 \\ 0 & \text{if} \quad |\ \eta\ | \quad > \quad 1 \end{cases} \qquad \eta = \frac{(M_3 - q_0 - k_0)^2 - q^2 - k^2 - m^2}{2kq}$$

Eliminate the principal value singularity by:

$$PV \int_0^\infty \frac{h(k_0)dk_0}{k_0^2 - \varepsilon_k^2} = \int_0^\infty dk_0 \frac{h(k_0) - h(\varepsilon_k)}{k_0^2 - \varepsilon_k^2}$$

using

$$PV \int_{-\infty}^0 \frac{h(\varepsilon_k)dk_0}{k_0^2 - \varepsilon_k^2} = PV \int_0^\infty \frac{h(\varepsilon_k)dk_0}{k_0^2 - \varepsilon_k^2} = 0.$$