High-energy QCD evolution beyond leading order

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GDR QCD

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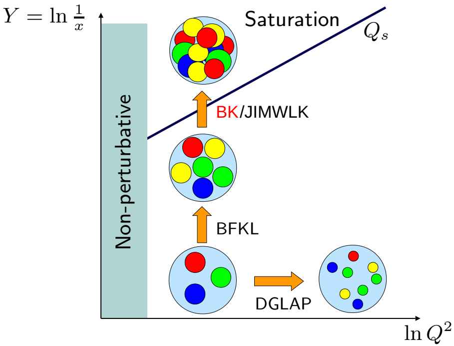
Motivations

QCD: theory difficult to study in the general case

Presence of a hard scale ($p_\perp$, M): possible to use perturbative expansion

One can then study the evolution of parton densities in hadrons:

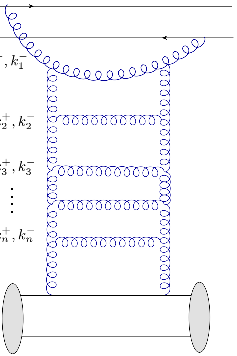
- as a function of Q^2: DGLAP
- as a function of x: BFKL (dilute) / BK, JIMWLK (dense)



Our goal here is to study the dense limit of QCD (saturation)

The LO BK equation

At high energy, DIS can be viewed as a virtual photon (virtuality Q^2, flying almost along P^+) splitting into a $q\bar{q}$ pair which then interacts eikonally with the target (transverse size Q_0^2, flying almost along P^-)



Kinematics of interest: $Q^2 \gg Q_0^2 \gg \Lambda^2_{\text{QCD}}$

Leading logarithmic approximation: resum any number of gluons strongly ordered in longitudinal momentum (rapidity)

Can look at the evolution

- in p^-: $q_0^- \gg k_n^- \gg \cdots \gg k_1^- \gg q^-$
  "η evolution": resum $(\alpha_s \eta)^n$
- in p^+: $q^+ \gg k_1^+ \gg \cdots \gg k_n^+ \gg q_0^+$
  "Y evolution": resum $(\alpha_s Y)^n$

The corresponding rapidity intervals are:

$$\eta = \ln \frac{q_0^-}{q^-} = \ln \frac{s}{Q^2} = \ln \frac{1}{x_{\text{Bj}}}$$

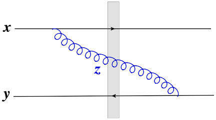
$$Y = \ln \frac{q^+}{q_0^+} = \ln \frac{s}{Q_0^2} = \eta + \ln \frac{Q^2}{Q_0^2} \equiv \eta + \rho > \rho$$

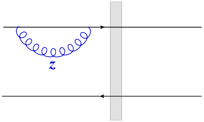
Note that the difference between Y and η is relevant only at NLO and beyond

Resummation of all soft emissions: Balitsky-Kovchegov (BK) equation:

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)$$

Possibility for a parent dipole with size $r = |\boldsymbol{x}-\boldsymbol{y}|$ to emit two daughter dipoles with sizes $|\boldsymbol{x}-\boldsymbol{z}|, |\boldsymbol{z}-\boldsymbol{y}|$ or to remain intact



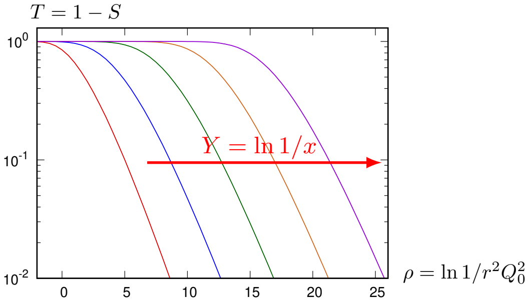


Starting with a given initial condition at $Y = 0$ (e.g. the simple GBW model $S^{(0)}_{\boldsymbol{xy}} = e^{-(\boldsymbol{x}-\boldsymbol{y})^2 Q_0^2}$), solve the BK equation numerically to larger rapidities

Can then compute standard DIS structure functions, e.g. $F_L(x_{\mathrm{Bj}}, Q^2) = \frac{Q^2}{4\pi^2\alpha_{em}}\sigma_L(x_{\mathrm{Bj}}, Q^2)$

with $\sigma_L(x_{\mathrm{Bj}}, Q^2) = \frac{4N_c\alpha_{em}}{\pi^2}\frac{\sigma_0}{2}\sum_f e_f^2 \int \mathrm{d}z_1 \mathrm{d}^2\mathbf{r}\, Q^2 z_1^2 (1-z_1)^2 K_0^2\left(Q\sqrt{z_1(1-z_1)\mathbf{r}^2}\right)(1 - S_{\mathbf{r}})$

Numerical solution of LO BK:



When we go to larger rapidities: saturation front moving to the right

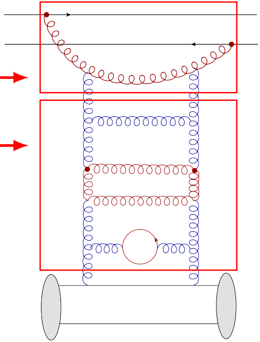
Saturation scale $Q_s(Y)$: defined such that $T(Y, r = 1/Q_s) \sim 1/2$

- Speed of the front: saturation exponent $\lambda_s = \dfrac{\mathrm{d} \ln Q_s^2(Y)}{\mathrm{d}Y}$
- Steepness of the front: anomalous dimension γ_s , $T(Y,\rho) \approx \exp[-\gamma_s(\rho - \lambda Y)]$

LO BK: $\lambda_s \approx 4.88 \bar{\alpha}_s$, $\gamma_s \approx 0.63$. What about NLO?

Sources of NLO corrections in this formalism:

- Corrections to the hard part
  DIS: Chirilli; Beuf
- Corrections to the BK evolution
  Balitsky, Chirilli



In this talk we focus on BK evolution

At NLO: take into account contributions where two successive emissions are not strongly ordered in longitudinal momentum

Numerical solutions show that the NLO corrections to BK are very large and negative, making the evolution unstable
Lappi, Mäntysaari

The NLO BK equation in Y

NLO BK for Y evolution as derived by Balitsky, Chirilli:

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$- \frac{\bar{\alpha}_s^2}{4\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\,\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\,[S_{\boldsymbol{xz}}(Y)S_{\boldsymbol{zy}}(Y) - S_{\boldsymbol{xy}}(Y)]$$

$$+ \bar{\alpha}_s^2 \times \text{“regular”}.$$

The second line is the source of the instability at large daughter dipole sizes:

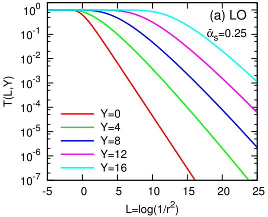
$$-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \simeq -\frac{1}{2}\ln^2\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{r^2} \quad \text{when } |\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg |\boldsymbol{x}-\boldsymbol{y}| = r$$

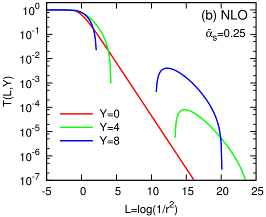
In this limit one evolution step yields (neglecting the “regular” $\bar{\alpha}_s^2$ terms)

$$\Delta T(Y,r) = \bar{\alpha}_s Y r^2 Q_s^2 \ln\frac{1}{r^2Q_s^2}\left(1 - \frac{\bar{\alpha}_s}{6}\ln^2\frac{1}{r^2Q_s^2}\right)$$

The $\bar{\alpha}_s^2$ term can be negative and larger in magnitude than LO $\rightarrow$ instability

The origin of the instability is confirmed by numerical calculations (Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos)





("NLO": LO+second line of NLO BK)

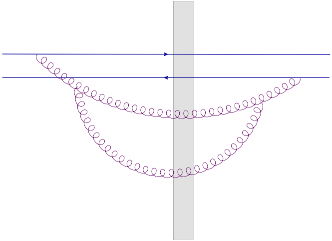
This issue is not surprising: the instability of NLO BFKL was observed long time ago and solved by resumming double logs to all orders (Salam et al.)

Physical origin of the instability: BK evolution in Y enforces ordering in p^+ but not in lifetime $\tau \sim 1/p^-$

Need to additionally impose $\tau_p > \tau_k \Leftrightarrow \frac{p^+}{p_\perp^2} > \frac{k^+}{k_\perp^2}$ for two successive emissions $\boldsymbol{p}$, $\boldsymbol{k}$

Also called "kinematical constraint" (Beuf)

If we worked in η, we would have the opposite problem: automatic ordering in p^- but not in p^+



We are interested in the regime of large collinear logarithms

$\rightarrow$ First consider the double-logarithmic approximation (DLA)

Only keep powers of $\bar{\alpha}_s$ enhanced by $Y\rho$ or ρ^2 $\quad (\rho = \ln(1/r^2 Q_0^2))$

At DLA the evolution equation for $\mathcal{A} \equiv T(Y, r)/r^2 Q_0^2$ reads

$$\mathcal{A}(q^+, r^2) = \mathcal{A}^{(0)}(r^2) + \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2} \int_{q_0^+}^{q^+} \frac{\mathrm{d}k^+}{k^+} \mathcal{A}(k^+, z^2)$$

Notice that k^+ can take any value between q_0^+ (target) and q^+ (projectile)

Now imposing time ordering: $\frac{q_0^+}{Q_0^2} \ll k^+ z^2 \ll q^+ r^2$,

This becomes: $\mathcal{A}(q^+, r^2) = \mathcal{A}^{(0)}(r^2) + \bar{\alpha}_s \int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2} \int_{q_0^+/z^2 Q_0^2}^{q^+ r^2/z^2} \frac{\mathrm{d}k^+}{k^+} \mathcal{A}(k^+, z^2)$

This time-ordered DLA equation can be rewritten using logarithmic variables ($\rho = \ln(1/r^2 Q_0^2)$, $\rho_1 = \ln(1/z^2 Q_0^2)$, $Y = \ln(q^+/q_0^+)$, $Y_1 = \ln(k^+/q_0^+)$)

$$\mathcal{A}(Y,\rho) = \mathcal{A}^{(0)}(\rho) + \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1 \int_{\rho_1}^{Y-\rho+\rho_1} \mathrm{d}Y_1 \mathcal{A}(Y_1,\rho_1)$$

- Non-local in rapidity (not a big issue): $\dfrac{\partial \mathcal{A}(Y,\rho)}{\partial Y} = \bar{\alpha}_s \int_0^{\rho} \mathrm{d}\rho_1\, \mathcal{A}(Y-\rho+\rho_1, \rho_1)$
- Boundary value problem (more serious): $\mathcal{A}^{(0)}(\rho) = \mathcal{A}(Y=\rho,\rho)$

Can be extended to full BK as:

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \Theta\,(Y-\rho_{\min})[S_{\boldsymbol{xz}}(Y-\Delta_{\boldsymbol{xyz}})S_{\boldsymbol{zy}}(Y-\Delta_{\boldsymbol{xyz}}) - S_{\boldsymbol{xy}}(Y)]$$

with $\rho_{\min} = \ln \dfrac{1}{\min\{(\boldsymbol{x}-\boldsymbol{y})^2, (\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{y}-\boldsymbol{z})^2\} Q_0^2}$, $\Delta_{\boldsymbol{xyz}} = \max\left\{0, \ln \dfrac{\min\{(\boldsymbol{x}-\boldsymbol{z})^2, (\boldsymbol{z}-\boldsymbol{y})^2\}}{(\boldsymbol{x}-\boldsymbol{y})^2}\right\}$

Similar to the collinear-improved BK proposed by Beuf

The non-local DLA equation is mathematically equivalent to a local equation:
(Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos)

$$\mathcal{A}(Y,\rho) = \mathcal{A}^{(0)}(\rho) + \bar{\alpha}_s \int_0^Y \mathrm{d}Y_1 \int_0^\rho \mathrm{d}\rho_1 \mathcal{K}_{\mathrm{DLA}}(\rho-\rho_1)\mathcal{A}(Y_1,\rho_1)\ , \quad \mathcal{K}_{\mathrm{DLA}}(\rho) = \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)}{\sqrt{\bar{\alpha}_s\rho^2}}$$

- Local in rapidity
- An initial value problem: $\mathcal{A}^{(0)}(\rho) = \mathcal{A}(Y=0,\rho)$

Recall that only values $Y > \rho$ are physical ($\Leftrightarrow x_{\mathrm{Bj}} < 1$)

- $Y < \rho$: analytic continuation to the unphysical regime
- $Y > \rho$: coincides with the physical solution

(Unphysical) initial condition $\mathcal{A}(Y=0,\rho) =$?

At the DLA level, one can construct $\mathcal{A}(Y=0,\rho)$ from $\bar{A}(\eta=0,\rho)$
e.g. in the GBW model $\bar{A}(\eta=0,\rho)=1$ and $\mathcal{A}(Y=0,\rho) = \mathrm{J}_0\big(2\sqrt{\bar{\alpha}_s\rho^2}\big)$

Collinear-improved BK in Y

This was just at the DLA level. Can be extended to full BK:

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\, \mathcal{K}_{\mathrm{DLA}}(\rho_{\boldsymbol{xyz}})\,(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})\,, \quad \rho^2_{\boldsymbol{xyz}} = \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

This is collinear-improved BK in Y. Expansion in powers of $\bar{\alpha}_s$:

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left(1 - \frac{\bar{\alpha}_s\rho^2}{2} + \frac{\bar{\alpha}_s^2\rho^4}{12} - \cdots\right)(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})$$

LO BK — collinear double logs

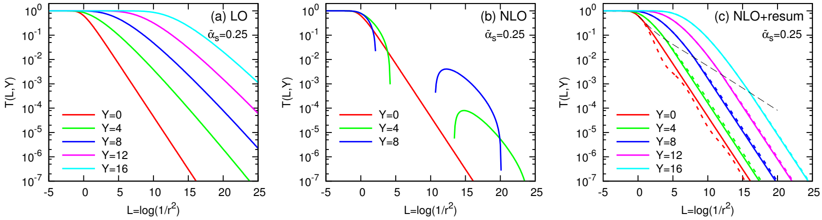
$\mathcal{K}_{\mathrm{DLA}}$ suppresses the large daughter dipoles $|\boldsymbol{z}-\boldsymbol{x}| \simeq |\boldsymbol{z}-\boldsymbol{y}| \gg |\boldsymbol{x}-\boldsymbol{y}|$

In principle rather straightforward procedure:

- Choose physical initial condition at $\eta = 0$
- Construct unphysical initial condition at $Y = 0$
- Solve the equation with $\mathcal{K}_{\mathrm{DLA}}$

Collinear-improved BK in Y

The resummation of the double logs indeed makes the evolution stable:
(Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos)



Similar results when including the other $\bar{\alpha}_s^2$ corrections (Lappi, Mäntysaari)
$\rightarrow$ Use this equation for phenomenology?

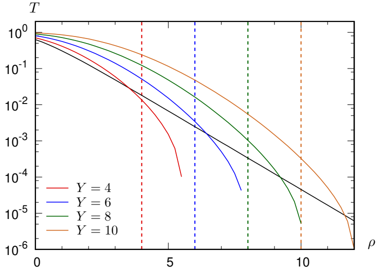
Good fits to HERA data obtained with collinear-improved BK
(Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos; Albacete)

But not consistent:

- Use the "wrong" rapidity interval η instead of $Y = \eta + \rho$
- Problem with the initial condition

At DLA we can construct the initial condition at $Y = 0$ from the one at $\eta = 0$

Not possible (at least exactly) with full BK:



Straight black line: physical initial condition $\bar{T}(\eta = 0, \rho) = \exp(-\rho)$

Colored lines: solution of collinear-improved BK with $T(Y = 0, \rho) = \exp(-\rho)\, \mathrm{J}_0\left(2\sqrt{\bar{\alpha}_s \rho^2}\right)$
Should match the black line at $\rho = Y$

In practice: quite large deviations already at small rapidities, gets worse as Y increases

Because of these issues, choosing Y as the evolution variable is not practical

It turns out that it is much more convenient to only work in η

NLO BK: derived for Y evolution but we can easily change variable to η

At NLO accuracy:

- Such a change only affects the LO piece. In the $\mathcal{O}(\bar{\alpha}_s^2)$ terms we can simply replace $Y \to \eta$
- We can use LO BK to evaluate $\partial \bar{S}_{\boldsymbol{xz}}(\eta)/\partial\eta$ in

$$S_{\boldsymbol{xz}}(Y) = S_{\boldsymbol{xz}}(\eta+\rho) \equiv \bar{S}_{\boldsymbol{xz}}\left(\eta + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right) \simeq \bar{S}_{\boldsymbol{xz}}(\eta) + \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\frac{\partial \bar{S}_{\boldsymbol{xz}}(\eta)}{\partial\eta}$$

1) Start with non-local equation in Y

$$\frac{\partial S_{\boldsymbol{xy}}(Y)}{\partial Y} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \Theta\,(Y-\rho_{\min})[S_{\boldsymbol{xz}}(Y-\Delta_{\boldsymbol{xyz}})S_{\boldsymbol{zy}}(Y-\Delta_{\boldsymbol{xyz}})-S_{\boldsymbol{xy}}(Y)]$$

2) Change variable from Y to $\eta = Y - \rho$

3) Extract $\mathcal{O}(\bar{\alpha}_s^2)$ contribution and subtract it

4) Add $\bar{\alpha}_s^2$ corrections from Balitsky, Chirilli

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \Theta(\eta-\delta_{\boldsymbol{zxy}})\Theta(\eta-\delta_{\boldsymbol{zyx}})\left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{zxy}})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zyx}})-\bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$- \frac{\bar{\alpha}_s^2}{4\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$+ \frac{\bar{\alpha}_s^2}{2\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}^2\boldsymbol{u}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \left[\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} + \delta_{\boldsymbol{uyx}}\right] \bar{S}_{\boldsymbol{xu}}(\eta) \left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta) - \bar{S}_{\boldsymbol{uy}}(\eta)\right]$$

$$+ \bar{\alpha}_s^2 \times \text{“regular”}$$

1) Start with non-local equation in Y

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$$\boxed{-\frac{\bar{\alpha}_s^2}{4\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{xy}}(\eta)\right]}$$

double logs in Balitsky-Chirilli

$$+\frac{\bar{\alpha}_s^2}{2\pi^2}\int \frac{\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}^2\boldsymbol{u}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\left[\boxed{\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}}+\boxed{\delta_{\boldsymbol{uyx}}}\right]\bar{S}_{\boldsymbol{xu}}(\eta)\left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{uy}}(\eta)\right]$$

change $Y\to\eta$ rapidity shift

$$+\,\bar{\alpha}_s^2\times\text{“regular”}$$

Collinear-improved BK in η

This collinear-improved NLO BK in η is our main result

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\Theta(\eta-\delta_{\boldsymbol{zxy}})\Theta(\eta-\delta_{\boldsymbol{zyx}})\left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{zxy}})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zyx}})-\bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$-\frac{\bar{\alpha}_s^2}{4\pi}\int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\left[\bar{S}_{\boldsymbol{xz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

$$+\frac{\bar{\alpha}_s^2}{2\pi^2}\int \frac{\mathrm{d}^2\boldsymbol{z}\,\mathrm{d}^2\boldsymbol{u}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{u}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}\left[\ln\frac{(\boldsymbol{u}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}+\delta_{\boldsymbol{uyx}}\right]\bar{S}_{\boldsymbol{xu}}(\eta)\left[\bar{S}_{\boldsymbol{uz}}(\eta)\bar{S}_{\boldsymbol{zy}}(\eta)-\bar{S}_{\boldsymbol{uy}}(\eta)\right]$$

$$+\,\bar{\alpha}_s^2\times\text{"regular"},$$

By construction:

- Exactly matches the NLO BK equation when expanded to $\mathcal{O}(\bar{\alpha}_s^2)$
- Can be solved knowing the initial condition at $\eta=0$
- Free of large double logs

However it is difficult to solve in practice: cancellation of double logs between the second and third terms + all difficulties with "pure" NLO BK

Because of the difficulties related to solving the full NLO equation, we don't consider here the $\bar{\alpha}_s^2$ terms (which are now expected to be truly NLO)

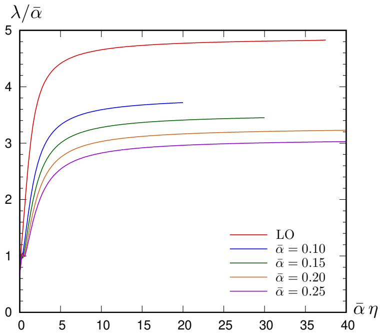
We are left with the following equation:

$$\frac{\partial \bar{S}_{\boldsymbol{xy}}(\eta)}{\partial \eta} = \frac{\bar{\alpha}_s}{2\pi} \int \frac{\mathrm{d}^2\boldsymbol{z}\,(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \Theta(\eta-\delta_{\boldsymbol{zxy}})\Theta(\eta-\delta_{\boldsymbol{zyx}}) \left[\bar{S}_{\boldsymbol{xz}}(\eta-\delta_{\boldsymbol{zxy}})\bar{S}_{\boldsymbol{zy}}(\eta-\delta_{\boldsymbol{zyx}}) - \bar{S}_{\boldsymbol{xy}}(\eta)\right]$$

- Contains LO BK evolution + resummation of double logs to all orders
- Not much more difficult to solve than standard BK (initial condition problem)

  Only difference: shifted rapidity arguments

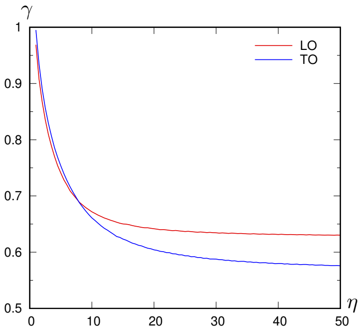
Saturation exponent as a function of $\bar{\alpha}_s \eta$:
(LO: no $\bar{\alpha}_s$ dependence for this quantity)



- The non-locality slows down the evolution compared to LO
- The deviation compared to LO is significant but not huge, increases with $\bar{\alpha}_s$

Numerical results

Anomalous dimension as a function of η ($\bar{\alpha}_s = 0.25$):



- Asymptotically the anomalous dimension of the non-local equation is smaller than at LO
- But no significant difference at rapidities relevant for HERA data ($\eta \lesssim 10$)

Conclusions

The large double logarithms appearing in the NLO BK equation must be resummed to avoid instabilities

- Can be formally done in Y, but not practical
- We propose an equation in η which is free of double logs and matches full NLO BK when expanded to $\mathcal{O}(\bar{\alpha}_s^2)$
- LO BK with resummation of double logs: stable, slower evolution than LO

What remains to be done:

- Add running coupling
- Resum remaining, single (DGLAP) logarithms
  Iancu, Madrigal, Mueller, Soyez, Triantafyllopoulos
- Implement "regular" NLO corrections