Mellin Approximants to Hadronic Vacuum Polarization

David Greynat
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Non affliated

In collaboration with Jérôme Charles and Eduardo de Rafael

Based on E. de Rafael, Phys. Lett. **B 736** 522 (2014)
E. de Rafael, Phys. Rev. D **96** (2017)
J. Charles, D.G. and E. de Rafael, Phys. Rev. **D97** (2018)
Work in progress with Jérôme Charles and Eduardo de Rafael

Anatomy of the muon anomaly

From the experimental value,

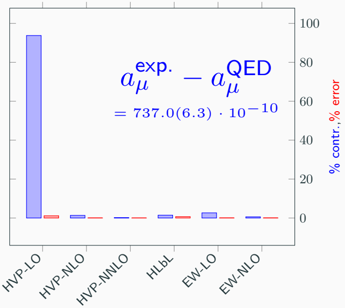
$$a_\mu^{\text{exp.}} = 11659208.9(6.3) \cdot 10^{-10} (0.54\,\text{ppm})$$

there is a persistent discrepancy about $3\sigma - 4\sigma$ with the SM evaluation

$$\Delta a_\mu = a_\mu^{\text{exp.}} - a_\mu^{\text{SM}} = 31.3 \cdot 10^{-10}$$

HVP-LO	692.6 ± 3.3	M. Davier (2016)
	688.8 ± 4.3	F. Jegerlehner (2017)
	693.3 ± 2.5	A. Keshavarzi *et al.* (2018)
HVP-NLO	-9.84 ± 0.07	K. Hagiwara *et al.* (2011)
	-9.93 ± 0.07	F. Jegerlehner (2017)
HVP-NNLO	$+1.24 \pm 0.01$	A. Kurz *et al.* (2014)
	$+1.22 \pm 0.01$	F. Jegerlehner (2017)
HLbL	$+10.5 \pm 2.6$	J. Prades *et al.* (2009)
	$+10.3 \pm 2.8$	F. Jegerlehner (2017)
EW-LO	19.48 ± 0.01	Collected papers (1972)
EW-NLO	-4.12 ± 0.1	C. Gnendiger *et al.* (2013)

All in 10^{-10} units



$$\Delta a_\mu \simeq a_\mu^{\text{QED}}(\alpha^4) \simeq 2a_\mu^{\text{EW}} \simeq 3a_\mu^{\text{HLbL}} \simeq 60\, a_\mu^{\text{QED}}(\alpha^5) \simeq 0.05\, a_\mu^{\text{HVP-LO}}$$

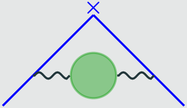
HVP contribution to the anomaly

HVP and a_μ

The two-point correlator Π obeys to the one sub. disp. rep. for $Q^2 = -q^2 > 0$

$$= \Pi(Q^2) = \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left[\frac{-Q^2}{t+Q^2} \right] \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$

which gives a contribution



$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{\pi} \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)} \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$

Determination of a_μ^{HVP}

$$a_\mu^{\mathrm{HVP}} = \int_0^1 dx (1-x) \int_{4m_\pi^2}^{\infty} \frac{dt}{t} \left[\frac{\frac{x^2}{1-x} m_\mu^2}{t + \frac{x^2}{1-x} m_\mu^2} \right] \frac{1}{\pi} \mathrm{Im}\,\Pi(t) \qquad \longleftarrow \frac{t}{4\pi^2\alpha} \sigma_{e\bar{e} \xrightarrow[\gamma*]{} \text{Hadr.}}(t)$$

$$= -\int_0^1 dx\,(1-x)\,\Pi\left(\frac{x^2}{1-x} m_\mu^2\right) \qquad \longleftarrow \text{LQCD, Models,...}$$

B. Lautrup , E. de Rafael (1969), E. de Rafael (1994), T. Blum (2003)

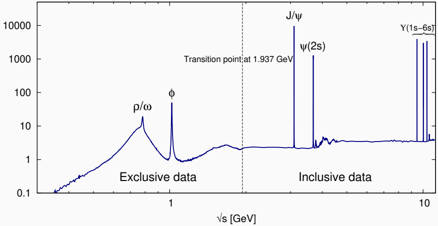
Spectral Function Moments

$$a_\mu^{\text{HVP}} = \int_0^1 \mathrm{d}x(1-x)\int_{4m_\pi^2}^\infty \frac{\mathrm{d}t}{t}\left[\frac{\frac{x^2}{1-x}m_\mu^2}{t+\frac{x^2}{1-x}m_\mu^2}\right]\frac{1}{\pi}\mathrm{Im}\,\Pi(t) = -\int_0^1 \mathrm{d}x\,(1-x)\,\Pi\left(\frac{x^2}{1-x}m_\mu^2\right)$$

Both integrals are related through the moments of $\frac{1}{\pi}\mathrm{Im}\,\Pi$ $(n \in \mathbf{N})$

$$\mathcal{M}(-n) = \int_{4m_\pi^2}^\infty \frac{\mathrm{d}t}{t}\left(\frac{4m_\pi^2}{t}\right)^n \frac{1}{\pi}\mathrm{Im}\,\Pi(t) = \frac{(-4m_\pi^2)^n}{n!}\left.\frac{\partial^n \Pi(t)}{\partial t^n}\right|_{t=0}$$

It is a Stieljes moments problem: How to get $\Pi(t)$ for any t knowing its moments?



From A. Keshavarzi *et al.* Phys. Rev. D **97** (2018)

Bell - de Rafael Bound

The first moment is an upper bound

$$a_\mu^{\text{HVP}} \leqslant \frac{\alpha}{\pi}\frac{m_\mu^2}{4m_\pi^2}\frac{1}{3}\mathcal{M}(0)$$

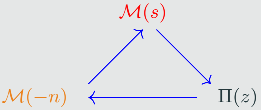
Actually a good approximation for a_e^{HVP}.

J. Bell and E. de Rafael Nucl. Phys. B **11** (1969)

The Mellin Approximants

Ramanujan Master Theorem $z \doteq Q^2/4m_\pi^2$

$$\int_0^\infty \mathrm{d}z z^{s-1}\left[-\frac{1}{z}\Pi(z) \underset{z\to 0}{\sim} \mathcal{M}(0)-\mathcal{M}(-1)z+\mathcal{M}(-2)z^2+\cdots\right] = \Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)$$



The discrete moments $\mathcal{M}(-n)$ coincides at $s=-j$ with $\mathcal{M}(s)$ for $s \in \mathbf{C}$. $\mathcal{M}(s)$ provides the analytic continuation of $\Pi(z)$ for $z \in \mathbf{C}$.

Mellin transform and transform

The inverse Mellin representation of Π $(0 < c < 1)$

$$\Pi(z) = -z \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2i\pi} z^{-s}\Gamma(s)\Gamma(1-s)\,\mathcal{M}(s)$$

$$\Gamma(s)\Gamma(1-s)\,\mathcal{M}(s) \underset{\mathrm{Re}\, s\leqslant 0}{\asymp} \frac{\mathcal{M}(0)}{s} - \frac{\mathcal{M}(-1)}{s+1} + \frac{\mathcal{M}(-2)}{s+2} + \cdots$$

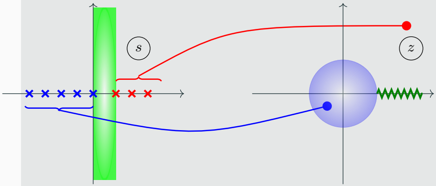
$$\underset{\mathrm{Re}\, s\geqslant 1}{\asymp} -\frac{\alpha}{\pi}\frac{5}{3}\frac{N_c}{3}\frac{1}{(s-1)^2} + \cdots$$

from the first order perturbative QCD result: $\Pi(z) \underset{z\to\infty}{\sim} -\frac{\alpha}{\pi}\frac{5}{3}\frac{N_c}{3} z \ln z$.

Inverse Mellin transform and the anomaly

Mellin transform

$$\mathcal{M}(s) = \int_{4m_\pi^2}^{\infty} \frac{\mathrm{d}t}{t} \left(\frac{t}{4m_\pi^2}\right)^{s-1} \frac{1}{\pi} \mathrm{Im}\,\Pi(t)$$



Singularities in s-plane drives behaviours in the full z-plane.

HVP anomaly contribution and Inverse Mellin transform

E. de Rafael, Phys. Lett. **B 736** 522 (2014)

$$a_\mu^{\mathrm{HVP}} = -\frac{\alpha}{\pi} \frac{m_\mu^2}{4m_\pi^2} \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2i\pi} \left(\frac{m_\mu^2}{4m_\pi^2}\right)^{-s} \Gamma(3-2s)\, \Gamma(-3+s)\, \Gamma(1+s) \mathcal{M}(s)$$

$\mathcal{M}(s)$ and its singularities drive completely the contribution to the anomaly.

Mellin Approximants

The Marichev - de Rafael Approximation

One claims that $\mathcal{M}(s)$ can be written as (λ, a, b, c and d parameters)

$$\mathcal{M}(s) \approx \mathcal{M}_N(s) = \sum_{n=1}^{N} \lambda_n \prod_{i,j,k,l} \frac{\Gamma(a_i - s)\Gamma(c_j + s)}{\Gamma(b_k - s)\Gamma(d_l + s)}$$

Recall of Mellin-Barnes integrals properties

M. Passare et al. , Cont. Comp. An.Geo., Aspects of Mathematics, vol. E**26** (1994)

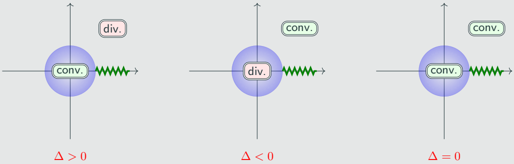
M. Passare et al., Theor. Math. Phys. **109** (1997) 1544

$$I(z) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, z^{-s} \frac{\prod_{j=1}^{m} \Gamma(A_j s + B_j)}{\prod_{k=1}^{n} \Gamma(C_k s + D_k)} \quad \text{and} \quad \begin{cases} \Delta \doteq \sum_{j=1}^{m} A_j - \sum_{k=1}^{n} C_k \\ \alpha \doteq \sum_{j=1}^{m} |A_j| - \sum_{k=1}^{n} |C_k| \end{cases}$$

$I(z)$ converges for $|\arg z| < \frac{\pi}{2}\alpha$

- If $\Delta > 0$ or $\Delta < 0$ the series expansions are not convergent everywhere.
- If $\Delta = 0$ the two expansions for $|z| < 1$ and $|z| > 1$ converges.
- If $\Delta = 0$ and $\alpha > 0$ each of the two series is an analytic continuation of the other.

The Mellin Approximants



If we impose to

- The fundamental strip is invariant: the first pole on the right is at $s = 1$.
- Let unchanged the values of $\mathcal{M}(-n)$ for $n \in \mathbf{N}$ then $j = 0$ and $d_l \notin \mathbf{N}$
- The Mellin representation is convergent for any value of z: $\Delta = k + 1 - i - l = 0$.
- The analyticity is unchanged *i.e.* the expansion for $|z| < 1$ is the analytic continuation of the expansion at $|z| > 1$: $\alpha = i - k - l > 0$ or $l = 0$

Therefore the **The Mellin Approximants written as**

$$\mathcal{M}_{(N,N')}(s) = \sum_{n=1}^{N} \lambda_n \prod_{k=1}^{N'} \frac{\Gamma(a_{n,k} - s)}{\Gamma(b_{n,k} - s)}$$

are a convergent integral representation for all Q^2.

In practice

Identification

- The matching of the first $N + 2N' - 2$ moments of the spectral function:

$$\forall j \in \{0, 1, \ldots, N + 2N' - 2\}, \quad \mathcal{M}_{(N,N')}(-j) = \mathcal{M}(-j)$$

- The first order of perturbative QCD residue fixes 2 constraints:

$$\mathcal{M}_{(N,N')}(s) \underset{s \to 1}{\sim} \frac{\alpha}{\pi} \frac{5}{3} \frac{N_c}{3} \frac{1}{s-1}$$

Therefore constants λ, a and b's are fixed $(\lambda_1 = 1$ and $b_{1,1} \geqslant 2)$

$$\mathcal{M}_{(N,N')}(s) = \frac{\alpha}{\pi} \frac{5}{3} \frac{N_c}{3} \sum_{n=1}^{N} \lambda_n \frac{\Gamma(1-s)}{\Gamma(b_{1,1}-s)} \prod_{k=2}^{N'} \frac{\Gamma(a_{n,k}-s)}{\Gamma(b_{n,k}-s)}$$

Physics constraints

But they are not unique, because of the polynomial relation: $\Gamma(z+1) = z\Gamma(z)$

The positivity of $\operatorname{Im}\Pi$ implies that $\mathcal{M}(-x)$ for $x \in \mathbf{R}_+$ is a completely monotonic function. We choose among all the solutions the set preserving this property.

Reconstruction of Π

Since

$$\Pi_{\text{Approx.}}(Q^2) = -\frac{Q^2}{4m_\pi^2} \int_{c-i\infty}^{c+i\infty} \frac{\mathrm{d}s}{2i\pi} \left(\frac{Q^2}{4m_\pi^2}\right)^{-s} \Gamma(s)\Gamma(1-s)\, \mathcal{M}_{(N,N')}(s)$$

this expression corresponds to a combination of Meijer's G - functions

$$\Pi_{\text{Approx.}}(Q^2) = -\frac{\alpha}{\pi}\frac{5}{3}\frac{N_c}{3}\frac{Q^2}{4m_\pi^2} \sum_{n=1}^{N} \lambda_n G^{1,N'+1}_{N'+1,N'+1}\left(\begin{matrix} 0, 0, \ldots, 1-a_{n,N'} \\ 0, 1-b_{n,1}, \ldots, 1-b_{n,N'} \end{matrix} \middle| \frac{Q^2}{4m_\pi^2} \right)$$

Meijer's G-functions are an extension of the usual hypergeometric functions. Their analytic continuation is well-defined.

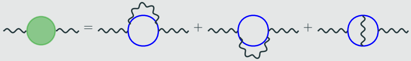
Example:

$$G^{1,2}_{2,2}\left(\begin{matrix} 0, 1 \\ 0, 1 \end{matrix} \middle| z \right) = \begin{cases} \ln(1-z) & |z| < 1 \\ \ln(z-1) + i\pi & |z| > 1 \end{cases}$$

The QED Vacuum Polarization example

QED 4^{th} vaccum polarization contribution

Källen and Sabry (1954), B. Lautrup and E. de Rafael (1968)



$$
= \left(\frac{\alpha}{\pi}\right)^2 \left\{ \frac{5\delta}{8} - \frac{3\delta^3}{8} - \left(\frac{\delta}{2} - \frac{\delta^3}{6}\right) \ln\left[\frac{64\delta^4}{(1-\delta^2)^3}\right] \right.
$$

$$
+ \left(\frac{11}{16} + \frac{11\delta^2}{24} - \frac{7\delta^2}{48} + \left(\frac{1}{2} + \frac{\delta^2}{3} - \frac{\delta^4}{6}\right) \ln\left[\frac{(1+\delta)^3}{8\delta^2}\right] \right) \ln\left(\frac{1+\delta}{1-\delta}\right)
$$

$$
\left. + \left(1 + \frac{2\delta^2}{3} - \frac{\delta^4}{6}\right) \left[2\mathrm{Li}_2\left(\frac{1-\delta}{1+\delta}\right) + \mathrm{Li}_2\left(\frac{\delta-1}{1+\delta}\right) \right] \right\}
$$

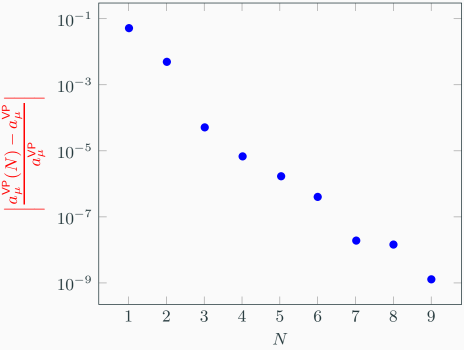
with $\delta^2 = 1 - {}^1/_z$

All discrete moments $\mathcal{M}(-j)$ can be generated (even $\mathcal{M}(s)$).

J. Mignaco and E. Remiddi (1969)

$$
a_\mu^{\mathrm{VP}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ \frac{673}{108} - \frac{41}{81}\pi^2 - \frac{4}{9}\pi^2 \log(2) - \frac{4}{9}\pi^2 \log^2(2) + \frac{4}{9}\log^4(2) \right.
$$

$$
\left. - \frac{7}{270}\pi^4 + \frac{13}{18}\zeta(3) + \frac{32}{3}\mathrm{Li}_4\left(\frac{1}{2}\right) \right\} = \left(\frac{\alpha}{\pi}\right)^3 0.0528707\cdots
$$



$$a_\mu^{\mathrm{VP}} = \left(\frac{\alpha}{\pi}\right)^3 0.0528707\cdots$$

All in $(\alpha/\pi)^3$ units

Input $\mathcal{M}(-j)$	$a_\mu^{\mathsf{VP}}(N)$	Accuracy
$j = 0$	0.0500007	5%
$j = \{0, 1\}$	0.0531447	0.5%
$j = \{0, 1, 2\}$	0.0528678	0.004%
$j = \{0, \ldots, 3\}$	0.0528711	0.00075%
$j = \{0, \ldots, 4\}$	0.0528706	0.00018%

Convergence

The very illustrative example of QED shows that the Mellin Approximation converges very quickly $(\Delta \sim 10^{-N})$.

Unfortunately the degree of complexity (the number of solutions for the matching) increases too $(\sim N^2)$.

The QCD HVP Contribution

Experimental analysis

A. Keshavarzi *et al.* Phys. Rev. D **97** (2018)

Experimental evaluation of the anomaly

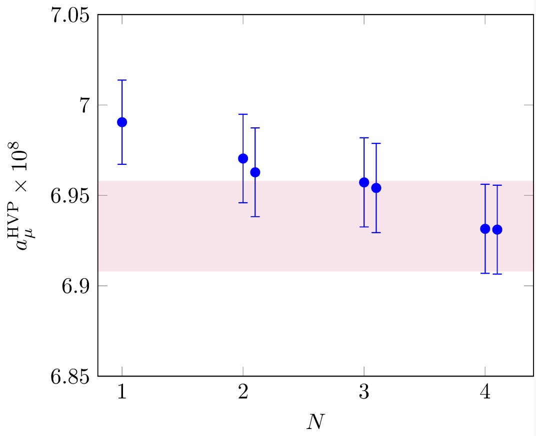
$$a_\mu^{\text{HVP}}(\text{exp.}) = (\mathbf{693.3} \pm \mathbf{2.5}) \times 10^{-10}$$

Moment $\mathcal{M}(-j)$	Experimental Value $\times 10^3$	Relative Error
0	0.7176 ± 0.0026	0.36%
1	0.11644 ± 0.00063	0.54%
2	0.03041 ± 0.00029	0.95%
3	0.01195 ± 0.00017	1.4%
4	0.00625 ± 0.00011	1.8%
5	0.003859 ± 0.000078	2.0%
$\dots$	$\dots$	$\dots$

Kindly provided by Alex Keshavarzi and Thomas Teubner

Application of Mellin Approximation

Input	Approximant	Central Value	Stat. Uncert.
0	1	**699.1**	**2.3**
$\{0, 1\}$	$1 + 1$	**697.0**	**2.4**
$\{0, 1, 2\}$	$2 + 1$	**695.7**	**2.5**
$\{0, \ldots, 3\}$	$2 + 1 + 1$	**693.2**	**2.5**



The simplest case: The Beta approximant

The Beta approximant case

One considers a simple case $(N, N' = 1)$ and $a_n = n$, one has

$$\mathcal{M}_N(s) = \frac{\alpha}{\pi}\frac{5}{3}\sum_{n=1}^{N}\lambda_n\Gamma(b_n - n)\frac{\Gamma(n-s)}{\Gamma(b_n - s)}$$

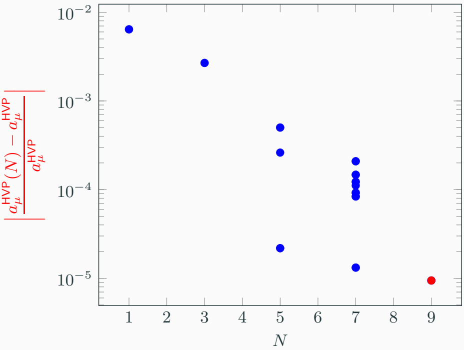
with $\lambda_1 = 1$ and $b_n \geqslant n+1$ for guaranteeing the positivity. One gets

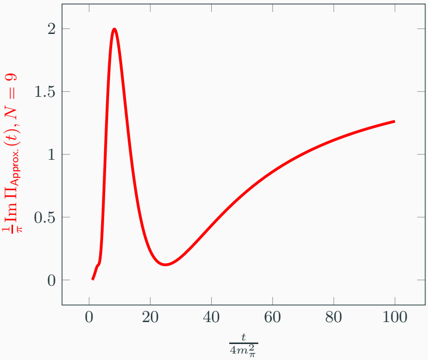
$$\Pi_{\text{Approx.}}(Q^2) = \frac{\alpha}{\pi}\frac{5}{3}\sum_{n=1}^{N}\lambda_n\frac{\Gamma(b_n - n)}{\Gamma(b_n)}\ {}_2\mathrm{F}_1\left(\begin{matrix}1, n\\ b_n\end{matrix}\middle| -\frac{Q^2}{4m_\pi^2}\right)$$

and an associated spectral function

$$\frac{1}{\pi}\mathrm{Im}\,\Pi_{\text{Approx.}}(t) = \sum_{n=1}^{N}\lambda_n\left(\frac{4m_\pi^2}{t}\right)^{n-1}\left(1-\frac{4m_\pi^2}{t}\right)^{b_n-n-1}\vartheta(t-4m_\pi^2)$$

The simplest case: The Beta approximant





Convergence for QCD

The simplest case of Mellin Approximants for the QCD HVP converges quickly to the experimental value for the anomaly.

The associated imaginary part shows a resonance structure.

Conclusion

Conclusion

- We have built the Mellin Approximation to the spectral function from few of its moments. This approximation is
  - valid for any value of Q^2.
  - preserving the analytic structure (cuts, singularities,...) – contrary to Padé approximants.
  - converging fast (for a moderate complexity).
- The Mellin Approximation is well adapted to QCD because only asymptotic behaviours are used (no extra scales).
- Regarding the HVP contribution to the muon anomaly, we are able to reproduce experimental evaluation with 4 moments.
- We are working on adapting this method to Lattice QCD evaluation: not by matching on moments because they are difficult to obtain but on values of $\Pi(Q^2)$ directly, more precisely on the Time Momentum Representation of it.