Diocotron instability of a hollow electron beam in the external magnetic field

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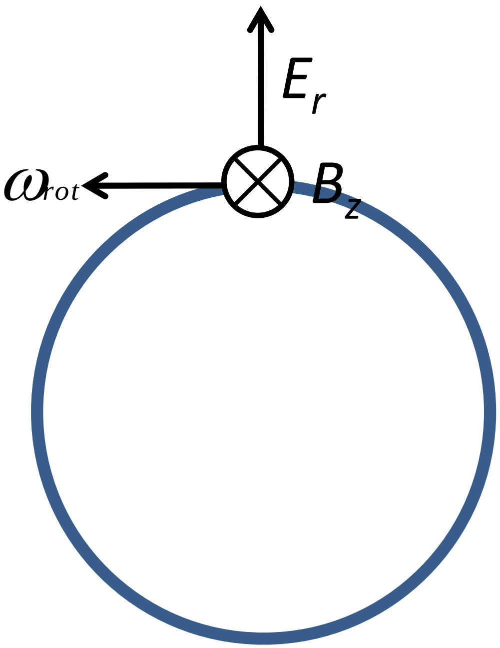
Origin of the diocotron instability

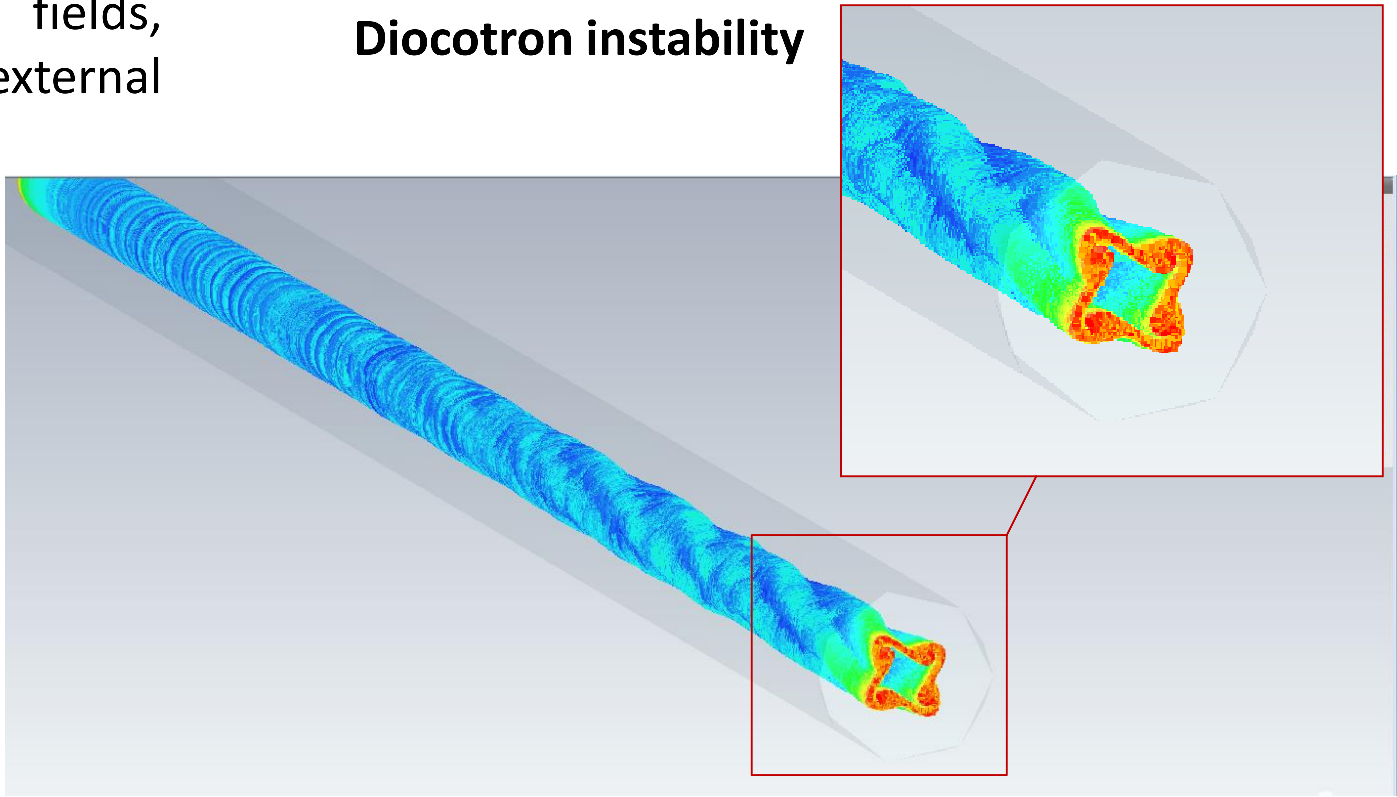
$$\omega_{rot}(r) = \frac{E_r(r)}{rB_z}$$

Angular velocity for the given radius r (arises in crossed electric and magnetic fields, beam field $E_r(r)$ and external magnetic field B_z

Different angular velocities for different radii provide **relative motion of layers.** It **may** lead to the density equilibrium violation and cluster origin

Diocotron instability





Theoretical consideration (Davidson, Physics of nonneutral plasma)

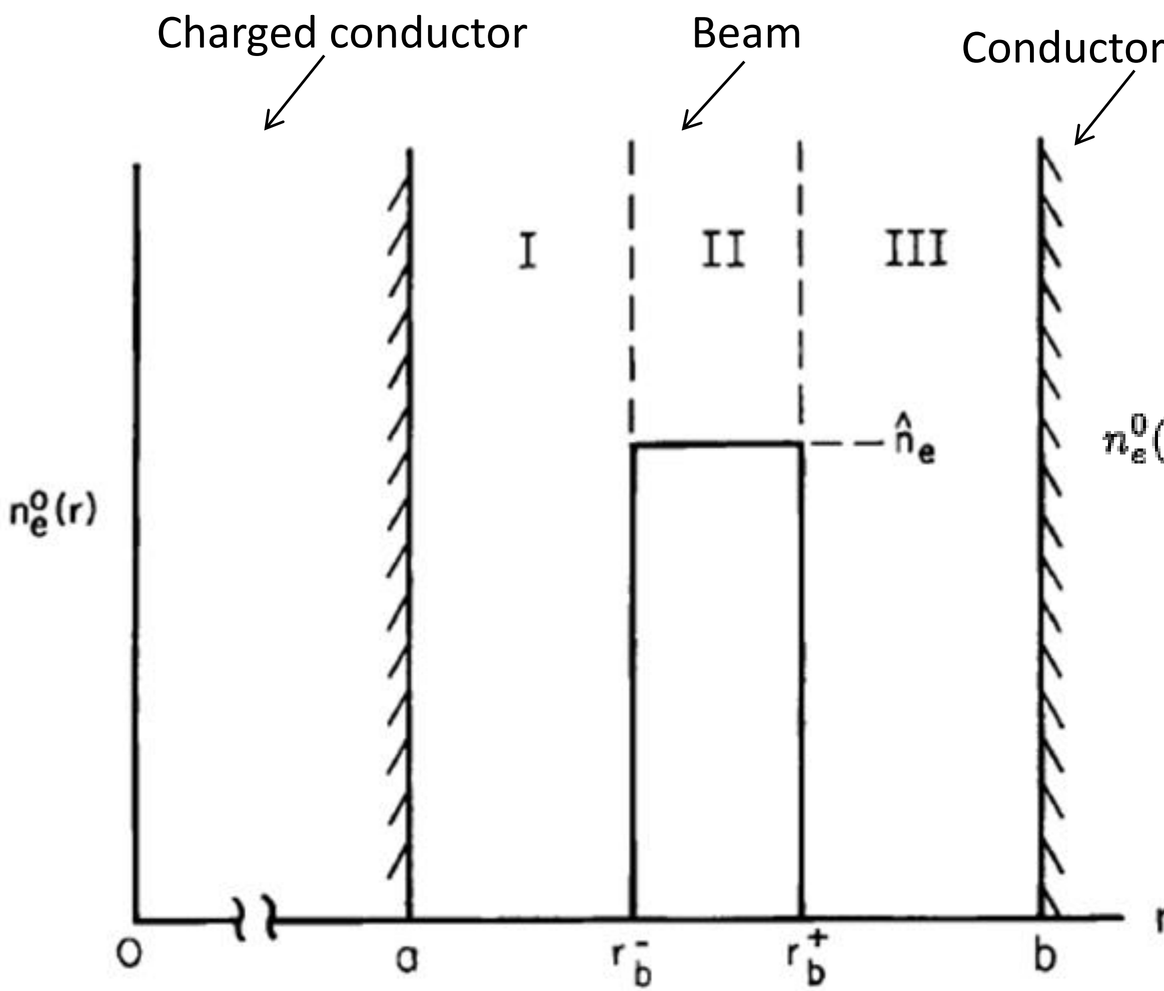


Figure 6.2. Annular electron density profile $n_e^0(r)$ assumed in Eq.(6.30). The inner conductor at $r = a$ carries a charge Q per unit length.

$$n_e^0(r) = \begin{cases} 0 , & a \le r < r_b^- , \\ \hat{n}_e = \text{const.} , & r_b^- < r < r_b^+ , \\ 0 , & r_b^+ < r \le b , \end{cases}$$

Theoretical consideration

To investigate stability properties, we assume small-amplitude perturbations of an azimuthally symmetric equilibrium (characterized by the density profile). These perturbations can be found from the Poisson's equation:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\phi_0(r) = 4\pi e n_e^0(r)$$

We shall find perturbed density and potential as Fourier series:

$$n_e(r,\theta,t) = n_e^0(r) + \sum_{\ell=-\infty}^{\infty} \delta n_e^{\ell}(r)\exp(i\ell\theta - i\omega t)$$

$$\phi(r,\theta,t) = \phi_0(r) + \sum_{\ell=-\infty}^{\infty} \delta\phi^{\ell}(r)\exp(i\ell\theta - i\omega t)$$

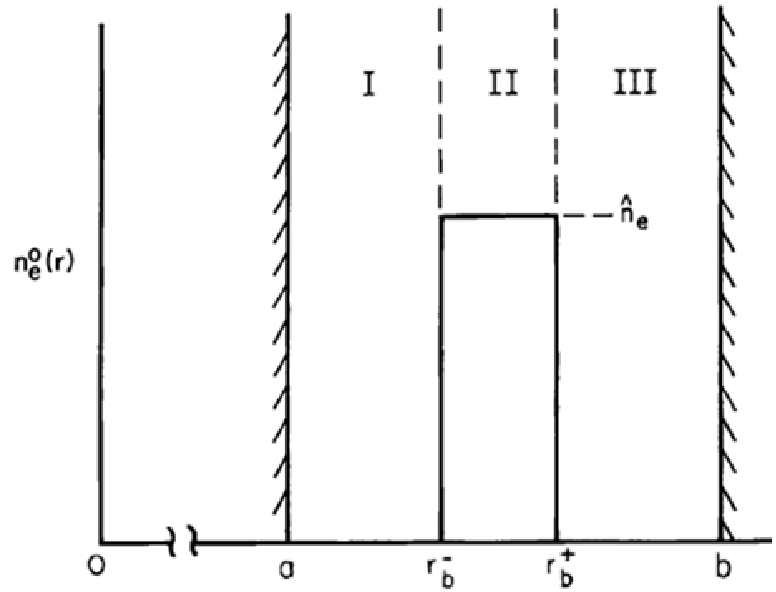
L here means number of the eigenfunction;
in terms of diocotron instability *L* is the number of clusters to be formed

We derive the dispersion relation:

$$(\omega/\omega_D)^2 - b_\ell\,(\omega/\omega_D) + c_\ell = 0$$

$$b_\ell = \left(\ell\left\{\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2\right] + \frac{\omega_q}{\omega_D}\left[1+\left(\frac{r_b^-}{r_b^+}\right)^2\right]\right\}\left[1-\left(\frac{a}{b}\right)^{2\ell}\right] + \left[1-\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\right]\left[\left(\frac{r_b^+}{b}\right)^{2\ell} - \left(\frac{a}{r_b^-}\right)^{2\ell}\right]\right)\left[1-\left(\frac{a}{b}\right)^{2\ell}\right]^{-1}$$

$$\begin{aligned} c_\ell = \Bigg(&\ell^2\frac{\omega_q}{\omega_D}\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2 + \frac{\omega_q}{\omega_D}\left(\frac{r_b^-}{r_b^+}\right)^2\right]\left[1-\left(\frac{a}{b}\right)^{2\ell}\right] \\ &- \ell\frac{\omega_q}{\omega_D}\left[1-\left(\frac{a}{r_b^+}\right)^{2\ell}\right]\left[1-\left(\frac{r_b^+}{b}\right)^{2\ell}\right] \\ &+ \ell\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2 + \frac{\omega_q}{\omega_D}\left(\frac{r_b^-}{r_b^+}\right)^2\right]\left[1-\left(\frac{r_b^-}{b}\right)^{2\ell}\right]\left[1-\left(\frac{a}{r_b^-}\right)^{2\ell}\right] \\ &- \left[1-\left(\frac{r_b^+}{b}\right)^{2\ell}\right]\left[1-\left(\frac{a}{r_b^-}\right)^{2\ell}\right]\left[1-\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\right]\Bigg)\left[1-\left(\frac{a}{b}\right)^{2\ell}\right]^{-1} \end{aligned}$$



$$\omega_q = -\frac{2Qc}{B_0(r_b^-)^2} \qquad \omega_D = \frac{2\pi\hat{n}_e ec}{B_0}$$

Dispersion relation solution

$$(\omega/\omega_D)^2 - b_\ell(\omega/\omega_D) + c_\ell = 0$$

$$\omega = \frac{1}{2}\omega_D\left[b_\ell \pm (b_\ell^2 - 4c_\ell)^{1/2}\right]$$

$$b_\ell^2 \geq 4c_\ell$$

Real frequencies, **stable** state

Stability condition

$$4c_\ell > b_\ell^2$$

Complex frequencies

$$\mathrm{Re}\,\omega = \frac{1}{2}b_\ell\omega_D\ ,$$

$$\mathrm{Im}\,\omega = \pm\frac{1}{2}\left(4c_\ell - b_\ell^2\right)^{1/2}\omega_D$$

$\mathrm{Im}\,\omega > 0$ corresponds to the **instability**, growth of the oscillation amplitude

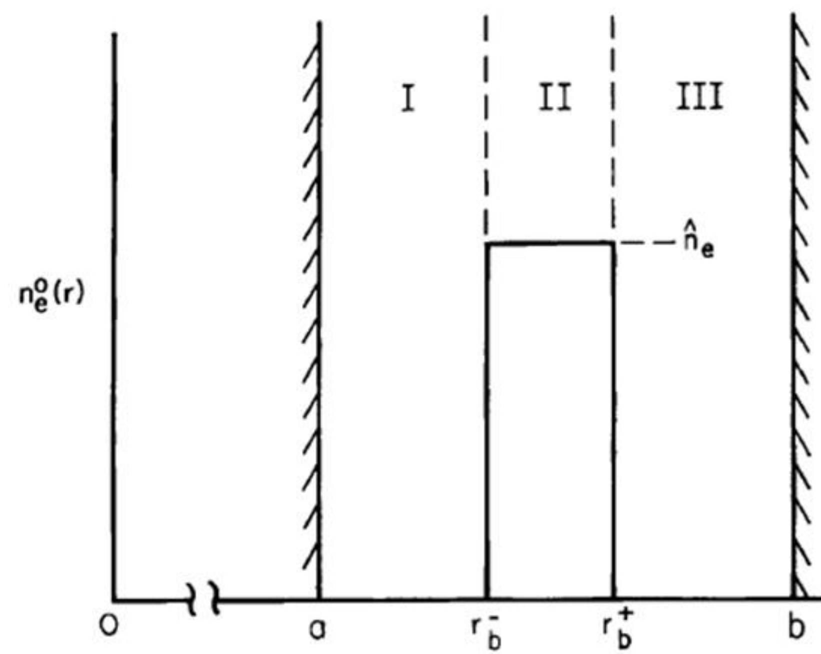
T = 1/Im ω – characteristic time of instability growth

Stability condition

Substituting b_l and c_l into stability condition, we obtain:

$$\left\{ -\ell\left(1-\frac{\omega_q}{\omega_D}\right)\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2\right]\left[1-\left(\frac{a}{b}\right)^{2\ell}\right] + 2\left[1+\left(\frac{a}{b}\right)^{2\ell}\right] - \left[1+\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\right]\left[\left(\frac{a}{r_b^-}\right)^{2\ell}+\left(\frac{r_b^+}{b}\right)^{2\ell}\right]\right\}^2$$

$$\geq 4\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\left[1-\left(\frac{r_b^+}{b}\right)^{2\ell}\right]^2\left[1-\left(\frac{a}{r_b^-}\right)^{2\ell}\right]^2$$

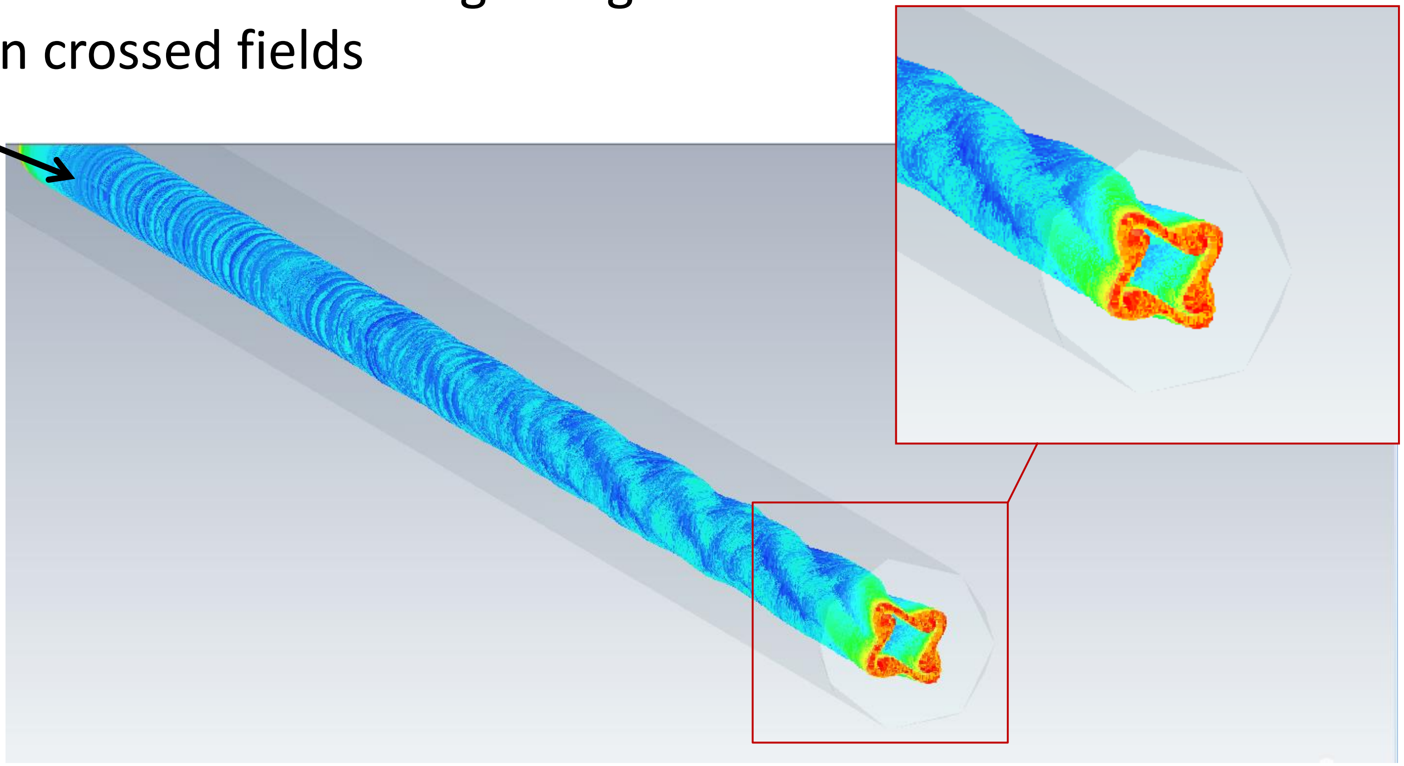


Diocotron instability

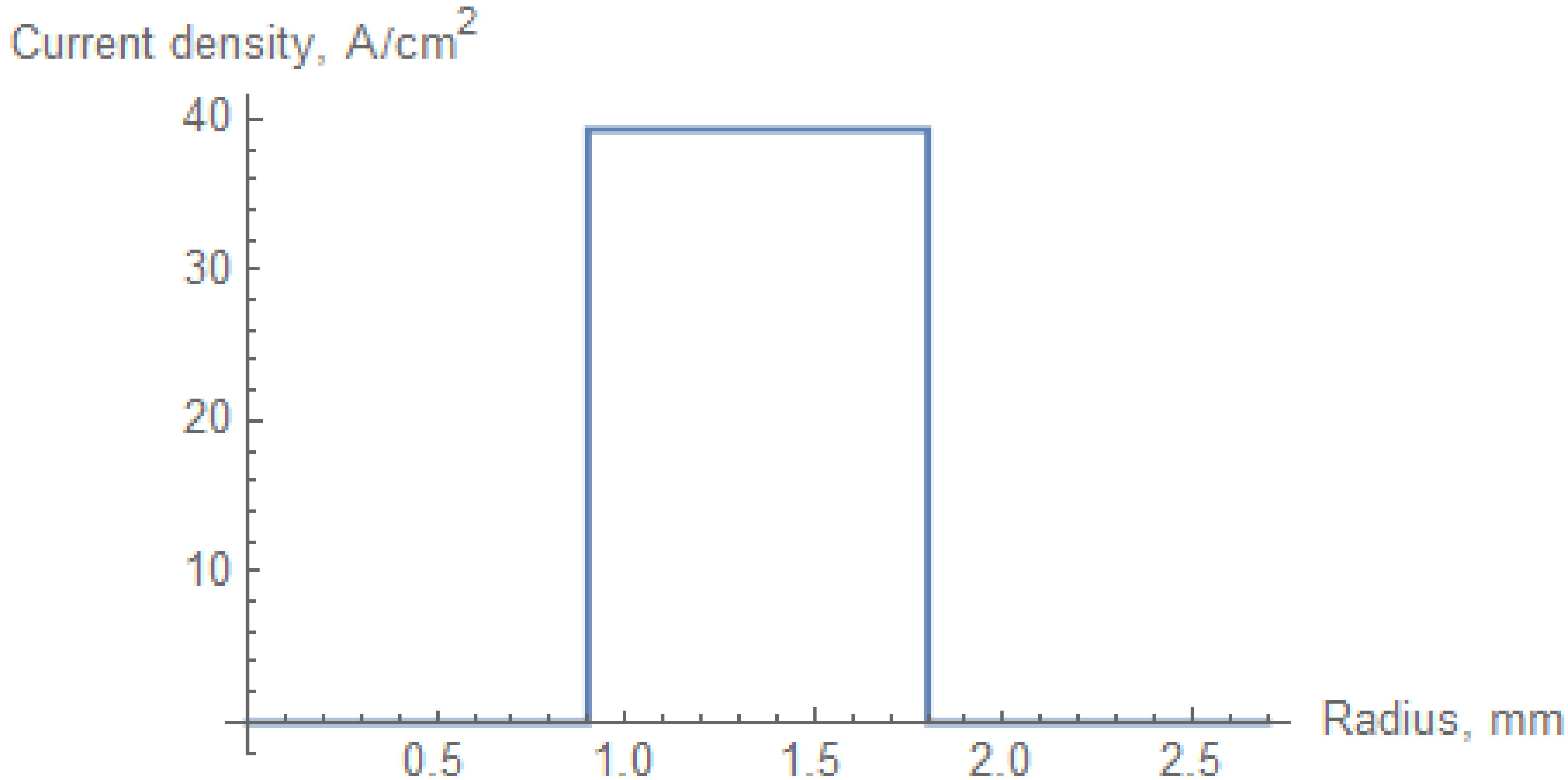
Current I = 3 A
Voltage U =12 kV
Magnetic field B=0.3 T

Inner radius r1 = 0.8 mm
Outer radius r2 = 1 mm
Tube radius b = 40 mm
Distance L=0.6 m

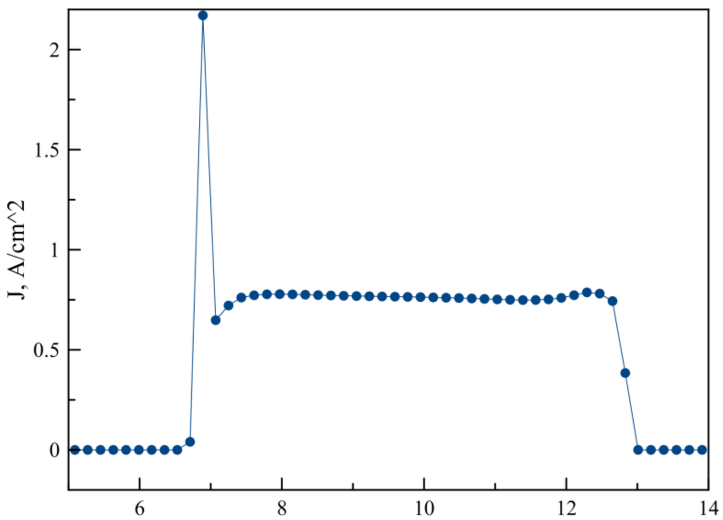
Beam shape is round at the beginning of motion in crossed fields



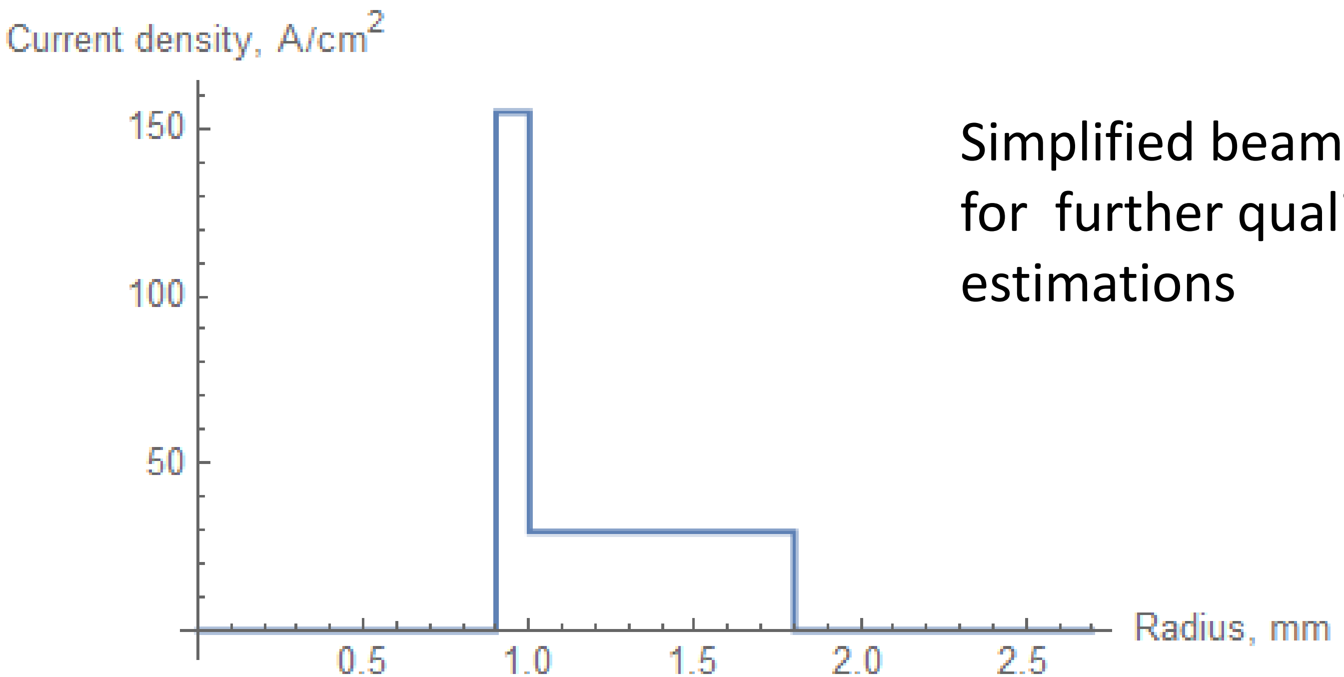
Proposed beam profiles: beam with a uniform density distribution



Proposed beam profiles: beam with a density peak

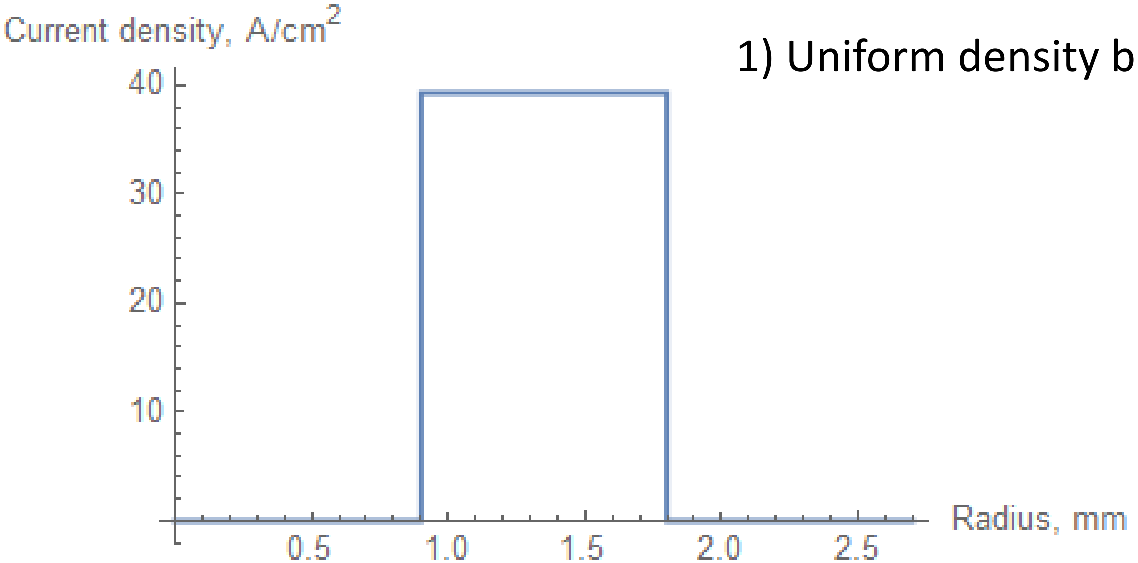


Beam density profile
from the Fermilab's gun

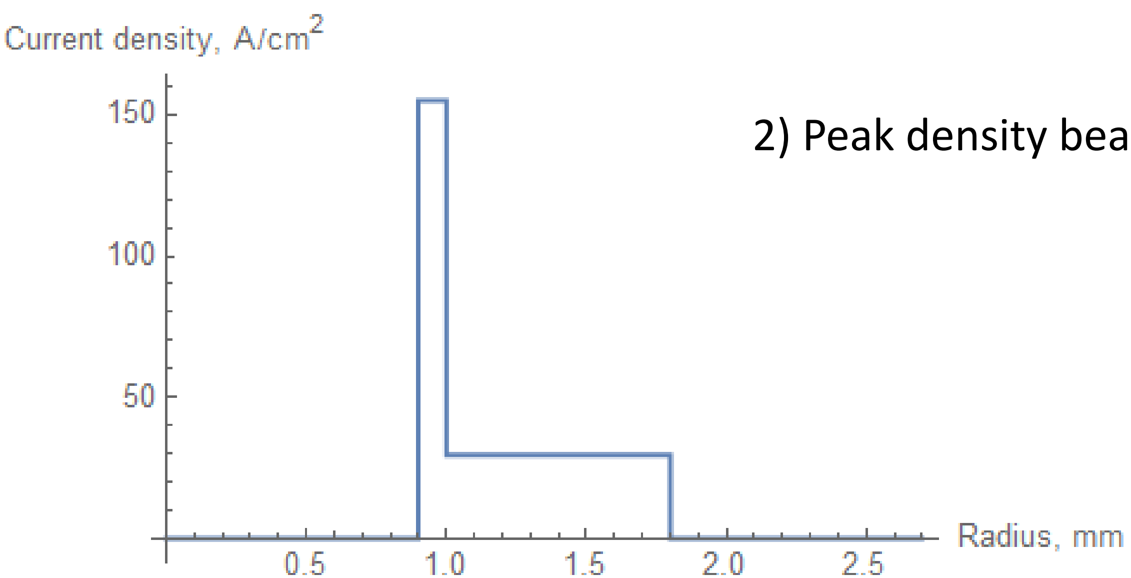


Simplified beam density profile
for further qualitative
estimations

Study of two beam profiles with the stability condition



1) Uniform density beam

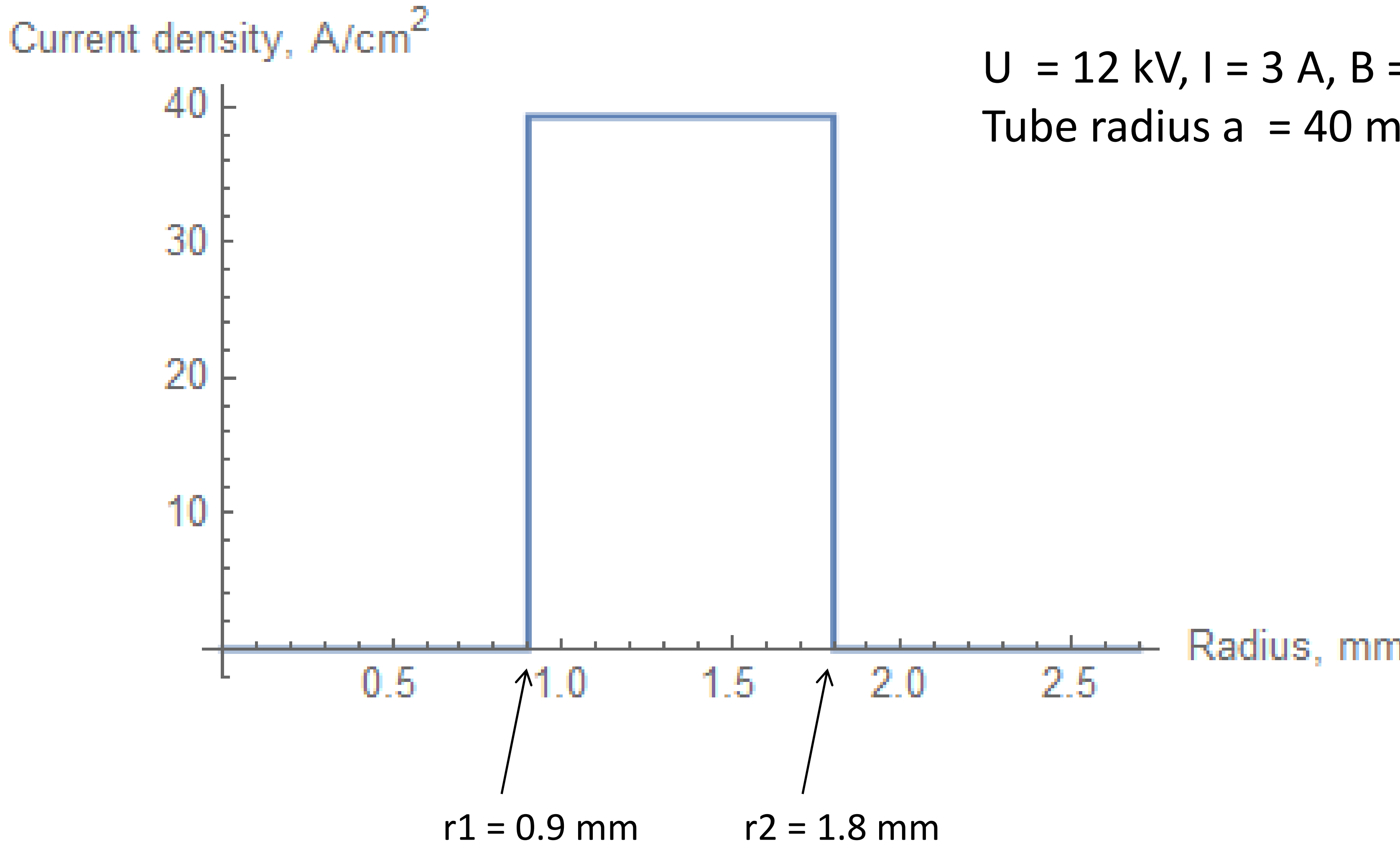


2) Peak density beam

The **same parameters**:
U = 12 kV, I = 3 A, B = 4 T
Inner radius r1 = 0.9 mm,
Outer radius r2 = 1.8 mm
Tube radius a = 40 mm

What beam is more stable?

Beam with a uniform density distribution



U = 12 kV, I = 3 A, B = 4 T
Tube radius a = 40 mm

Stability condition (6.43) for the uniform density beam

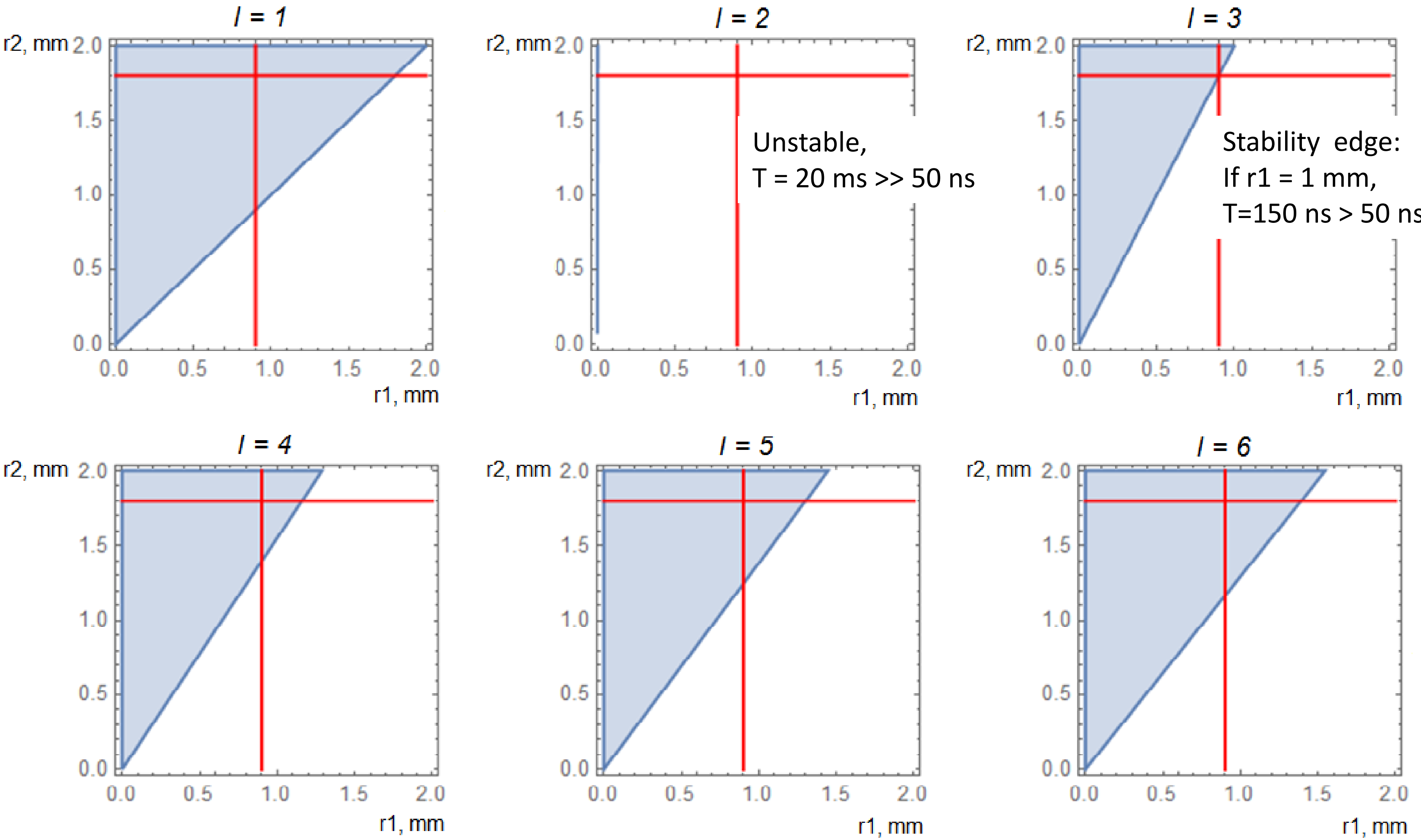
$$\left\{-\ell\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2\right]+2-\left[\left(\frac{r_b^+}{b}\right)^{2\ell}+\left(\frac{r_b^-}{b}\right)^{2\ell}\right]\right\}^2$$

$$\geq 4\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\left[1-\left(\frac{r_b^+}{b}\right)^{2\ell}\right]^2 . \qquad (6.44)$$

r_b^- − inner radius (r1)
r_b^+ − outer radius (r2)
b – tube radius (we use 40 mm)

Note: for the uniform density **beam stability condition depends on geometry** only.
Beam density and magnetic field affect rate of the instability growth

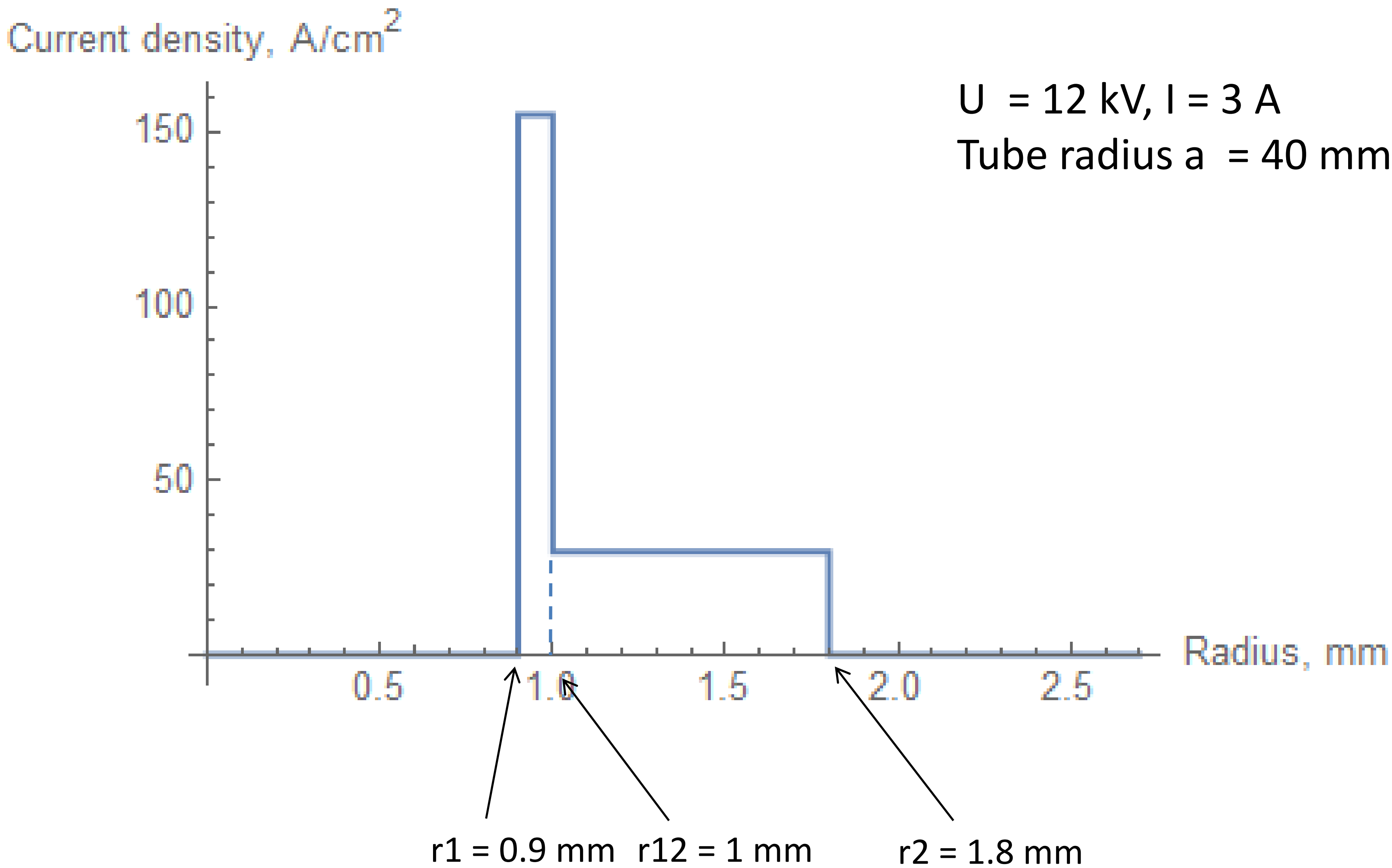
Stability charts for uniform density beam distribution



Shaded region correspond to the beam stable state. Red lines correspond to r1 = 0.9 mm, r2 = 1.8 mm. I. e., if line intersection lies in the shaded region, beam is stable

Beam with a peak density distribution

U = 12 kV, I = 3 A
Tube radius a = 40 mm



Stability condition (6.43) for the peak density beam

$$\left\{-\ell\left(1-\frac{\omega_q}{\omega_D}\right)\left[1-\left(\frac{r_b^-}{r_b^+}\right)^2\right]+2-\left[\left(\frac{r_b^+}{b}\right)^{2\ell}+\left(\frac{r_b^-}{b}\right)^{2\ell}\right]\right\}^2 \geq 4\left(\frac{r_b^-}{r_b^+}\right)^{2\ell}\left[1-\left(\frac{r_b^+}{b}\right)^{2\ell}\right]^2$$

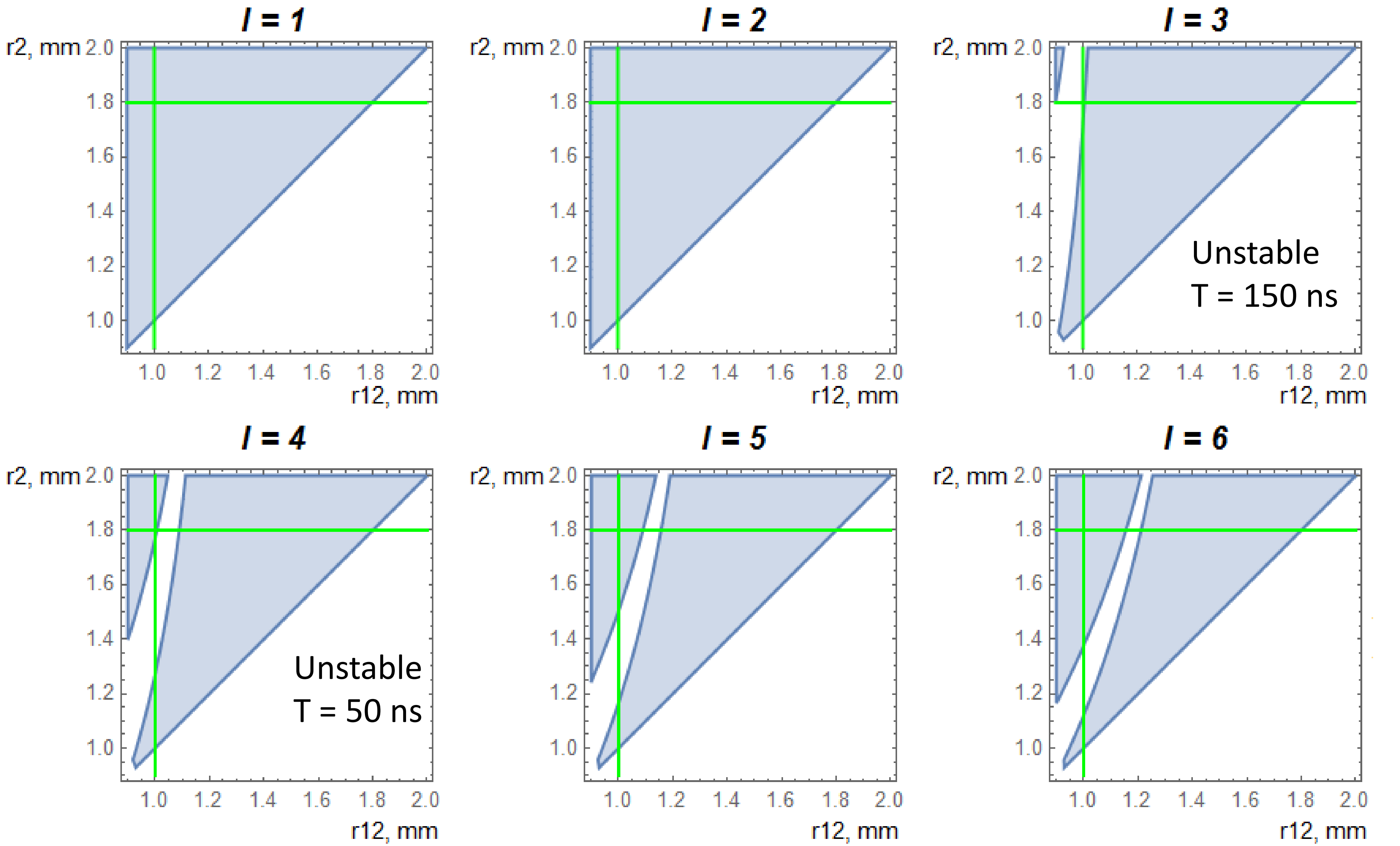
↑

Difference from the uniform density beam

$$\left(1-\frac{j1\,(r12^2-r1^2)}{j2\,r12^2}\right)$$

r_b^- – inner radius (r1)
r_b^+ – outer radius (r2)
b – tube radius (we use 40 mm)

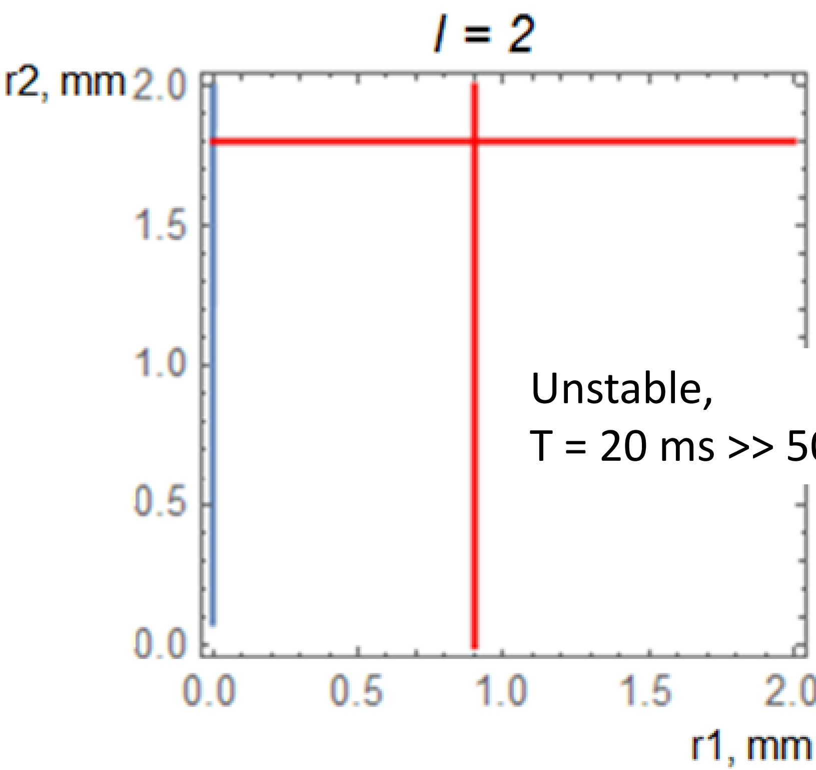
Stability charts for the beam with a density peak



Inner radius is fixed 0.9 mm, shaded region correspond to the beam stable state. Red lines correspond to r1 = 0.9 mm, r2 = 1.8 mm. I. e., if line intersection lies in the shaded region, beam is stable

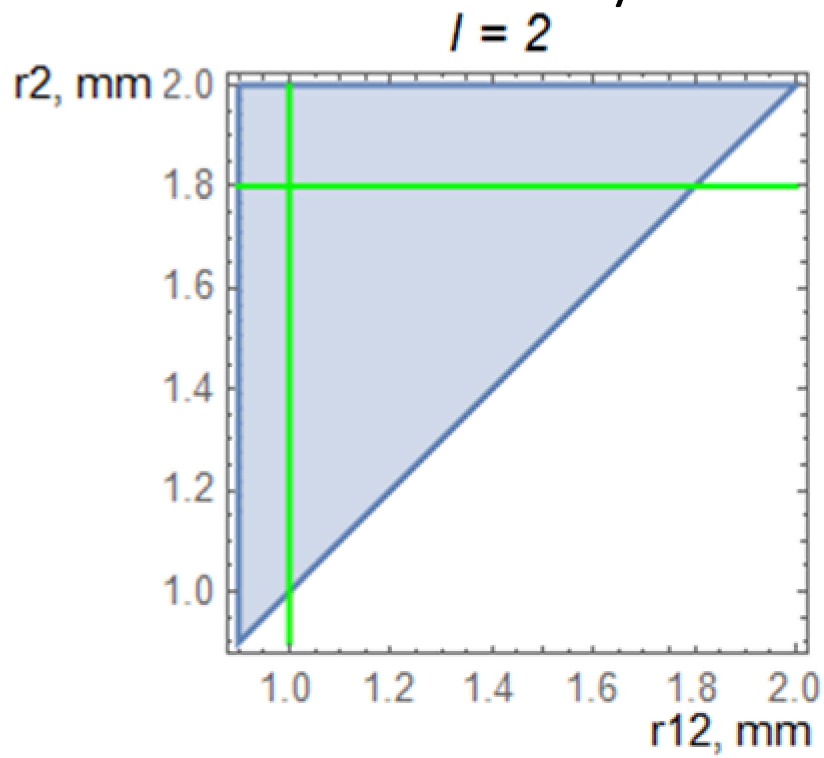
Stability comparison for mode *l* =2

Uniform density



Unstable,
T = 20 ms >> 50 ns

Peak density



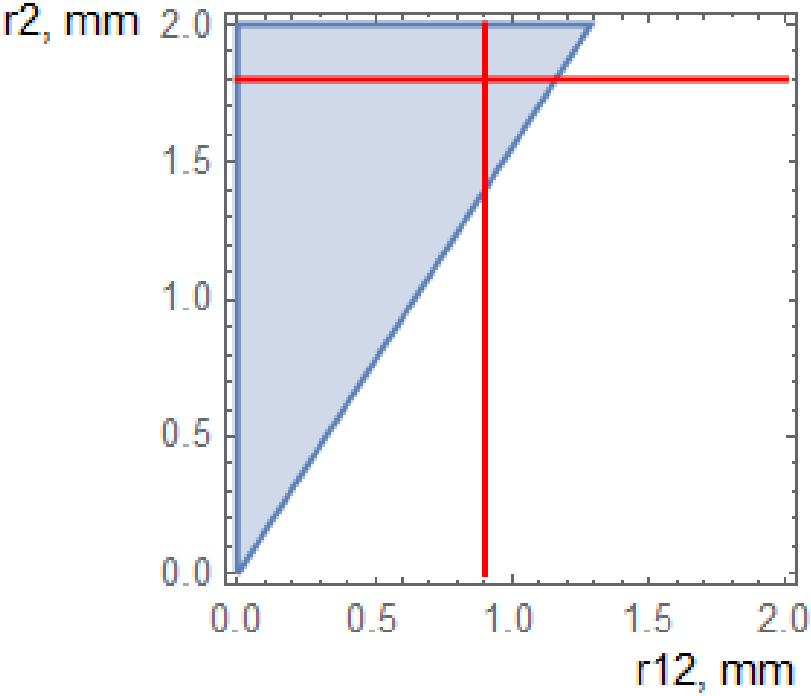
Unstable, but characteristic time significantly exceeds time of flight, **instability has no enough time to develop**

In general, for peak density beam stability region is larger

Stability comparison for mode *l = 4*

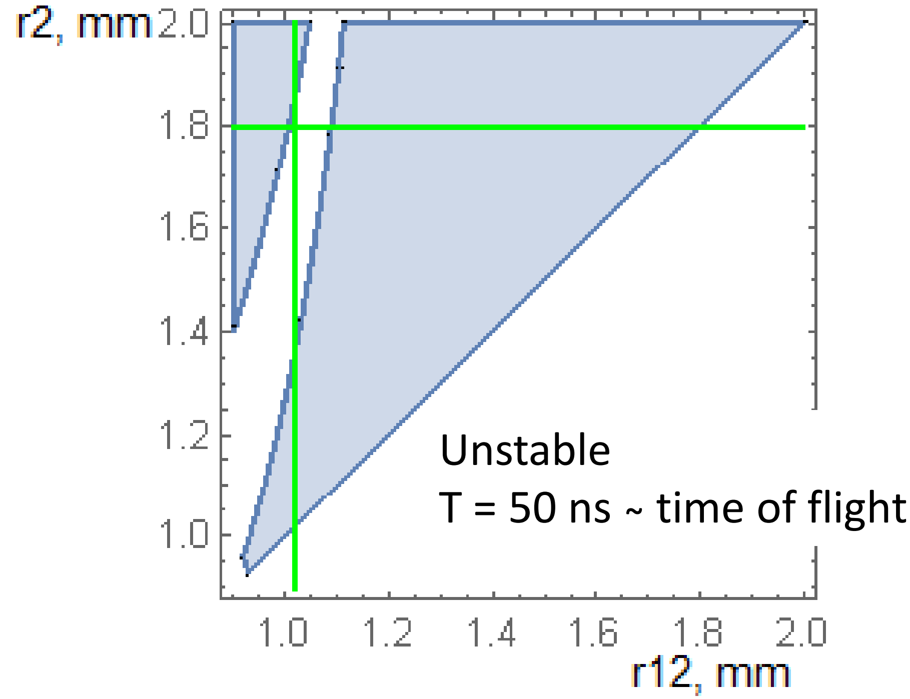
Uniform density

***l* = 4**



Peak density

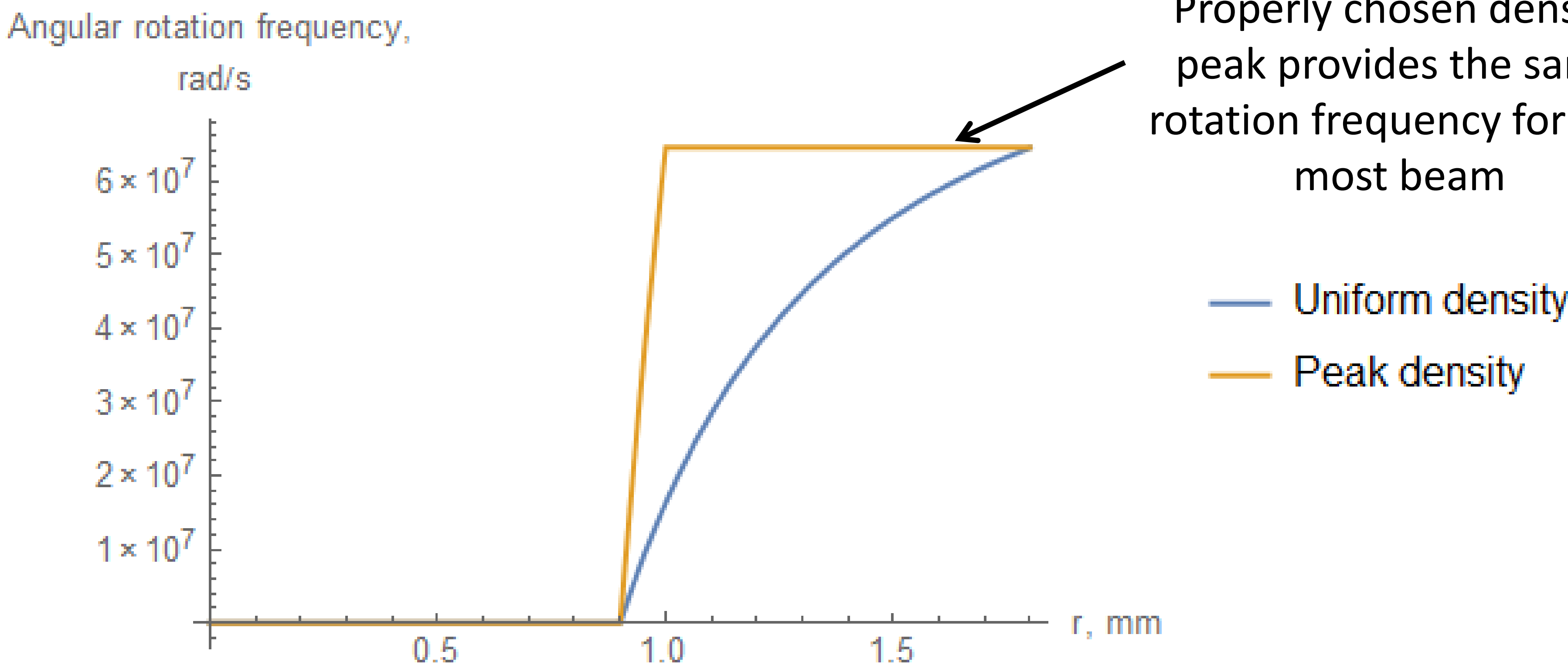
***l* = 4**



Unstable, characteristic time of instability is comparable with the time of flight

Also, for peak density beam stability region is larger, but there is an **instability region** which is very **sensitive to the density peak parameters**: small parameter deviation may lead to the instability

Why we obtain peak density beam more stable



Properly chosen density peak provides the same rotation frequency for the most beam

The same rotation frequency for different radii

↓

no relative motion of layers

↓

stability

Comparison of stability of two beams

Note: with the same beam size and the same current peak density beam is more stable (i.e. shaded region is larger)

But: in peak density beam there is non-shaded region. It is very sensitive to the beam sizes and current densities j1 and j2. We need **very precisely** choose peak density profile (j1, j2, r1, r12, r2). Otherwise, we **risk to fall into non-stable** region

Conclusion : behavior of the **uniform density beam is better predictable**. Uniform density beam is not so sensitive to the size changing

It is more preferable to work with uniform beam distribution