Dynamic Aperture and Momentum Acceptance in Low Emittance Storage Rings

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Alexander Papash*
IBPT KIT, 76344, Karlsruhe, Germany

* alexander.papash@kit.edu
“Original incentive to study the stability of the motion in non-linear dynamic systems has been prompted by development of Celestial Mechanics in XIX century to describe the orbital motion of Planets” (W.Scandale. “DYNAMIC APERTURE”, CAS CERN-95-06)

We discuss here single particle Beam Dynamics. Collective effects are out of scope of talk.

Trajectory of charge particles in magnetic fields of storage ring, composed of \((N)\) periodic cells each of length \((L)\) and total ring circumference \(C = L \cdot N\), is described by Lorentz equation

\[
m\dddot{\mathbf{R}} = -\frac{e}{c} \left[ \mathbf{R}' \times \mathbf{B} \right]
\]

Restoring forces are periodic \(K_Y(s + L) = K_Y(s)\) and particles oscillate around closed ("reference") orbit \(\mathbf{R}_0\). Here \((Y)\) stays for horizontal \((X)\) or vertical \((Z)\) components of transverse coordinates. K.Steffen. “Basic Course on accelerator Optics”. CAS CERN-85-19.

Oscillations of particles in transverse to beam motion direction - fast with respect to slow synchrotron oscillations of energy and phase

-- Transverse and longitudinal planes might be split

--- Motion in the transverse phase space might be considered independently on motion in longitudinal phase space

Solving Lorentz equation in cylindrical system of coordinates \((R, \theta, z)\) and

---assuming PARAXIAL beam conditions:

-- the deviation of horizontal \(X(s) = (R(s) - R_0(s))\)

and vertical \((Z)\) position of particle from reference orbit \(R_0\) much less than curvature radius \(X \ll R_0\)

--- angular deviations \(Y'(s) = dY(s)/ds \ll 1\)

---- one can NEGLECT high order terms and

--- derive LINEAR equations of harmonic oscillations

A.Wolski. US PAC School-2013
If periodic restoring forces acting on particle with reference momentum $p_0$ are perfectly \textbf{LINEAR}, the particle oscillations are \textbf{STABLE} and described by second order homogeneous differential equations (\textit{quasi-harmonic oscillator}) with periodic restoring force (\textbf{HILL equations}) (longitudinal coordinate $s$ is independent variable) E.Courant and H.Snyder Theory of of the alternating-gradient synchrotron. Ann.Phy, 3 (1958).

\[
\frac{d^2X(s)}{ds^2} + k_x(s)X(s) = 0 \quad \frac{d^2Z(s)}{ds^2} + k_z(s)Z(s) = 0
\]

- Radial Focusing term $1/\rho_0$ in magnets should be included into consideration $K_x(s) = \frac{1}{\rho_0(s)} - k_x(s)$
- Flat sector bend is a \textbf{drift} in axial direction $k_z(s) = 0$
- Focusing strength of quadrupole
  
  \[k_z(s) = -k_x(s) = \frac{1}{B \cdot \rho} \left( \frac{\partial B_x}{\partial x} \right)\]

- in bending plane $X_D = D \cdot (\Delta p/p_0) = D\delta$ and equations for off-momentum particles are

\[
\frac{d^2X(s)}{ds^2} + K_x(s)X(s) = \frac{1}{\rho_0(s)} \frac{\delta p}{p_0} \quad \text{and} \quad \frac{d^2D(s)}{ds^2} + K_x(s)D(s) = \frac{1}{\rho_0(s)}
\]


\[
\vec{Y}(s) = M(s/s_0) \cdot \vec{Y}(s_0)
\]

where $\vec{Y}(s_0)$ is vector of initial coordinates and momentums in 6D phase space

\[
\vec{Y}(s_0) = (x_0, p_{x0}, z_0, p_{z0}, \varphi_0, \delta)
\]
• Linear Transfer matrixes \((M)\) follow multiplication rule \\
\[M(s_2/s_0) = M(s_2/s_1) \cdot M(s_1/s_0)\]

• Transfer matrix for full revolution must repeat itself in order for motion to be **stable**

\[M(s + L) = M(s + N \cdot L) = M(s)\]

• **Necessary and sufficient condition** of STABLE motion – transfer matrix \([M]^{n-N}\) is **BOUNDED** at any \(n \rightarrow \infty\)

• Solutions of linear equations of motion are Real part of periodic quasi-harmonic function of orbit trajectory \((s)\)

\[Y(s) = Re \left( \sqrt{\epsilon \beta_y(s)} \cdot \exp(\pm i \mu_y) \right) = \sqrt{\epsilon \beta_y(s)} \cdot \cos(\mu_y(s))\]

\[p_y(s) = -\frac{\epsilon}{\sqrt{\beta_y(s)}} \cdot \left[ \sin(\mu_y(s)) - \frac{\beta'_y(s)}{2} \cos(\mu_y(s)) \right]\]

A.Kolomensky and A.Lebedev. Theory of Cyclic Acceleratros (1966)

• Betatron function has periodicity of lattice Cell \(\beta_y(s + L) = \beta_y(s)\) No LIMIT on the amplitude \(Y_0(s) = \sqrt{\epsilon \beta_y(s)}\)

• **For the linear lattice** the dynamic aperture is **infinite**

• Phase advance \((\mu)\) and **Trace** of transfer matrix \((M)\) related as \\
\[\cos(\mu) = \frac{1}{2} Tr(M) = \frac{1}{2} (m_{11} + m_{22})\]

• Betatron oscillations are **BOUNDED** if phase advance \((\mu)\) has REAL values, that is \(\cos(\mu) < 1\)

Thus, **main condition** of STABLE motion in linear approximation is 
\(|Tr(M)| \leq 2\)

Integrable motion of particles in accelerators and rings should be **stable**. Deviations in all 3D planes with respect to central trajectory are **finite** and **limited**. Particles oscillate around reference orbit. **Asymptotic unbounded growth** of particle coordinates represents **unstable** behavior.
• Betatron tunes are defined as number of betatron oscillations per turn ($\mu(s)$ is phase advance per Cell -- element of periodicity)

$$Q_y = \frac{\omega_y}{\omega_{rot}} = \frac{N \cdot \mu_y(s)}{2\pi} = \frac{1}{2\pi} \int ds \frac{d\beta_y(s)}{s}$$

• Transfer Matrix should satisfy SYMPLECTIC conditions

$$M^T \cdot S \cdot M = S$$


• Symplectic matrix ($S$) is composed of UNIT matrixes

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$J \equiv S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and in 1D phase space

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

• SYMPLECTIC conditions are realized “if” and “when”

$$\text{det} \; M = m_{11}m_{22} - m_{12}m_{21} = 1$$

• Equations of harmonic oscillations might be derived also from the unperturbed Hamiltonian function ($H_0$) where second order terms ($x \cdot \delta$) and ($\delta^2$) etc. are omitted

$$H_0 = \frac{p_x^2}{2} + \frac{p_z^2}{2} + \frac{K(s)}{2} (x^2 - z^2)$$

• Hamiltonian of system as function of canonical variables ($\dot{q}_k = \frac{\partial H}{\partial p_k}, \; \dot{p}_k = -\frac{\partial H}{\partial q_k}$) is Integral of motion

$$H(q, p, t) = \frac{\partial H}{\partial t} + \sum_k \left( \frac{\partial H}{\partial q_k} \dot{q} + \frac{\partial H}{\partial p_k} \dot{p} \right) = \text{const}(t)$$

(B. Montague. CAS-95-06)

• providing there is no EXPLICIT dependence of Hamiltonian on time ($t$) ($\frac{\partial H}{\partial t} = 0$) ($s = \omega t$) ($\frac{\partial H}{\partial s} = 0$)

• (energy conservation law)
• **Lie Operator** denoted by symbol $f$: acts on function $g$ as Poisson brackets in 2n Phase Space $\{q_1, q_2, \ldots q_n, p_1, p_2, \ldots p_n\}$ (A.Dragt. „An Overview of Lie Methods for Accelartrio Physics“. Proc. PAC-2012. USA)

\[ f: g = \{f, g\} = \sum_{i=1}^{n} \left\{ \left( \frac{\partial f}{\partial q_i} \right) \left( \frac{\partial g}{\partial p_i} \right) - \left( \frac{\partial f}{\partial p_i} \right) \left( \frac{\partial g}{\partial q_i} \right) \right\} \]

• “Solving of Hamiltonian equations of motion and finding symplectic maps are equivalent tasks“

• Transfer map for an element of length ($L$) in **symplectic** Lie operator form is

\[ M_{s0 \rightarrow s1} = \exp \left( - \int_{s0}^{s1} H(s) ds \right) = \exp( -LH : ) \]

• Lie generators applied to describe non-linear kicks

\[ \exp : f : = 1 + \frac{1}{2!} (f :)^2 + \cdots = \sum_{k=0}^{\infty} \frac{(f :)^k}{k!} \]

• Canonical transformations simplify problems by choice of proper coordinate system from original $\{q_k, p_k, t\}$ variables to new canonical variables $\{Q_k, P_k, t\}$. (E.Wilson, CAS CERN 95-06)

\[ H'(Q_k, P_k, t) = H(q_k, p_k, t) + \frac{\partial F}{\partial t} \]

• New Hamiltonian preserves form of Hamiltonian equations

\[ \dot{Q}_k = \frac{\partial H'}{\partial P_k} \quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k} \]

• Different types of **GENERATING functions** ($F$) are applied in order to eliminating EXPLICIT dependence of Hamiltonian on time ($\frac{\partial H'}{\partial t} = 0$) and build new Integral of motion

\[ dF = \sum_k (p_k \, dq_k - P_k \, dQ_k) + (H - H') \, dt \]
• "Introduction of non-linear elements into ring lattice will cause oscillations about the closed (reference) orbit to grow in amplitude for particular tunes" (A.Wolski. "Beam Dynamics in High Energy Particle Accelerators". 2014.

• High order terms of magnetic fields cause perturbations of linear lattice and leads to restriction on beam stability: 1–Resonances, 2–Momentum dependence of betatron tunes, 3–Amplitude dependent tune shifts.

• **Canonical perturbation theory** to deal with non-linear BD AWAY of resonances \((mQ_x + nQ_z ≠ k)\) (L.Nadolski, Non-Linear Beam Dynamics. NPA-2011-2012. (V2.2).

• Mechanism to change betatron tunes with momentum offset (linear, quadratic and cubic chromaticity)

• Mechanism to SHIFT betatron tunes with Amplitude of oscillations (ADTS) –linear approximation

• Resonance conditions, resonance width, stopbands

2D Hamiltonian includes linear part \((H_0) +\) Perturbation terms (In Light Sources kinematic term \(\frac{x}{\rho} ≪ 1\) is ignored)

\[ H = \left(1 + \frac{x}{\rho}\right) \frac{p_x^2 + p_z^2}{2(1 + \delta)} - \frac{x}{\rho} \frac{p_x^2}{2\rho^2} + \frac{k_1}{2} (x^2 - z^2) + \frac{k_2}{3} (x^3 - 3xz^2) + \frac{k_3}{4} (x^4 - 6x^2y^2 + y^4) \]

Quads strength - \(K_Q \equiv k_1 = \frac{1}{B\rho} \left(\frac{\partial B_z}{\partial x}\right)\) Sextupole -- \(K_S \equiv k_2 = \frac{1}{B\rho} \left(\frac{\partial^2 B_z}{\partial x^2}\right)\) Octupole -- \(K_{OCT} \equiv k_3 = \frac{1}{B\rho} \left(\frac{\partial^3 B_z}{\partial x^3}\right)\)

• After canonical transformations Hamiltonian of a ring might be represented in Action-Angle variable by combination of linear part and NON-linear contributions (kicks) of quads \((H_2)\), sextupoles \((H_3)\) G.Guignard. Part.Acc. V.18 (1986), J.Bengtsson.SLS Note 9/97.

\[ H(\psi, J_x, J_y, J_z, s) \propto Q_xJ_x + Q_zJ_z + \int [H_2(s) + H_3(s)] \]
Phase space ellipse.
Linear motion

\[ p_x = -\frac{\gamma_x}{\alpha_x} \]

\[ x = -\frac{\alpha_x}{\beta_x} \]

Courant-Snyder Invariant
\[ 2J_x = \varepsilon_x = \gamma x^2 + 2\alpha xx' + \beta x'^2 \]

Amplitude of Stable Motion is limited and Phase Space is distorted at large Amplitudes

\[ H_{SXT} \sim \frac{J^{3/2}\beta^{3/2}(s)}{3!} K_S \cos^3 \psi \rightarrow K_S \cos 3\psi \]

\[ DA_x = \sqrt{\beta_x 2J_{\text{max}}} = \sqrt{\beta_x A_x} \]

\( DA_x \) is cross-section of ring acceptances \( A_{x,z} \) at position \( s \)

Acceptance is limited by „Separatrix“ connecting UFP and dividing stable area around reference orbit and area of unstable motion (hyperbolic curves with asymptotic behavior)

Figure taken from A.Wolski. US PAC School/2013.

Figure taken from E.Wilson. „Non-linear Resonances“. CAS CERN 95-06.
Period of Pendulum oscillations depends on angle (Amplitude of oscillations).

\[ \Delta T(\theta) \approx \frac{1}{16} \theta^2 T_0 \]

\[ T(\theta) \approx 2\pi \sqrt{\frac{l}{g}} \left(1 + \frac{1}{16} \theta^2 \right) + \ldots \]

\[ \theta_0 \ll 1 \text{ (Rad)} \]

Unstable
Fixed
Point
Stable
Fixed
Point

Approximation of first order term of Tailor expansion is valid for SMALL Amplitudes…

Acceptance is limited by „Separatrix“ dividing stable area around remote (external) Fixed points from central area of stable motion around reference orbit.

\[ H_{OCT} \sim \frac{(2J_\beta (s))^{4/2}}{4!} K_{OCT} \cos^4 \psi \rightarrow K_{OCT} \cos 4 \psi \]

Figure taken from E. Wilson. „Non-linear Resonances“. CAS CERN 95-06.
Note that there are two conditions for the action of a particle to be driven to large values by the sextupoles in a lattice:

1. The tune of the lattice must be close to an integer, or a third integer.

2. The resonant driving term must be significantly large.

Horizontal phase space in the ALS, close to a third-integer resonance, produced by tracking in a model of the lattice.


Dynamic aperture for ON-momentum particles is a cross-section of Phase space Acceptance at $(s)$

$A_y$ is a ring Acceptance = maximum area of stable oscillations limited by Separatrix

$$DA_x(s) = \sqrt{2J_{max} \beta_x(s)} = A_x \beta_x(s)$$

### Analytical expressions


- **Single Sextupole** of strength $(K_S)$ located at position $(s_1)$ limits horizontal Dynamic Aperture at position $(s)$

$$DA_x^{sxt}(s) = \sqrt{2J_{max} \beta_x(s)} = \frac{\sqrt{2\beta_x(s)}}{\beta_x(s_1)^{3/2}} \left( \frac{1}{|K_S|L_S} \right)$$

- **Single Octupole** of strength $(K_{OCT})$ located at position $(s_2)$ limits horizontal DA at position $(s)$

$$DA_x^{oct}(s) = \sqrt{2J_{max} \beta_x(s)} = \frac{\beta_x(s)}{\beta_x(s_2)} \sqrt{\frac{1}{|K_{OCT}|L_{OCT}}}$$

- Estimation of vertical DA (Courant Snyder Invariant)

$$DA_y^{sxt}(s) = \sqrt{\frac{\beta_x(s_1)}{\beta_y(s_1)} (DA_x^2 - x^2)}$$

- $(|K_S|L_S)$ and $(|K_{OCT}|L_{OCT})$- integrated strengths of Sextupole and Octupole. Higher strength – less DA
- High value of beta-function at position $(s)$ (Septum) preferable for injection
- High Dispersion $(D_x)$ at SXT location allows to reduced SXT strength -- DA is improved
- Small $\beta_y$ at location of SXT (OCT) might improve DA but Higher strength of non-linear element will be required
• Dynamic Aperture in horizontal plane of a single (2m) multipole (m>3)

\[ DA_x^{2m}(s) = \sqrt{2} \beta_x(s) \left( \frac{1}{m \beta_x^{2m}(s)} \right)^{m-2} \left( \frac{1}{|K_{m-1}|L} \right) \]

• If non-linear elements are independent -- no special phase and amplitude relations between SXT (OCT)

\[ \frac{1}{DA_{total}} = \sum_i^{N_{SXT}} \frac{1}{DA_{SXT}} + \sum_i^{M_{OCT}} \frac{1}{DA_{OCT}} + \ldots \]

Scaling


• In approximation of small phase advance between two sextupoles one can estimate reduction of horizontal Dynamic Aperture for ultra-low emittance Synchrotron Light Sources according to scaling Law

\[ DA_x(s) \sim \sqrt{\frac{\epsilon_{nat}^{x}}{\zeta_{cell}^x}} \sim \frac{D_{X}^{max}}{\zeta_{cell}^{x}} \sim \frac{1}{|K_S| \cdot L_S} \]

• and for vertical Dynamic Aperture

\[ DA_z(s) \sim \frac{DA_x(s)}{\zeta_{cell}^{x}} \sqrt{\frac{\beta_x(s)}{\beta_z(s)}} \]

• \( \epsilon_{nat}^{x} \) -- natural emittance after SR damping  \( \zeta_{cell}^{x} \) -- natural chromaticity per cell  \( D_{X}^{max} \) -- Dispersion (max)

High natural CHR + reduced Dispersion -- main factors limiting DA of ultra-low emittance Light Sources
Small Dispersion -- to keep ultra-low emittance, High chromaticity – due to strong Quads and SXT
RESONANCE DRIVING TERMS


• **Symplectic** form of a lattice Hamiltonian integral is contributed by individual Hamiltonians of **each** Quadrupole $H_{2m}$, Sextupole $H_{3n}$ and might be approximated by a Sum of different **modes** – RDT

\[
\int [H_2(s) + H_3(s)] \, ds = h_3 \propto \sum h_{jklmp}
\]

• Hamiltonian coefficients $h_{jklm}$ contain the contribution from all the multipoles of order $(n = j + k + l + m)$. Even sum $(j + k)$ corresponds to **normal** multipoles ($B_n$) while odd sum $(l + m)$ - to **skew** multipoles ($A_n$)

• Driving terms are derivatives of Lie operator over action variable

\[
h_{jklmp} = \frac{\partial h_3}{\partial J_{x,y}}
\]

• “**LINEAR**” (first order) Hamiltonian modes and their complex conjugates (*) are proportional to the **first order** of integrated sextupole strength ($K_S L$)
First order RDT and their complex conjugative (*) is a **SUM** of integrated strengths of $N_{SXT}$ sextupoles and $M_{QUAD}$ quadrupoles with **scaling factors** ("lever of arm") over different frequencies multiple to betatron phase advance $\mu_{xn}, \mu_{zn}$ at element location

$$h_{jklmp} = -h_{jklmp}^* = - \sum_n^{N_{SXT}} (K_S \cdot L_S)_n \beta_{xn}^{2j+k} \cdot \beta_{yn}^{2l+m} \cdot D_p^p \cdot e^{i(j-k)\mu_{xn} + i(l-m)\mu_{yn}}$$

$$\begin{align*}
M_{QUAD} & - \sum_n (K_Q \cdot L_Q)_n \beta_{xn}^{2j+k} \cdot \beta_{yn}^{2l+m} \cdot e^{i(j-k)\mu_{xn} + i(l-m)\mu_{yn}}
\end{align*}$$

Quadrupoles at **dispersive** sections ($D \neq 0$) – contribute to **chromatic** Hamiltonian driving modes ($p \neq 0$)

Quads in **achromatic** sections ($D = D' = 0$) do **NOT** contribute to RDT (in theory)

Sextupoles contribute to both **chromatic** RDT ($p \neq 0$) in dispersive sections ($D \neq 0$) and **geometric** RDT ($p = 0$) in **achromatic** sections ($D = D' = 0$)

**Chromatic** sextupoles compensate natural negative chromaticity and its strengths are fixed

Additional **harmonic** sextupoles in achromatic sections compensate shrinking of area of stable oscillations (reduction of Dynamic Aperture) caused by main chromatic sextupoles

Driving modes ideally should be **SUPPRESSED** either **CANCELLED** because they are the source of ALL the resonances considering long term behavior by multiple repetition of the lattice structure

$$|h_{jklmp}^\infty| = \frac{|h_{jklmp}|}{2\sin\{\pi[(j-k)Q_x^{cell} + (l-m)Q_y^{cell}]\}}$$
Table 1. LINEAR Resonance Driving Terms (RDT) and their effects

<table>
<thead>
<tr>
<th>N</th>
<th>Linear Driving terms</th>
<th>Effect</th>
<th>Comments</th>
<th>Phase dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( h_{11001} ) (CrX _ lin)</td>
<td>( \partial Q_x / \partial \delta )</td>
<td>Linear Chromaticity (hor) ( \xi_x^{(1)} )</td>
<td>chromatic</td>
</tr>
<tr>
<td>2</td>
<td>( h_{00111} ) (CrY _ lin)</td>
<td>( \partial Q_y / \partial \delta )</td>
<td>Linear Chromaticity (vert) ( \xi_y^{(1)} )</td>
<td>chromatic</td>
</tr>
<tr>
<td>3</td>
<td>( h_{20001} = -h_{02001}^* )</td>
<td>( 2Q_x \pm Q_s )</td>
<td>parametric half-integer resonance ( e^{i\cdot2\mu_x} )</td>
<td>chromatic</td>
</tr>
<tr>
<td>4</td>
<td>( h_{00201} = -h_{00021}^* )</td>
<td>( 2Q_y \pm Q_s )</td>
<td>parametric half-integer resonance ( e^{i\cdot2\mu_y} )</td>
<td>chromatic</td>
</tr>
<tr>
<td>5</td>
<td>( h_{10002} = -h_{01002}^* )</td>
<td>( \partial D / \partial \delta )</td>
<td>Second order dispersion ( D^{(1)} \equiv \partial D / \partial \delta )</td>
<td>chromatic</td>
</tr>
<tr>
<td>6</td>
<td>( h_{21000} = -h_{12000}^* )</td>
<td>( Q_x )</td>
<td>( e^{i\cdot\mu_x} )</td>
<td>geometric</td>
</tr>
<tr>
<td>7</td>
<td>( h_{10110} = -h_{01110}^* )</td>
<td>( Q_x )</td>
<td>( e^{i\cdot\mu_x} )</td>
<td>geometric</td>
</tr>
<tr>
<td>8</td>
<td>( h_{30000} = -h_{03000}^* )</td>
<td>( 3Q_x )</td>
<td>( e^{i\cdot3\mu_x} )</td>
<td>geometric</td>
</tr>
<tr>
<td>9</td>
<td>( h_{10200} = -h_{01020}^* )</td>
<td>( Q_{xy} + 2Q_y )</td>
<td>( e^{i\cdot(\mu_x + 2\mu_y)} )</td>
<td>geometric</td>
</tr>
<tr>
<td>10</td>
<td>( h_{10020} = -h_{01200}^* )</td>
<td>( Q_{xy} - 2Q_y )</td>
<td>( e^{i\cdot(\mu_x - 2\mu_y)} )</td>
<td>geometric</td>
</tr>
</tbody>
</table>

Linear chromaticity is INDEPENDENT on phase advance \( \mu \)

\[
h_{11001} = \xi_x^{(1)} = -\frac{1}{4\pi} \sum_n^{N_{QUAD}} (K_Q L_Q)_n \beta_{yn} + \frac{1}{4\pi} \sum_m^{M_{SXT}} (K_S L_S)_m \beta_{ym} D_m
\]
SECOND ORDER RESONANCE DRIVING TERMS

\[ h^{(2)} \propto (K_{SXT} \cdot L_{SXT})^2 \]

- Second order RDT drives synchrotron sidebands of LINEAR Resonance modes. When Linear RDT are vanished either reduced the second order terms i.e. corresponding sidebands are week and ignored
- It is mandatory to suppress (minimize) first order terms in order to reduce second order resonance modes

**Amplitude Dependent betatron Tune Shifts (ADTS)**

- Cross-talks of first order RDT in sextupoles as well as between different sextupoles produce phase independent second order RDT and cause ADTS. In linear approximations ADTS are

\[ \delta Q_{x}^{ADTS} = \frac{\partial h^{(2)}}{\partial J_x} \approx \alpha_{xx}J_x + \alpha_{yx}J_y \]
\[ \delta Q_{y}^{ADTS} = \frac{\partial h^{(2)}}{\partial J_y} \approx \alpha_{xy}J_x + \alpha_{yy}J_y \]

- ADTS originated from an amplitude or momentum dependent shift of the closed orbit in the sextupole. Orbits must be corrected to the magnetic axis of a sextupole in order to minimize ADTS. Beam Position Monitors must be located as close as possible to each sextupole in order to control beam centroid

- Octupols applied to COMPENSATE ADTS caused by Sextupoles

\[ \alpha_{yy} = + \frac{1}{16\pi} \left( \sum_{n} (K_{oct} \cdot L_{oct}) \cdot \beta_{yn}^2 \right) \]
\[ \alpha_{xz} = - \frac{1}{8\pi} \left( \sum_{n} (K_{oct} \cdot L_{oct}) \cdot \beta_{xn} \cdot \beta_{zn} \right) \]
One might consider RDT as a sum of complex vectors

Each complex vector represents local Hamiltonian of sextupole either quadrupole

Phase of each vector is multiple of betatron phase advance at element location

**Ten** first order Hamiltonian terms are **linear** in sextupole strength

**Two** real RDT drive horizontal and vertical Chromaticies \((j=k)\) and \((l=m)\)

\[
h_{11001} = \xi_{11001} = -\frac{1}{4\pi} \sum_n (K_Q L_Q)_n \beta_{yn} + \frac{1}{4\pi} \sum_m (K_S L_S)_m \beta_{ym} D_m
\]

Only SXT at **dispersive** sections \((D \neq 0)\) control linear chromaticiticy \(\xi^{(1)}\) and their settings are fixed

Hamiltonian RDT of chromatic SXT accumulated **ADDITIVELY** regardless of phase \((\mu)\).

Amplitudes of chromatic SXT added / subtracted depending on sign of element strength and sign of \((D)\)

Harmonic SXT and quadrupoles located at achromatic sections \((D = D' = 0)\) **do not** contribute to \(\xi^{(1)}\)

Strengths and positions of HARMONIC SXT adjusted to compensate geometric aberrations from CHR SXT

**Eight** first order modes and their complex conjugative are phase dependent -- resonance behavior

**9** families of SXT could be enough to eliminate all excitation modes but at phase advance of cell close to 
\(\mu_x = 180^\circ\) Hamiltonian RDT proportional to \((2\mu_x)\) will be amplified coherently by all SXT

Linear system of **9** equations degenerates down to **rank 8** and no solution to suppress ALL RDT

Lattice optics --adjusted to cancel / minimize RDT  by proper phase advance between sections of the lattice

Figure taken from A. Streun. CAS CERN-2006.
• applying **mirror symmetry** conditions and phase advance between CELLS close to quarter of integer

\[ \mu_y \approx (k \pm 0.25) \cdot 2\pi \]

• resonances associated with \(2\mu_y\) phase dependent RDT could be **cancelled** between cells \((2\mu_y = \pi)\), resonances \((\mu_y, 3\mu_y)\) and coupling resonances \((\mu_x \pm 2\mu_z)\) – between TWO pairs of cells

• Periodicity \(\Delta Q_{x,z} = N \cdot \mu_{cell}^{x,z} [2\Delta Q_{x,z} 3\Delta Q_x] \rightarrow integer\) (N=5)

• Figure taken from A. Streun. CAS CERN-2006.

• To compensate linear CHR and cancel some RDT "− I" condition is applied. Non-interleaved pairs of sextupoles are located at mirror symmetry points of lattice sections with phase advance multiple to

\[ \mu_y = (2n \pm 1)\pi \]

• Twiss

\[ \alpha_{S1} = -\alpha_{S2} \hspace{1cm} \beta_{S1} = \beta_{S2} \]

• parameters

\[ D(s_1) = D(s_2) \hspace{1cm} D'(s_1) = -D'(s_2) \]

• In this case RDT of sextupoles proportional to \(2\mu_y\) will amplify itself (rank of matrix is reduced – degradation problem)

• To minimize RDT proportional to ODD tunes, one need to provide phase advance between lattice sections close to \(\mu_y = (2n \pm 1)\pi\). Complex Hamiltonian modes with phase proportional to \(2\mu_y\) will be added to each other and amplified instead of to be cancelled.

• Location of SXT in a sequence of **EQUAL** phase advance steps \((\Delta\mu_x), (\Delta\mu_z)\) might help to limit RDT

• **General symmetry** conditions full cancellation of non-linear terms of two sextupoles (thin lens approx.)

\[ \Delta\mu_{2-1} = n\pi \]

\[ \beta_{Z1}/\beta_{X1} = \beta_{Z2}/\beta_{X2} \]

\[ (S_{X1})\beta_{X1}^{3/2} = -(-1)^n (S_{X2})\beta_{X2}^{3/2} \]

\[ D_1 = -(-1)^n D_2 \]

Diffractive Limited Light Source (E. Levichev. BINP Proposals. 2013)

Figure 11: Five-cell superperiod with two horizontal sex-tupole pairs (denoted X) and two vertical ones (denoted Y).

DA of BINP DLLS ring (proposals)

Non-interleaved SXT pairs

On-momentum

DA_{BINP} > ±20 mm
MA_{BINP} = ±1.5%

Figure 13: Dynamic aperture of the 10 pm emittance storage ring at $\beta_{x,y} = 10$ m.

Figure 10: Split magnet TME cell for ultimate storage ring. A. Bogomyagkov, E. Levichev, P. Piminov. Proc. IPAC-2014.

$K_{S_x}L = +39 \text{ m}^{-2}$
$K_{S_y}L = -94 \text{ m}^{-2}$
$K_{S_x}^{\text{comp}}L = -2.7 \text{ m}^{-2}$
$K_{S_z}^{\text{comp}}L = +8.4 \text{ m}^{-2}$

On-momentum

only $K_{S_y}$
## Comparisoin of DLLS Projects

<table>
<thead>
<tr>
<th>MAX IV</th>
<th>7BA</th>
<th>3 GeV</th>
<th>528 m</th>
<th>320 pm</th>
<th>500 mA</th>
<th>DA 20 mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESRF Phase II</td>
<td>7BA</td>
<td>6</td>
<td>850</td>
<td>130</td>
<td>200</td>
<td>10 mm</td>
</tr>
<tr>
<td>Spring-8</td>
<td>6BA</td>
<td>6</td>
<td>1400</td>
<td>67.5</td>
<td>300</td>
<td>3 mm</td>
</tr>
<tr>
<td>Diamond</td>
<td>4-5-7BA</td>
<td>3</td>
<td>560</td>
<td>45-300</td>
<td>300</td>
<td>2 mm</td>
</tr>
<tr>
<td>ALS</td>
<td>5-7BA</td>
<td>2</td>
<td>200</td>
<td>50-100</td>
<td>500</td>
<td>2-3 mm</td>
</tr>
<tr>
<td>Our proposal</td>
<td>7BA</td>
<td>3</td>
<td>1300</td>
<td>15</td>
<td>200</td>
<td>40 mm</td>
</tr>
<tr>
<td>Pep-X</td>
<td>7BA</td>
<td>4.5</td>
<td>2200</td>
<td>11</td>
<td>200</td>
<td>10 mm</td>
</tr>
<tr>
<td>tUSR</td>
<td>7BA</td>
<td>9</td>
<td>6200</td>
<td>3</td>
<td>100</td>
<td>0.8 mm</td>
</tr>
</tbody>
</table>


Table taken from E.Levichev. „Diffraction Limited electron storage ring with large DA“. BINP Proposals. 2013.

It is commonly recognized that Required DA for Light Sources to ensure beam injection

\[
DA_x \geq \pm 10 \text{ mm}
\]

Momentum Acceptance should be

\[
MA = \frac{\delta P}{P_0} \geq \pm (2\div4)\%
\]

to provide reasonable Life time

\[
T_{1/2} \geq 10 \text{ hours}
\]
Parameters of "21_3,6,9" lattice are shown in brackets.

Circumference [m] 44.112 (41)
Horizontal Tune 5.8438 (11.31)
Vertical Tune 8.4605 (2.61)

Nat. hor. Chromaticity −16 (−27)
Nat. vert. Chromaticity −21 (−19)
Chr / Cell −4 / −5.2 (−6.8 / −4.8)

MAX-IV Chroma (h/v) −50 / −44
MAX-IV CHR / Cell −2.5 / −2.2
NSLS-II CHR / Cell < |3|
ESRF-100 pm / Cell −3.6 / −2.7

Mom compact. 6.034E-03 (3.6E-03)
Hor. damp. partition Jx 1.397 (0.97)

Energy 50 MeV
Rad. energy/turn [MeV] 0.00
Natural Emittance [nm] 0.18 (0.1)
Appr. vert. Emittance [pm] 0.31
Natural energy spread 4.24E-05
Hor. damping time [s] 24 (20)
Vert. damping time [s] 34 (19)
Long. damping time [s] 21 (9)

Radiation Integrals:
I1 [m] 2.662E-01
I2 [1/m] 4.935E+00
I3 [1/m2] 3.876E+00
I4 [1/m] -1.96E+00
I5 [1/m] 3.389E-01

Parameters of old "21_3,6,9" lattice are shown in brackets.

One Cell. Length \( L = 11.028 \text{ m} \)

**Quads strength \((m^2)\)**

- \( Q1 = -2.138 \pm 5.8 \)
- \( Q2 = +6.736 \pm 18.2 \)
- \( Q3 = -4.01 \pm 13.9 \)
- \( Q4A = +11.7 \pm 28.7 \)
- \( Q4B = +11.485 \)
- \( Q5A = -15.305 \pm 20 \)
- \( Q5B = -16.435 \)
- \( Q6A = +11.65 \pm 28.9 \)
- \( Q6B = +11.65 \)
- \( Q7 = -2.437 \pm 12.1 \)
- \( Q8 = +8.460 \pm 26.2 \)

**Quads string strength \((m^2)\)**

- \( |Kq| < 16 \text{ }m^2 \) (30)
- \( |G| < 3 \div 30 \text{ }T/m \)
- MAX-IV \( Kq < 4 \text{ }m^2 \)

**Chromatic SXT**

**Integ. Strength \((Ks \cdot L \text{ (m}^2)\)**

- \( 4A-4B= +2.2 / 2.2 / 2.2 (+33) \)
- \( 5A-5B= -11.3 / 11.3 / 11.3 (-35) \)
- \( 6A-6B= +2.1 / 2.1 / 2.1 (+34) \)

**SXT effective length**

15-10-15 cm (15)
Lattice “3Q_SPLIT_SHORT_5”.

Half-cell

Chromaticity
\[ \xi_x = -16 \quad \xi_y = -21 \]

Betatron tune
\[ Q_x = 5.8438 \quad Q_y = 8.4605 \]

Phase adv/cell
\[ \mu_x = 1.461 \cdot 2\pi = 2.922\pi \]
\[ \mu_y = 2.1151 \cdot 2\pi = 4.2303\pi \]

Ph adv (half-cell)
\[ \mu_x = 0.7305 \cdot 2\pi = 1.461\pi \theta_x \approx \frac{3}{4} \]
\[ \mu_y = 1.084 \cdot 2\pi = 2.1151\pi \]

\[ \Delta \mu_x (S6B-S4A) = (0.64 - 0.203) \cdot 2\pi = 0.44 \cdot 2\pi = 0.88 \pi \approx 1\pi \]
\[ \Delta \mu_x (S6-S4) = (0.636 - 0.208) \cdot 2\pi = 0.43 \cdot 2\pi = 0.86 \pi \]
\[ \Delta \mu_x (S6A-S4B) = (0.63 - 0.21) \cdot 2\pi = 0.42 \cdot 2\pi = 0.84 \pi \]

Minimization of linear RDT by proper choice of betatron phase advance and mirror symmetry.

Second and third order chromaticity terms are reduced.

Split quads and sextupoles applied to reduce overfocusing.

Horizontal and vertical betatron functions are WELL separated.

D = 25 cm
Lattice “3Q_SPLIT_SHORT_5” One cell

Betatron tune
Qx=5.8438
Qy=8.4605

Phase adv/cell
\( \mu_x=1.461 \cdot 2\pi=2.922 \pi \)
\( \mu_y=2.1151 \cdot 2\pi=4.2303 \pi \)

D=25 cm

Mirror symmetry

Betafunctions [m]

Dispersion [m]
DYNAMIC APERTURE.  Lattice “3Q_SPLIT_SHORT_5”

**Dynamic Aperture. Early version lattice “LWFA_21_3,6,9”**

**Chromatic SXT**

- DAy = ±5 mm (±50 $\sigma_y$)
- Axial injection might be possible

**Chromatic SXT**

- $4A-4B=+2,+2,+2\ m^2$
- $5A-5B=-11,-11,-11\ m^2$
- $6A-6B=+2,+2,+2\ m^2$

- SXT int. strength S·L

- $D_{Ax} = -14...+18\ mm (±100 \ \sigma_x)$
- $Ax ≈ 120\ mm\cdotmr$

- With errors
- $Ax ≈ 50\ mm\cdotmr$
- $Ax (\text{MAX-IV}) < 10\ mm\cdotmr$

**Off-momentum $\delta=+3\%$**

**Chamber 60 x 40 mm**

**Off-momentum $\delta=-3\%$**

**Chamber 60 x 40 mm**

**Axial injection might be possible**

**Dynamic Aperture. Early version lattice “LWFA_21_3,6,9”**

**CHROM+HRM SXT + OCT**

- SXT int-str
- $S4L=+33\ m^2$
- $S5L=+55\ m^2$
- $S6L=+34\ m^2$

- $D_{Ay} = ±5\ mm (±50 \ \sigma_y)$

- $Ax≈120\ mm\cdotmr$

- $Ax(\text{MAX-IV})<10\ mm\cdotmr$

- $Ax ≈ 50\ mm\cdotmr$

- $Ax (\text{MAX-IV}) < 10\ mm\cdotmr$
Off-momentum dynamic aperture as function of energy deviation

Lattice “3Q_SPLIT_SHORT_5”

early versions of VLA-cSR lattice. strong settings of quads cause over-focusing and non-linear distortions
Betatron tune diagram. Lattice “3Q_SPLIT_SHORT_5”

By adjusting chromatic and harmonic sextupole families one can compensate linear as well as second order chromaticity.
Tune deviation for off-momentum particles is reduced and momentum acceptance is improved.
Betatron tune deviation for off-momentum particles

Chromatic and Harmonic Sextupoles as well as octupoles located in the dispersion sections of a ring compensate first and second order chromaticity. Octupoles to adjust ADTS are located in the achromatic sections of a ring and switched OFF.

Integrated octupole strength is limited to $K_3 L \leq \pm 20 \text{ m}^3$ in order to minimize high-order non-linear distortions and preserve dynamic aperture.

Betatron tune deviation $< \delta Q_{x,y} \leq 0.03$ for $\delta \leq \pm 10\%$

Lattice “3Q_SPLIT_SHORT_5”.
Amplitude dependent tune shift. Lattice “3Q_SPLIT_SHORT_5”.
Octupoles located in achromat section of a ring compensate ADTS

In order to suppress ADTS the integrated octupole strength is increased to $K_3L \approx \pm 250 \text{ m}^3$
As a consequence the DA is **shrunk** in horizontal and vertical planes
Early versions of the VLA-cSR lattice suffer from non-linear distortion caused by strong over-focusing

- It was necessary to relax strong settings of early versions (lattice „21.3,6,9“ as an example) and find compromises between lattice parameters
- Quadrupole strength of „21_3,6,9“ lattice $Kq>30$ m$^{-2}$ i.e. ~10 times MORE of MAX-IV quads ($Kq<4$ m$^{-2}$)...
- Chromaticity/cell HIGH, dispersion SMALL $D_{\text{max}}<12$ cm and required SXT strength is HIGH ($S\cdot L\approx40$ m$^{-2}$)
- Dynamic aperture (on-momentum) is SMALL $D_{\text{Ax}}\approx-5..+7$ mm NOT enough for stable circulation

**Merit of the „3Q_SPLIT_SHORT_5“ lattice**

- Ring dimensions are fitted to existing FLUTE Bunker while main parameters are improved
- Radius of bending magnets is increased, gradient and edge focusing in vertical direction are applied
- Quads are splitted in doublets QA-QB, distances between elements are increased
- Quads strength reduced from $Kq=30$ m$^{-2}$ to $Kq<14$ m$^{-2}$ (still few times higher those for MAX-IV)
- The compensation of horizontal chromaticity is done by sets of splitted SXT triplets SXA–SX–SXB
- Ring lattice is modelled to satisfy „–I“ condition for mirror symmetry non-interleaved sextupoles
- Max of Dispersion is doubled $D=25$ cm, integrated SXT strength reduced $S\cdot L<20$ m$^{-2}$ (MAX-IV $S\cdot L\approx20$ m$^{-2}$)
- Side effect – decreasing of Geometric Momentum Acceptance to $MA<\pm7\%$ (60×30 mm chamber)
**Merit of the “3Q_SPLIT_SHORT_5” lattice**

- Chromatic SXT S4 and S6 are flanked between quads and located at position of $D_{\text{max}}$ and $\beta_{\text{Xmax}}$
- Phase advance between S4 and S6 is close to $\pi$ and Symmetry conditions are applied

\[
\begin{align*}
\beta_x(S6) &= \beta_x(S4) \\
\alpha_x(S6) &= -\alpha_x(S4) \\
D(S6) &= D(S4) \\
D'(S6) &= -D'(S4) \\
\mu_x(S6-S4) &\approx \pi
\end{align*}
\]

- S5 is positioned in the middle of dispersion section at MAX of vertical $\beta_y(S5)$ and MIN $\beta_x(S5) \approx 5$ cm
- Vertical SXT S5 has little influence on SXT non-linear terms in horizontal plane
- Magnetic rigidity $B \cdot R = 1.67 \, \text{T} \cdot \text{m}$ (500 MeV) and quads gradient is relaxed $\partial B/\partial R < 25 \, \text{T} \cdot \text{m}$ (MAX-IV $\approx 40 \, \text{T} \cdot \text{m}$)
- Half-cell betatron phase advance close to $\mu_x \approx 3.5$ ($v_x \approx 1.75$) and Resonance Driving Terms ($Q_x, 2Q_x, 3Q_x, Q_x \pm 2Q_y$) are reduced due to lattice symmetry and periodicity
- Quadratic ($\xi_{2x}$), Cubic ($\xi_{3x}$) CHROMA and ADTS are minimized by lattice geometry and harmonic SXT
- Dynamic aperture $\pm 5 \, \text{mm}$ of early lattice versions (“21_3.6.9“) is OPENED to $-14..+18 \, \text{mm}$ (3Q_SPLIT_SHORT_5)
- DA for off-momentum particles ($\delta \leq 5\%$) is enough for stable circulation of wide momentum spread beam
- Octupoles and decapoles should be added to VLA-cSR lattice to suppress ADTS, quadratic and cubic chromaticities etc. but its strength must be limited in order to preserve stability of betatron motion
- Full suppression of ADTS by octupoles leads to reduction of DA. Full suppression of second order chromaticity by octupoles leads to increase of ADTS and cubic chromaticity terms. As a result the MA is improved but DA is reduced
Outcome of cSTART Feasibility Studies

- Extensive studies of possible geometry and lattice of the very large acceptance compact storage ring operating in the energy range 50 to 500 MeV have been provided.

- The main objective of feasibility studies was to create ring model suitable to store the beam after Laser Wake field accelerator with wide momentum spread ($\sigma \sim 1\%$) as well as ultra-short electron bunches in a “$\sim$fs” range.

- More than 40 models of compact ring lattice based on DBA, DBA-FDF, TBA, 5BA cells etc. have been composed, simulated, analyzed and merit of different configurations has been carefully studied.

- The DBA-FDF Lattice with relaxed settings and optimized parameters could be accepted as a basis for further Detailed Design studies of the Very Large Acceptance compact Storage Ring.

- Proposed VLA-cSR lattice model compromises contradictory conditions:
  -- Small circumference of the ring $C \leq 50$ m
  -- Small dispersion $D < 15\%$ cm
  -- Large Dynamic Aperture in the dispersion plane $D_{Ax} > 15$ mm
  -- Large Acceptance in both planes $A_{x,y} > 20$ mm-mr
  -- Wide Momentum Acceptance $M_A \sim 5\%$ -10%
  -- Chromaticity / cell should be limited to $|\xi/\text{cell}| < 5\%$ (NSLS-II sets limits for SXT $|\xi/\text{cell}| < 3$)

- The “−I” condition is provided. The mirror symmetry at position of horizontal chromatic sextupoles is satisfied.
- Local maximums of horizontal beta-function and dispersion at position of main chromatic sextupoles help to restrict sextupole strength. Dynamic Aperture is opened significantly from $D_A = -5..+6$ mm to $D_A = -15..+20$ mm.

- Phase advance per cell is adjusted to minimize leading Resonance Driving Terms including high order chromaticity terms. The dynamic aperture for ON- and OFF-momentum particles is enough to store wide energy spread beam.

- Harmonic sextupole and octupole families should be used for non-linear studies in particular to operate the ring at negative compaction factor, to manipulate with bunch length and shape etc. but OCT strength must be limited.
Momentum Acceptance of KARA ring at 2.5 GeV (OPA simulations)

USER Lattice $\alpha = +9.3 \times 10^{-3}$

$T_{1/2} = 16 \text{ h (I}=100\text{mA)}$

Proceedings of the 2001 Particle Accelerator Conference, Chicago

UNDERSTANDING THE DYNAMIC MOMENTUM APERTURE OF THE ADVANCED LIGHT SOURCE$^a$

C. Steier, D. Robin, Y. Wu, LBNL, Berkeley, CA94720, U.S.A.;
W. Decking, DESY, Hamburg, Germany; J. Laskar, L. Nadolski, IMC-CNRS, Paris, France

Figure 1: Scan of the RF amplitude and therefore the longitudinal bucket height for three different sets of horizontal and vertical chromaticities (1.5 GeV).

Figure 5: Simulation of a frequency map with the energy offset and the horizontal oscillation amplitudes as the variables spanning the configuration space. The color code indicates the diffusion rate on a logarithmic scale.


Example of Diffusion map of ESRF

DA/tune diffusion map for EBS, ESRF upgrade
Electron beam Tracking codes

- So many codes that do the same thing
- Important to know what’s in the code you use, but not so useful to keep reinventing the wheel.
- Lattice conversion is a pain.
- Each lab has preferred code: challenge for those switching labs!
- My experience with Tracy and Accelerator Toolbox: can we create a viable multi-laboratory open source collaboration?
CONCLUSION

• It is possible to open on-momentum DA of Diffraction Limited Light Sources to more than ±10 mm even for ultimately low horizontal beam emittance
  • $\varepsilon \sim 15$ pm·rad
  • Optimization of Dynamic Aperture for ON-momentum particles might be realized by applying of “-l” condition for NON-interleaved pairs of sextupoles located in Dispersion sections of a ring

• MBA cell should be adapted to provide required phase advance
  • $\mu_y = (2n \pm 1)\pi$

• MBA Cell should be designed in such a way that pairs of chromatic sextupole are located at mirror symmetry points of a cell

• Dispersion Bump at position of chromatic sextupole helps to reduce SXT strength

• Optimization of beta-function at position of Sextupole could help to open DA

• Improving of Momentum Acceptance (off-momentum DA) might be achieved by families of harmonic sextupoles located in achromatic sections of a ring

• Octupoles to compensate ADTS should be incorporated in Achromat sections of ring. Octupole strengths must be limited