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Dynamic Aperture and Momentum Acceptance in Low Emittance Storage Rings

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Basics (Linear equations)

- "Original incentive to study the stability of the motion in non-linear dynamic systems has been prompted by development of Celestial Mechanics in XIX century to describe the orbital motion of Planets" (W.Scandale. "DYNAMIC APERTURE", CAS CERN-95-06)

- **We discuss here single particle Beam Dynamics. Collective effects are out of scope of talk**

- Trajectory of charge particles in magnetic fields of storage ring, composed of $(\boldsymbol{N})$ periodic cells each of length $(\boldsymbol{L})$ and total ring circumference $\boldsymbol{C = L \cdot N}$, is described by Lorentz equation

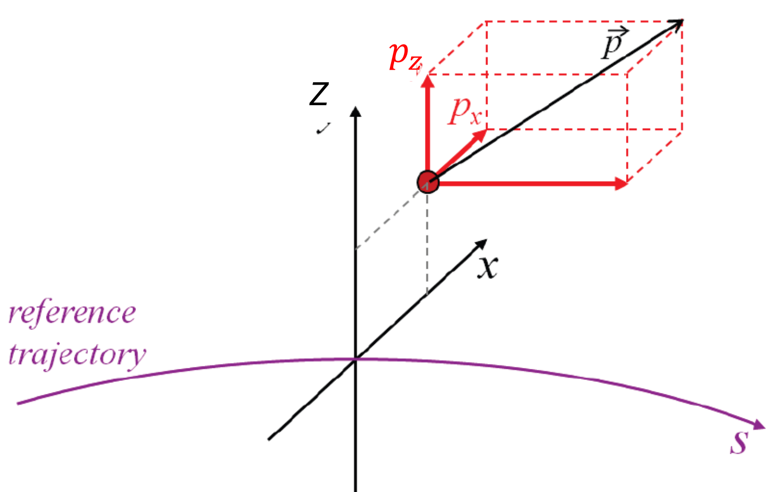
$$m\vec{R}'' = -\frac{e}{c}\left[\vec{R}' \times \vec{B}\right]$$

- Restoring forces are **periodic** $K_y(s+L) = K_y(s)$ and particles oscillate around closed ("reference") orbit $\overrightarrow{R_0}$ Here $(\boldsymbol{Y})$ stays for horizontal $(\mathbf{X})$ or vertical $(\boldsymbol{Z})$ components of transverse coordinates. K.Steffen. "Basic Course on accelerator Optics". CAS CERN-85-19.

Oscillations of particles in transverse to beam motion direction - fast with respect to slow synchrotron oscillations of energy and phase
-- Transverse and longitudinal planes might be ***split***
--- Motion in the transverse phase space might be considered ***independently*** on motion in longitudinal phase space

A.Wolski. US PAC School-2013

Solving Lorentz equation in cylindrical system of coordinates $(\boldsymbol{R}, \boldsymbol{\theta}, \boldsymbol{z})$ and
---assuming **PARAXIAL** beam conditions :
-- the deviation of horizontal $\boldsymbol{X(s) = \left(R(s) - R_0(s)\right)}$ and vertical $(\boldsymbol{Z})$ position of particle from reference orbit $\overrightarrow{R_0}$ **much less** than curvature radius $\boldsymbol{X \ll R_0}$
--- angular deviations $\boldsymbol{Y'(s) = dY(s)/ds \ll 1}$
---- one can NEGLECT high order terms and
--- derive **LINEAR** equations of harmonic oscillations



- If periodic restoring forces acting on particle with reference momentum p_0 are perfectly **LINEAR**, the particle oscillations are **STABLE** and described by second order homogeneous differential equations (**quasi-harmonic oscillator**) with periodic restoring force **(HILL equations)** (longitudinal coordinate (s) is independent variable) E.Courant and H.Snyder Theory of of the alternatiing-gradient synchrotron. Ann.Phy, 3 (1958).

$$\frac{d^2X(s)}{ds^2} + k_x(s)X(s) = 0 \qquad\qquad \frac{d^2Z(s)}{ds^2} + k_z(s)Z(s) = 0$$

- Radial Focusing term $1/\rho_0$ in magnets should be included into consideration $K_x(s) = \frac{1}{\rho_0^2(s)} - k_x(s)$
- Flat sector bend is a **drift** in axial direction $k_z(s) = 0$
- Focusing strength of quadrupole $k_z(s) = -k_x(s) = \frac{1}{B\cdot\rho}\left(\frac{\partial B_z}{\partial x}\right)$
- in bending plane $X_D = D\cdot(\Delta p/p_0) = D\delta$ and equations for off-momentum particles are

$$\frac{d^2X(s)}{ds^2} + K_x(s)X(s) = \frac{1}{\rho_0(s)}\frac{\delta p}{p_0} \qquad \text{and} \qquad \frac{d^2D(s)}{ds^2} + K_x(s)D(s) = \frac{1}{\rho_0(s)}$$

- Solution of linear differential equations is **LINEAR** combination of initial coordinates and momentums and might be written in matrix form J.Livingood. Principles of Cyclic Acceleratrs. (1961). K.Steffen. High energy Beam Optics (1965).

$$\vec{Y}(s) = M(s/s_0)\cdot\vec{Y}(s_0)$$

where $\vec{Y}(s_0)$ is vector of initial coordinates and momentums in 6D phase space

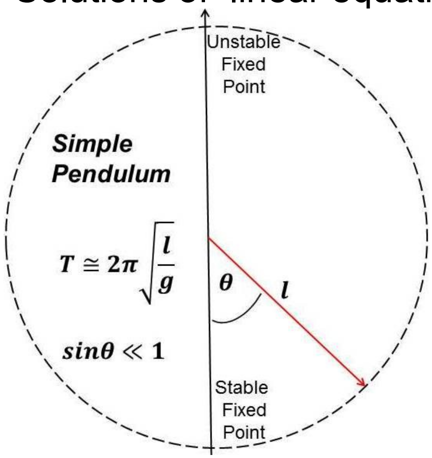
$$\vec{Y}(s_0) = (x_0, p_{x0}, z_0, p_{z0}, \varphi_0, \delta)$$

- Linear Transfer matrixes (M) follow multiplication rule $M(s_2/s_0) = M(s_2/s_1) \cdot M(s_1/s_0)$

- Transfer matrix for full revolution must repeat itself in order for motion to be **stable**

$$M(s+L) = M(s+N\cdot L) = M(s)$$

- Necessary and sufficient condition of STABLE motion – transfer matrix $[M]^{n\cdot N}$ is **BOUNDED** at any $n \to \infty$
- Solutions of linear equations of motion are Real part of periodic quasi-harmonic function of orbit trajectory (***s)***



$$Y(s) = Re\left(\sqrt{\epsilon\beta_y(s)} \cdot exp(\pm i\mu_y)\right) = \sqrt{\epsilon\beta_y(s)} \cdot cos(\mu_y(s))$$

$$p_y(s) = -\sqrt{\frac{\epsilon}{\beta_y(s)}} \cdot \left[sin(\mu_y(s)) - \frac{\beta_y'(s)}{2} cos(\mu_y(s))\right]$$

A.Kolomensky and A.Lebedev. Theory of Cyclic Acceleratros (1966)

- Betatron function has periodicity of lattice Cell $\beta_y(s+L) = \beta_y(s)$ No LIMIT on the amplitude $Y_0(s) = \sqrt{\epsilon\beta_y(s)}$
- For the **linear lattice** the dynamic aperture is **infinite**
- Phase advance (μ) and **Trace** of transfer matrix (M) related as $cos(\mu) = \frac{1}{2}Tr(M) = \frac{1}{2}(m_{11} + m_{22})$

- Betatron oscillations are **BOUNDED** if phase advance (μ) has **REAL** values, that is $cos(\mu) < 1$
  Thus, **main condition** of **STABLE** motion in linear approximation is (G.Guignard. Particle Accelerators, 1986, V.18)

$$|Tr(M)| \leq 2$$

Integrable motion of particles in accelerators and rings should be **stable.** Deviations in all 3D planes with respect to central trajectory are **finite** and **limited.** Particles **oscillate** around reference orbit. **Asymptotic unbounded growth** of particle coordinates represents **unstable** behavior

- Betatron tunes are defined as number of betatron oscillations per turn ($\mu(s)$ is phase advance per Cell -- element of periodicity)

$$Q_y = \frac{\omega_y}{\omega_{rot}} = \frac{N \cdot \mu_y(s)}{2\pi} = \frac{1}{2\pi}\oint \frac{ds}{\beta_y(s)}$$

- Transfer Matrix should satisfy SYMPLECTIC conditions $M^T \cdot S \cdot M = S$

J.Rees. "Symplecticity in Beam Dynamics: An Introduction". SLAC-PUB-9939 (2003).

- Symplectic matrix (S) is composed of UNIT matrixes $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$J \equiv S = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and in 1D phase space $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

- **<u>SYMPLECTIC</u>** conditions are realized "if" and "when" $\det M = m_{11}m_{22} - m_{12}m_{21} = 1$

- Equations of harmonic oscillations might be derived also from the unperturbed Hamiltonian function (H_0) where second order terms ($x \cdot \delta$) and (δ^2) etc. are **omitted**

$$H_0 = \frac{p_x^2 + p_z^2}{2} + \frac{K(s)}{2}(x^2 - z^2)$$

- Hamiltonian of system as function of canonical variables $\left(\dot{q}_k = \frac{\partial H}{\partial p_k},\ \dot{p}_k = -\frac{\partial H}{\partial q_k}\right)$ is **Integral of motion**

$$H(q,p,t) = \frac{\partial H}{\partial t} + \sum_k \left\{\frac{\partial H}{\partial q_k}\dot{q} + \frac{\partial H}{\partial p_k}\dot{p}\right\} = const(t)$$

(B.Montague. CAS-95-06)

- **<u>providing</u>** there is no EXPLICIT dependence of Hamiltonian on time (t) $\left(\frac{\partial H}{\partial t} = 0\right)$ $(s = \omega t)$ $\left(\frac{\partial H}{\partial s} = 0\right)$
- (energy conservation law)

- **Lie Operator** denoted by symbol $:\boldsymbol{f}:$ acts on function $(\boldsymbol{g})$ as Poisson brackets in **2n** Phase Space $\{q_1, q_2, \dots q_n, p_1, p_2, \dots p_n\}$ (A.Dragt. „An Overview of Lie Methods for Accelratro Physics“. Proc. PAC-2012. USA)

$$:f:g = [f,g] = \sum_{i=1}^{n}\left\{\left(\frac{\partial f}{\partial q_i}\right)\left(\frac{\partial g}{\partial p_i}\right) - \left(\frac{\partial f}{\partial p_i}\right)\left(\frac{\partial g}{\partial q_i}\right)\right\}$$

- „Solving of Hamiltonian equations of motion and finding symplectic maps are equivalent tasks“
- Transfer map for an element of length (L) in **symplectic** Lie operator form is

$$M_{s0\rightarrow s1} = exp:\left(:-\int_{s0}^{s1} H(s)ds\right) = exp(:-LH:)$$

- Lie generators applied to describe non-linear kicks $exp^{:f:} = 1 + :f: + \frac{1}{2!}(:f:)^2 + \cdots = \sum_{k=0}^{\infty}\frac{(:f:)^k}{k!}$

- Canonical transformations simplify problems by choice of proper coordinate system from original $\{q_k, p_k, t\}$ variables to new canonical variables $\{Q_k, P_k, t\}$. (E.Wilson, CAS CERN 95-06)

$$H'(Q_k, P_k, t) = H(q_k, p_k, t) + \frac{\partial F}{\partial t}$$

- New Hamiltonian preserves form of Hamiltonian equations $\dot{Q}_k = \frac{\partial H'}{\partial P_k}$ $\quad \dot{P}_k = -\frac{\partial H'}{\partial Q_k}$
- Different types of **GENERATING functions** (F) are applied in order to eliminating EXPLICIT dependence of Hamiltonian on time $\left(\frac{\partial H'}{\partial t} = 0\right)$ and build new Integral of motion

$$dF = \sum_k (p_k\, dq_k - P_k dQ_k) + (H - H')dt$$

NON-LINEAR BEAM DYNAMICS (NBD)

- "Introduction of non-linear elements into ring lattice will cause oscillations about the closed (reference) orbit to grow in amplitude for particular tunes" (A.Wolski. "Beam Dynamics in High Energy Particle Accelerators". 2014.
- High order terms of magnetic fields cause perturbations of linear lattice and leads to restriction on beam stability: 1–Resonances, 2–Momentum dependence of betatron tunes, 3–Amplitude dependent tune shifts

- **Canonical perturbation theory to deal with non-linear BD AWAY of resonances** $(mQ_x + nQ_z \neq k)$
  (L.Nadolski, Non-Linear Beam Dynamics. NPA-2011-2012. (V2.2).
- Mechanism to change betatron tunes with momentum offset (linear, quadratic and cubic chromaticity)
- Mechanism to SHIFT betatron tunes with Amplitude of oscillations (ADTS) –linear approximation
- Resonance conditions, resonance width, stopbands

2D Hamiltonian includes linear part (H_0) + Perturbation terms (In Light Soures kinematic term $\left(\frac{x}{\rho} \ll 1\right)$ is ignored)

A. Verdier. CAS-CERN 95-06. W.Herr. CAS 2013 (CERN-2014-009). P.Streun, CAS-2003 (CERN-2006-002). Y.Cai. NIM-A 645 (2011).

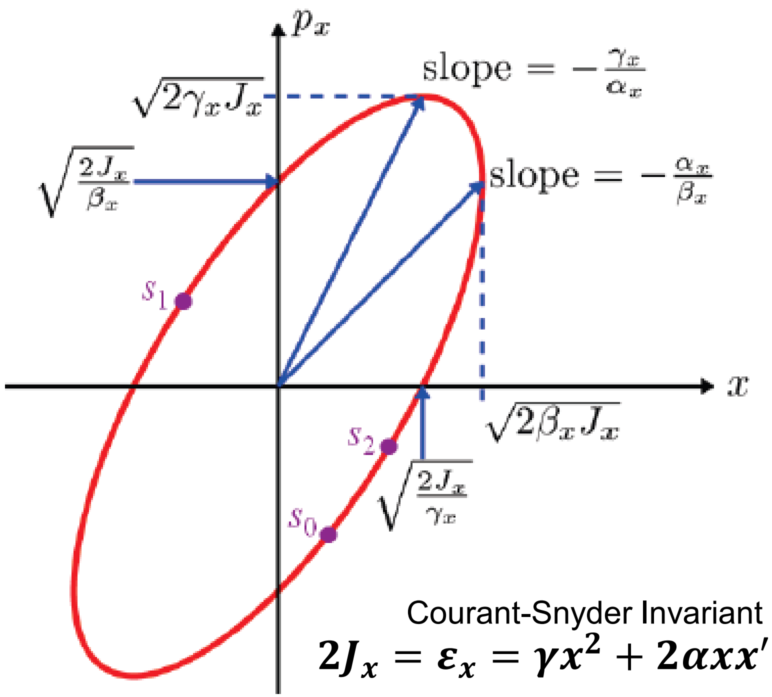
$$H = \left(1 + \frac{x}{\rho}\right)\overbrace{\frac{p_x^2 + p_z^2}{2(1+\delta)}}^{kinematic} - \overbrace{\frac{x}{\rho}\delta + \frac{x^2}{2\rho^2}}^{bend} + \overbrace{\frac{k_1}{2}(x^2 - z^2)}^{quad} + \overbrace{\frac{k_2}{3}(x^3 - 3xz^2)}^{sextupole} + \overbrace{\frac{k_3}{4}(x^4 - 6x^2y^2 + y^4)}^{octupole}$$

Quads strength - $K_Q \equiv k_1 = \frac{1}{B\cdot\rho}\left(\frac{\partial B_z}{\partial x}\right)$ Sextupole -- $K_S \equiv k_2 = \frac{1}{B\cdot\rho}\left(\frac{\partial^2 B_z}{\partial x^2}\right)$ Octupole -- $K_{OCT} \equiv k_3 = \frac{1}{B\cdot\rho}\left(\frac{\partial^3 B_z}{\partial x^3}\right)$

- After canonical transformations Hamiltonian of a ring might be represented in Action-Angle variable by combination of linear part and NON-linear contributions (kicks) of quads (H_2), sextupoles (H_3) G.Guignard. Part.Acc. V.18 (1986), J.Bengtsson.SLS Note 9/97.

$$H(\psi_x, J_x, \psi_z, J_z, s) \propto Q_x J_x + Q_z J_z + \int [H_2(s) + H_3(s)]$$

Phase space ellipse. Linear motion



Courant-Snyder Invariant

$$2J_x = \varepsilon_x = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

Figure taken from   A.Wolski. US PAC School/2013.

Linear motion: after canonical transformations. Phase space is circle in action-angle variables

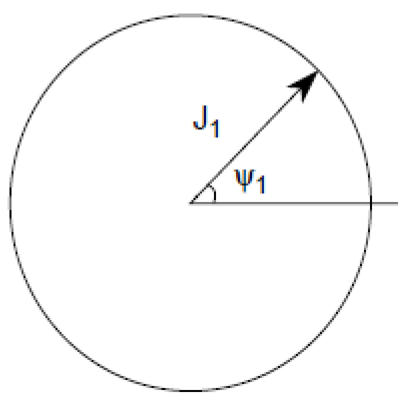
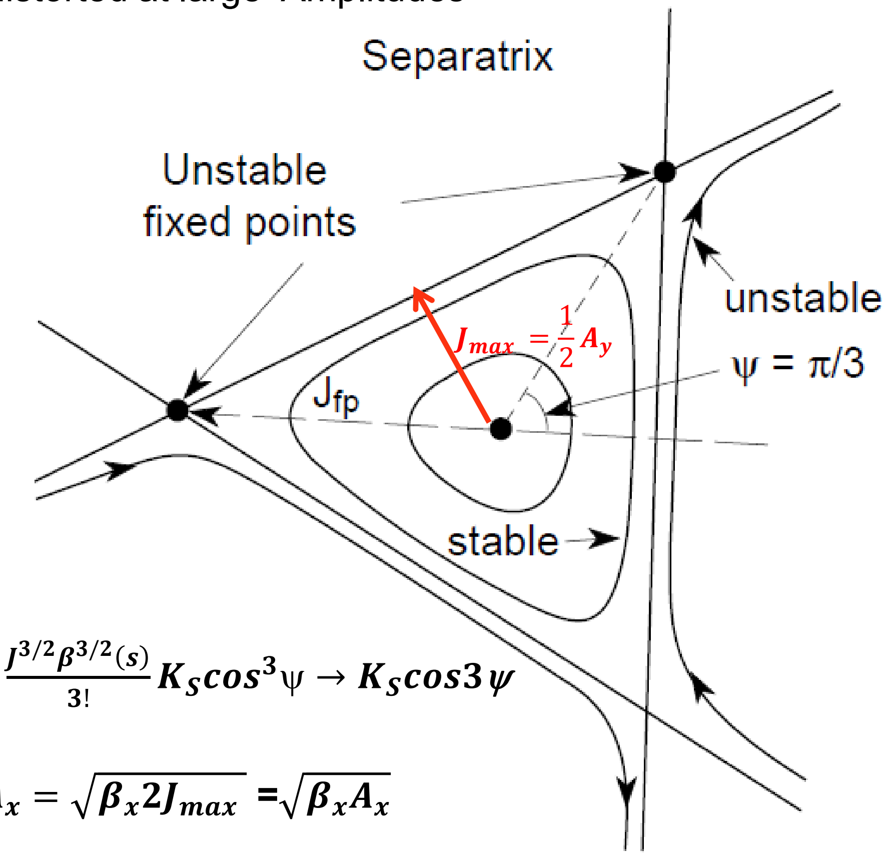


Figure taken from   E.Wilson. „Non-linear Resonances“. CAS CERN 95-06.

Phase space plot in Action-angle coordinates (J, ψ) with **non-linear** (**Sextupole**) term.

Amplitude of Stable Motion is limited and Phase Space is distorted at large Amplitudes



$$H_{SXT} \sim \frac{J^{3/2}\beta^{3/2}(s)}{3!} K_S \cos^3\psi \rightarrow K_S \cos 3\psi$$

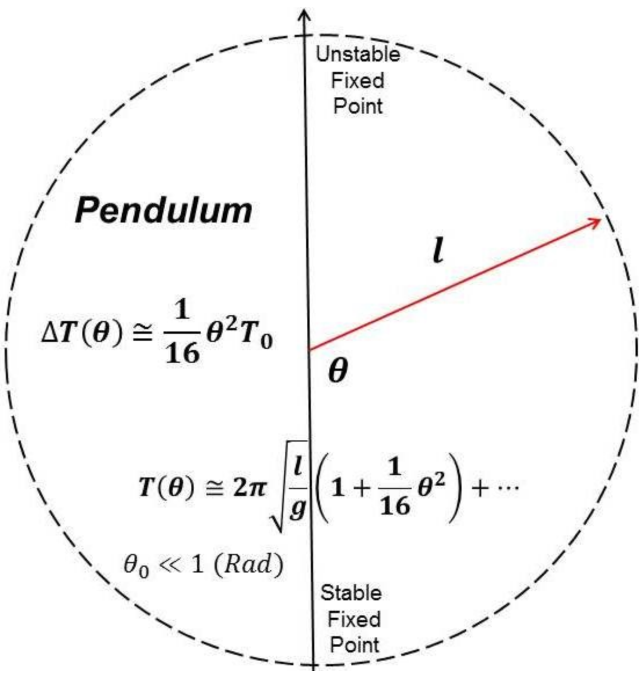
$$DA_x = \sqrt{\beta_x 2J_{max}} = \sqrt{\beta_x A_x}$$

DA_x is cross-section of ring acceptances $A_{x,z}$ at position (s)

Acceptance is limited by „Separatrix“ connecting UFP and dividing stable area around reference orbit and area of unstalble motion (hyperbolic curves with assymptotic behavior)

Figure taken from   E.Wilson. „Non-linear Resonances“. CAS CERN 95-06.

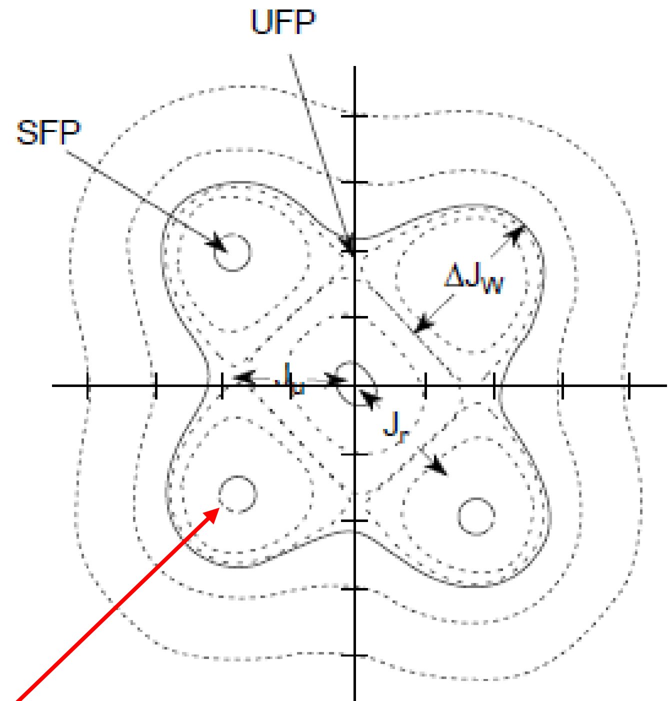
Period of Pendulum oscillations depends on angle (Amplitude of oscillations).



Approximation of first order term of Tailor expansion is valid for SMALL Amplitudes…

„Islands of stability“ are presented at very large Amplitudes, while we consider Ring Acceptance as an Area of closed curves around reference orbit.

Phase space plot in Action-angle coordinates (J, ψ) with non-linear (Octupole) terms.
Amplitude of Stable Motion is limited and Phase Space is distorted at large Amplitudes



$$H_{OCT} \sim \frac{\left(2J\beta_y(s)\right)^{4/2}}{4!} K_{OCT}\cos^4\psi \rightarrow K_{OCT}\cos 4\psi$$

Acceptance is limited by „Separatrix“ dividing stable area around remote (external) Fixed points from central area of stable motion around reference orbit

* L.Evans.. „The effect of an Octupole near a fourth-order resonance „ CAS CERN 84-15. p.319.
Figure taken from E.Wilson. „Non-linear Resonances“. CAS CERN 95-06.

Exciting of Third-Integer Resonance

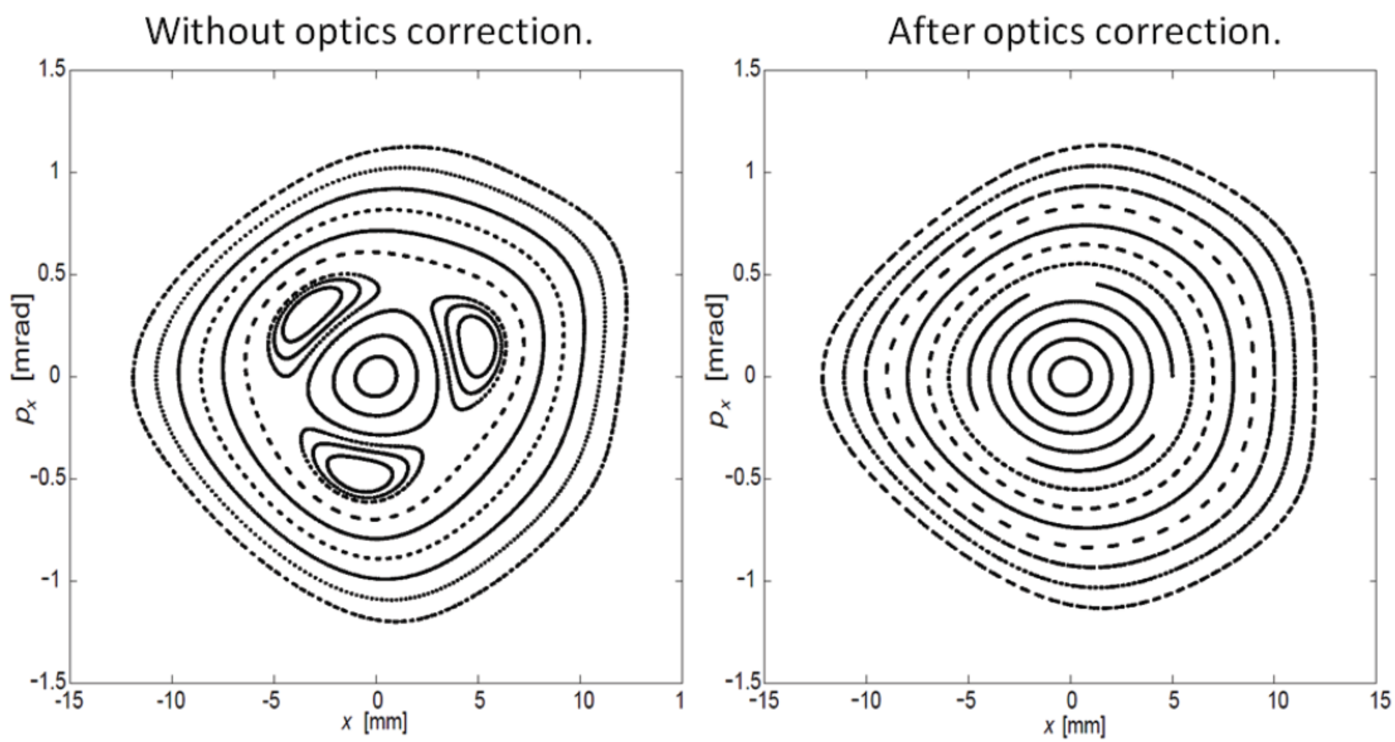
$$Q_x = N/3$$

is applied for Slow extraction of particles at Proton Synchrotrons

Note that there are two conditions for the action of a particle to be driven to large values by the sextupoles in a lattice:

1. The tune of the lattice must be close to an integer, or a third integer.

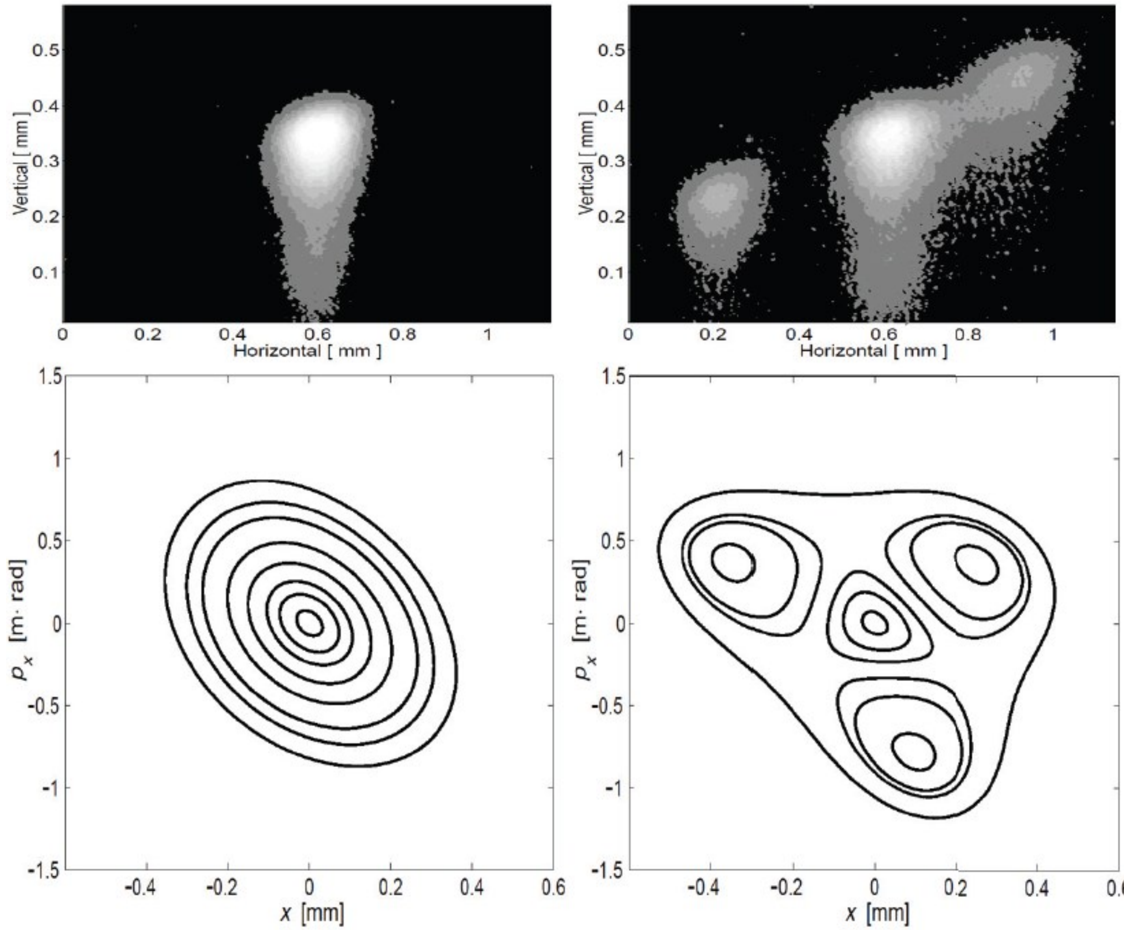
2. The resonant driving term must be significantly large.



Horizontal phase space in the ALS, close to a third-integer resonance, produced by tracking in a model of the lattice.

D. Robin, J. Safranek, W. Decking, "Realizing the benefits of restored periodicity in the Advance Light Source," PRST-AB, 2-044001 (1999).

Slide taken from presentation of D.Newton at KIT.2017.



Effect of third-integer resonance on beam distribution. Top: SR light monitor images. Bottom: (simulated) horizontal phase space.

D. Robin, J. Safranek, W. Decking, "Realizing the benefits of restored periodicity in the Advance Light Source," PRST-AB, 2-044001 (1999).

Slide taken from presentation of D.Newton at KIT.2017.

- Dynamic aperture for ON-momentum particles is a cross-section of Phase space Acceptance at (s)
- (A_y) is a ring Acceptance = maximum area of stable oscillations limited by Separatrix

$$DA_x(s) = \sqrt{2J_{max}\,\beta_x(s)} = \sqrt{A_x\,\beta_x(s)}$$

Analytical expressions

J.Gao. Analytical Estimation of the Dynamic Aperture of Circular Acceleratros. LAL/RT 00-02 (2000).

- **Single Sextupole** of strength (K_S) located at position (s_1) limits horizontal Dynamic Aperture at position (s)

$$DA_x^{sxt}(s) = \sqrt{2J_{max}\,\beta_x(s)} = \frac{\sqrt{2\beta_x(s)}}{\sqrt{3}\,\beta_x(s_1)^{3/2}}\left(\frac{1}{|K_S|L_S}\right)$$

- **Single Octupole** of strength (K_{OCT}) located at position (s_2) limits horizontal DA at position (s)

$$DA_x^{oct}(s) = \sqrt{2J_{max}\,\beta_x(s)} = \frac{\sqrt{\beta_x(s)}}{\beta_x(s_2)}\sqrt{\frac{1}{|K_{OCT}|L_{OCT}}}$$

- Estimation of vertical DA (Courant Snyder Invariant)

$$DA_y^{sxt}(s) = \sqrt{\frac{\beta_x(s_1)}{\beta_y(s_1)}\left(DA_x^2 - x^2\right)}$$

- $(|K_S|L_S)$ and $(|K_{OCT}|L_{OCT})$- integrated strengths of Sextupole and Octupole. Higher strength– less DA
- High value of beta-function at position (s) (Septum) preferable for injection
- High Dispersion (D_x) at SXT location allows to reduced SXT strength --- DA is improved
- Small β_y at location of SXT (OCT) might improve DA but Higher strength of non-linear element will be required

- Dynamic Aperture in horizontal plane of a single (**2m**) **multipole** (**m>3**)

J.Gao. Analytical Estimation of the Dynamic Aperture of Circular Acceleratros. LAL/RT 00-02 (2000).

$$DA_x^{2m}(s) = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x^m(s_{2m})}\right)^{\frac{1}{2(m-2)}}\left(\frac{1}{|K_{m-1}|L}\right)^{\frac{1}{(m-2)}}$$

- If non-linear elements are **independent** -- no special phase and amplitude relations between SXT (OCT)

$$\frac{1}{DA_{total}} = \sum_i^{N_{SXT}} \frac{1}{DA_{SXT}} + \sum_i^{M_{OCT}} \frac{1}{DA_{OCT}} + \cdots$$

Scaling

R.Bartolini. „Review of lattice design for low emittance rings“. BD Newsletter, No.57, 2012.
E.Levichev, A.Bogomyagkov, P.Piminov. Proc.IPAC-2014, Dresden. Private Comm.

- In approximation of small phase advance between two sextupoles one can estimate reduction of horizontal Dynamic Aperture for ultra-low emittance Synchrotron Light Sources according to scaling Law

$$DA_X(s) \sim \frac{\sqrt{\varepsilon_x^{nat}}}{|\xi_x^{cell}|} \sim \frac{D_X^{max}}{|\xi_x^{cell}|} \sim \frac{1}{|K_S| \cdot L_S}$$

- and for vertical Dynamic Aperture

$$DA_Z(s) \sim \frac{DA_X(s)}{|\xi_z^{cell}|}\sqrt{\frac{\beta_x(s)}{\beta_z(s)}}$$

- ε_x^{nat} -- natural emittance after SR damping ξ_x^{cell} -- natural chromaticity per cell D_X^{max} -- Dispersion (max)

High natural CHR + reduced Dispersion -- main factors limiting DA of ultra-low emittance Light Sources

Small Dispersion -- to keep ultra-low emittance, High chromaticity – due to strong Quads and SXT

RESONANCE DRIVING TERMS

- The concept of **RESONANCE DRIVING TERMS** is applied when nonlinear components are **dependent** -- special phase and amplitude relations are introduced for cancellation… G.Guignard Part.Acc.1986,v.18, pp.129-165. (J.Bengtsson. [1] SLS Note 9/97. [2] Beam Dynamics Newsletter, No.57.2012. [3] Nucl.Inst.Meth. A404 (1998) 237-247.)

- **Symplectic** form of a lattice Hamiltonian integral is contributed by individual Hamiltonians of **each** Quadrupole H_{2m}, Sextupole H_{3n} and might be approximated by a Sum of different **modes** – **RDT**

$$\int [H_2(s) + H_3(s)]\, ds = h_3 \propto \sum h_{jklmp}$$

- Hamiltonian coefficients h_{jklm} contain the contribution from all the multipoles of order $(n = j + k + l + m)$. Even sum $(j + k)$ corresponds to **normal** multipoles (B_n) while odd sum $(l + m)$ - to **skew** multipoles (A_n)

- Driving terms are derivatives of Lie operator over action variable

$$h_{jklmp} = \frac{\partial h_3}{\partial J_{x,y}}$$

- "**LINEAR**" (first order) Hamiltonian modes and their complex conjugates (*) are proportional to the **first order** of integrated sextupole strength $(K_S L)$

- First order RDT and their complex conjugative (*) is a **SUM** of integrated strengths of $\boldsymbol{N_{SXT}}$ sextupoles and $\boldsymbol{M_{QUAD}}$ quadrupoles with **scaling factors** ("lever of arm") over different frequencies multiple to betatron phase advance $\boldsymbol{\mu_{xn}}, \boldsymbol{\mu_{zn}}$ at element location

$$h_{jklmp} = -h^{*}_{jklmp} = -\sum_{n}^{N_{SXT}} (K_S \cdot L_S)_n \, \beta_{xn}^{\frac{j+k}{2}} \cdot \beta_{yn}^{\frac{l+m}{2}} \cdot D_n^p \cdot e^{i(j-k)\mu_{xn}+i(l-m)\mu_{yn}}$$

$$+\left[= -\sum_{n}^{M_{QUAD}} \left(K_Q \cdot L_Q\right)_n \beta_{xn}^{\frac{j+k}{2}} \cdot \beta_{yn}^{\frac{l+m}{2}} \cdot e^{i(j-k)\mu_{xn}+i(l-m)\mu_{yn}}\right]_{p\neq 0}$$

- Quadrupoles at **dispersive** sections $(\boldsymbol{D \neq 0})$ – contribute to **chromatic** Hamiltonian driving modes $(\boldsymbol{p \neq 0})$
- Quads in **achromatic** sections $(\boldsymbol{D = D' = 0})$ do **NOT** contribute to RDT (in theory)

- Sextupoles contribute to both **chromatic** RDT $(\boldsymbol{p \neq 0})$ in dispersive sections $(D \neq 0)$
- AND **geometric** RDT$(\boldsymbol{p = 0})$ in **achromatic** sections $(\boldsymbol{D = D' = 0})$

- Chromatic sextupoles compensate natural negative chromaticity and its strengths are fixed

- Additional harmonic sextupoles in achromatic sections compensate shrinking of area of stable oscillations (reduction of Dynamic Aperture) caused by main chromatic sextupoles

- Driving modes ideally should be SUPPRESSED either CANCELLED because they are the source of ALL the resonances considering long term behavior by multiple repetition of the lattice structure

$$\left|\boldsymbol{h^{\infty}_{jklmp}}\right| = \frac{\left|\boldsymbol{h_{jklmp}}\right|}{\boldsymbol{2sin\{\pi[(j-k)Q_x^{cell} + (l-m)Q_y^{cell}]\}}}$$

- Table 1. LINEAR Resonance Driving Terms (RDT) and their effects

N	Linear Driving terms	Effect	Comments	Phase dependence	
1	$h_{11001} \quad (CrX\_lin)$	$\partial Q_x / \partial \delta$	Linear Chromaticity (hor) $\xi_x^{(1)}$		chromatic
2	$h_{00111} \quad (CrY\_lin)$	$\partial Q_y / \partial \delta$	Linear Chromaticity (vert) $\xi_y^{(1)}$		chromatic
3	$h_{20001} = -h^*_{02001}$	$2Q_x \pm Q_s$	parametric half-integer resonance synchro-betatron coupling	$e^{i \cdot 2\mu_x}$	chromatic
4	$h_{00201} = -h^*_{00021}$	$2Q_y \pm Q_s$	parametric half-integer resonance synchro-betatron coupling	$e^{i \cdot 2\mu_y}$	chromatic
5	$h_{10002} = -h^*_{01002}$	$\partial D / \partial \delta$	Second order dispersion $D^{(1)} \equiv \partial D / \partial \delta$	$e^{i \cdot \mu_x}$	chromatic
6	$h_{21000} = -h^*_{12000}$	Q_x		$e^{i \cdot \mu_x}$	geometric
7	$h_{10110} = -h^*_{01110}$	Q_x		$e^{i \cdot \mu_x}$	geometric
8	$h_{30000} = -h^*_{03000}$	$3Q_x$		$e^{i \cdot 3\mu_x}$	geometric
9	$h_{10200} = -h^*_{01020}$	$Q_{xy} + 2Q_y$		$e^{i \cdot (\mu_x + 2\mu_y)}$	geometric
10	$h_{10020} = -h^*_{01200}$	$Q_{xy} - 2Q_y$		$e^{i \cdot (\mu_x - 2\mu_y)}$	geometric

Linear chromaticity is INDEPENDENT on phase advance μ

$$\boldsymbol{h_{11001}} = \xi_y^{(1)} = -\frac{1}{4\pi} \sum_{n}^{N_{QUAD}} (K_Q L_Q)_n \beta_{yn} + \frac{1}{4\pi} \sum_{m}^{M_{SXT}} (K_S L_S)_m \beta_{ym} D_m$$

SECOND ORDER RESONANCE DRIVING TERMS

$$h^{(2)} \propto (K_{SXT} \cdot L_{SXT})^2$$

- Second order RDT d*ri*ves synchrotron sidebands of LINEAR Resonance modes. When Linear RDT are vanished either reduced the second order terms i.e. corresponding sidebands are week and ignored
- It is mandatory to suppress (minimize) first order terms in order to reduce second order resonance modes

Amplitude Dependent betatron Tune Shifts (ADTS)

- Cross-talks of first order RDT in sextupoles as well as between different sextupoles produce phase independent second order RDT and cause ADTS. In **linear** approximations ADTS are

$$\delta Q_X^{ADTS} = \frac{\partial h^{(2)}}{\partial J_X} \approx \alpha_{xx} J_X + \alpha_{yx} J_Y \qquad \delta Q_Y^{ADTS} = \frac{\partial h^{(2)}}{\partial J_Y} \approx \alpha_{xy} J_X + \alpha_{yy} J_Y$$

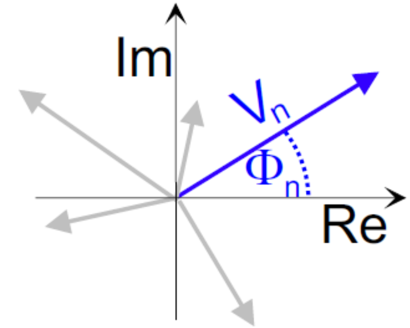
- ADTS originated from an amplitude or momentum dependent **shift** of the closed orbit in the sextupole. Orbits must be corrected to the magnetic axis of a sextupole in order to minimize ADTS. Beam Position Monitors must be located as close as possible to each sextupole in order to control beam **centroid**

- Octupols applied to COMPENSATE ADTS caused by Sextupoles

$$\alpha_{yy} = +\frac{1}{16\pi}\left(\sum_{n}^{N_{oct}} (K_{oct} \cdot L_{oct}) \cdot {\beta_{yn}}^2\right) \qquad \alpha_{xz} = -\frac{1}{8\pi}\left(\sum_{n}^{N_{oct}} (K_{oct} \cdot L_{oct}) \cdot \beta_{xn} \cdot \beta_{zn}\right)$$

- One might consider RDT as a sum of complex vectors
- Each complex vector represents local Hamiltonian of sextupole either quadrupole
- Phase of each vector is multiple of betatron phase advance at element location

Figure taken from
A. Streun. CAS CERN-2006.



- **Ten** first order Hamiltonian terms are **linear** in sextupole strength
- **Two** real RDT drive horizontal and vertical Chromaticies (***j=k***) and (***l=m***)

- $h_{11001} = \xi_y^{(1)} = -\frac{1}{4\pi}\sum_n^{N_{QUAD}}(K_Q L_Q)_n \beta_{yn} + \frac{1}{4\pi}\sum_m^{M_{SXT}}(K_S L_S)_m \beta_{ym} D_m$

- Only SXT at **dispersive** sections $(D \neq 0)$ control linear chromaticity $\xi^{(1)}$ and their settings are fixed
- Hamiltonian RDT of chromatic SXT accumulated **ADDITIVELY** regardless of phase (μ).
- Amplitudes of chromatic SXT added / subtracted depending on sign of element strength and sign of (D)
- Harmonic SXT and quadrupoles located at achromatic sections $(D = D' = 0)$ **do not** contribute to $\xi^{(1)}$
- Strengths and positions of HARMONIC SXT adjusted to compensate geometric aberrations from CHR SXT

- **Eight** first order modes and their complex conjugative are phase dependent -- resonance behavior

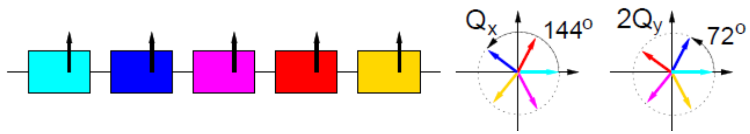
- **9** families of SXT could be enough to eliminate all excitation modes but at phase advance of cell close to $\mu_x = 180°$ Hamiltonian RDT proportional to $(2\mu_x)$ will be amplified coherently by all SXT
-
- Linear system of **9** equations degenerates down to **rank 8** and no solution to suppress ALL RDT

- Lattice optics --adjusted to cancel / minimize RDT by proper phase advance between sections of the lattice

- applying **mirror symmetry** conditions and phase advance between CELLS close to quarter of integer

$$\mu_y \approx (k \pm 0.25) \cdot 2\pi$$

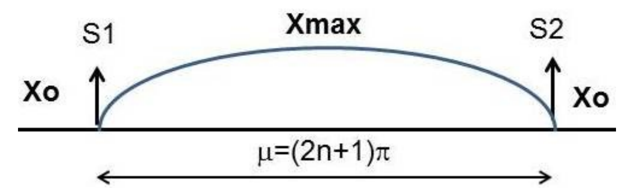
- resonances associated with $2\mu_y$ phase dependent RDT could be **cancelled** between cells ($2\mu_y = \pi$), resonances $(\mu_y, 3\mu_y)$ and coupling resonances $(\mu_x \pm 2\mu_z)$ – between TWO pairs of cells
- Periodicity $\Delta Q_{x,z} = N \cdot \mu_{x,z}^{cell} \left[2\Delta Q_{x,z}\ 3\Delta Q_x\right] \rightarrow integer$ **(N=5)**



- Figure taken from A. Streun. CAS CERN-2006.

- To compensate linear CHR and cancel some RDT "$-I$" condition is applied. Non-interleaved pairs of sextupoles are located at mirror symmetry points of lattice sections with phase advance multiple to

$$\mu_y = (2n \pm 1)\pi$$



- Twiss $\alpha_{S1} = -\alpha_{S2}$ $\beta_{S1} = \beta_{S2}$
- parameters $D(s_1) = D(s_2)$ $D'(s_1) = -D'(s_2)$

- In this case RDT of sextupoles proportional to $2\mu_y$ will amplify itself (rank of matrix is reduced – degradation problem)
- To minimize RDT proportional to ODD tunes, one need to provide phase advance between lattice sections close to $\mu_y = (2n \pm 1)\pi$. Complex Hamiltonian modes with phase proportional to $2\mu_y$ will be added to each other and amplified instead of to be cancelled.
- Location of SXT in a sequence of **EQUAL** phase advance steps $(\Delta\mu_x), (\Delta\mu_z)$ might help to limit RDT

- **General symmetry** conditions full cancellation of non-linear terms of two sextupoles (thin lens approx.)

$$\Delta\mu_{2-1} = n\pi \qquad \beta_{Z1}/\beta_{X1} = \beta_{Z2}/\beta_{X2}$$

$$(S_{X1})\beta_{X1}^{3/2} = -(-1)^n (S_{X2})\beta_{X2}^{3/2} \qquad D_1 = -(-1)^n D_2$$

- G.Xu. „General Conditions for self-cancellation of geometric aberrations in a lattice". Phys. Rev. A 8, 104002 (2005)

Difraction Limited Light Source (E.Levichev. BINP Proposals. 2013)

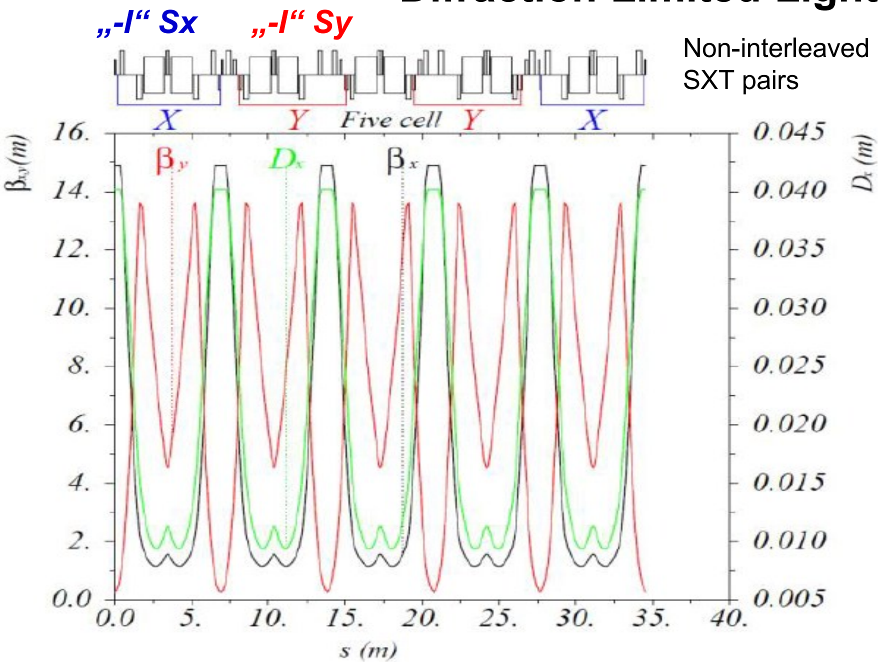


Figure 11: Five-cell superperiod with two horizontal sextupole pairs (denoted X) and two vertical ones (denoted Y).

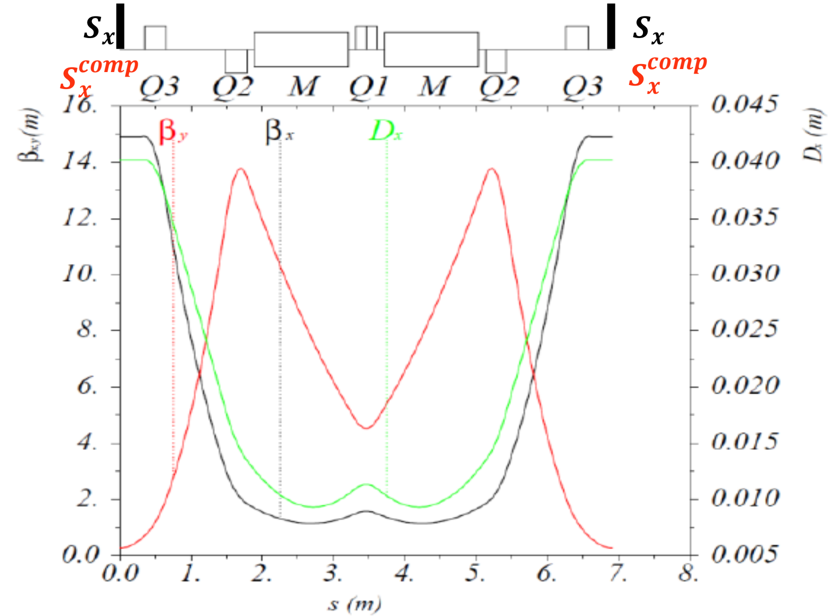


Figure 10: Split magnet TME cell for ultimate storage ring. A.Bogomyagkov, E.Levichev, P.Piminov. Proc. IPAC-2014.

DA of BINP DLLS ring (proposals)

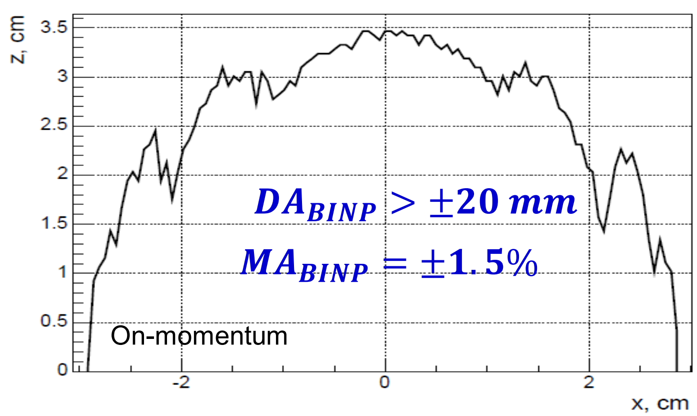
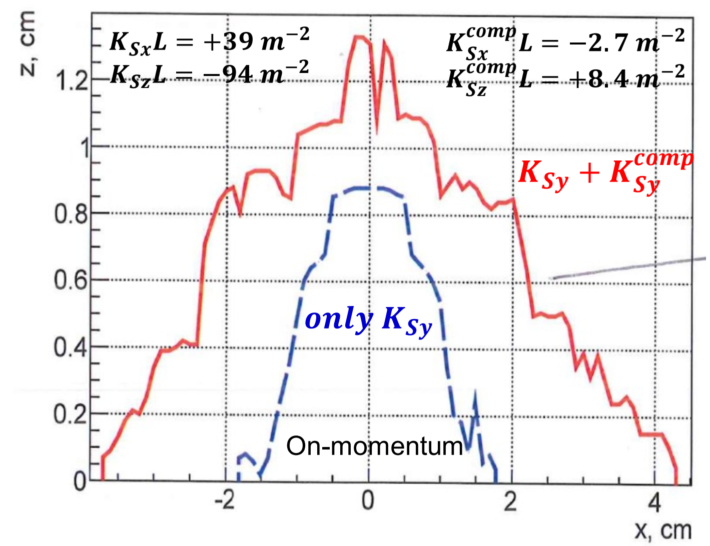


Figure 13: Dynamic aperture of the 10 pm emittance storage ring at $\beta_{x,y} = 10$ m.



E.Levichev. „Difraction Limited electron storage ring with large DA“. BINP Proposals. 2013.

Comparisoin of DLLS Projects

MAX IV	7BA	3 GeV	528 m	320 pm	500 mA	DA 20 mm
ESRF Phase II	7BA	6	850	130	200	10 mm
Spring-8	6BA	6	1400	67.5	300	3 mm
Diamond	4-5-7BA	3	560	45-300	300	2 mm
ALS	5 -7BA	2	200	50-100	500	2-3 mm
Our proposal	**7BA**	**3**	**1300**	**15**	**200**	**40 mm**
Pep-X	7BA	4.5	2200	11	200	10 mm
tUSR	7BA	9	6200	3	100	0.8 mm

Original table -- R.Bartolini. “Ultimate storage rings”. CLIC Workshop 2013. CERN.

Table taken from E.Levichev. „Difraction Limited electron storage ring with large DA“. BINP Proposals. 2013.

It is commonly recognized that Required DA for Light Sources to ensure beam injection

$$DA_X \geq \pm 10\ mm$$

Momentum Acceptance should be $MA = \frac{\delta P}{P_0} \geq \pm\,(2 \div 4)\%$

to provide reasonable Life time $T_{1/2} \geq 10\ hours$

Lattice of cSTART ring. Relaxed parameters

"3Q_SPLIT_SHORT_5" lattice version fitted to the FLUTE Bunker 15 × 14 m

Parameters of „21_3,6,9" lattice are shown in brackets

Circumference [m] **44.112** (41)
Horizontal Tune **5.8438** (11.31)
Vertical Tune **8.4605** (2.61)

Nat. hor. Chromaticity –16 (–27)
Nat. vert.Chromaticity –21 (–19)
Chr / Cell –4 / –5.2 (–6.8 / (–4.8)

MAX-IV Chroma (h/v) –50 / –44
MAX-IV CHR / Cell –2.5 / –2.2
NSLS-II CHR / Cell < |3|
ESRF-100 pm / Cell –3.6 / –2.7

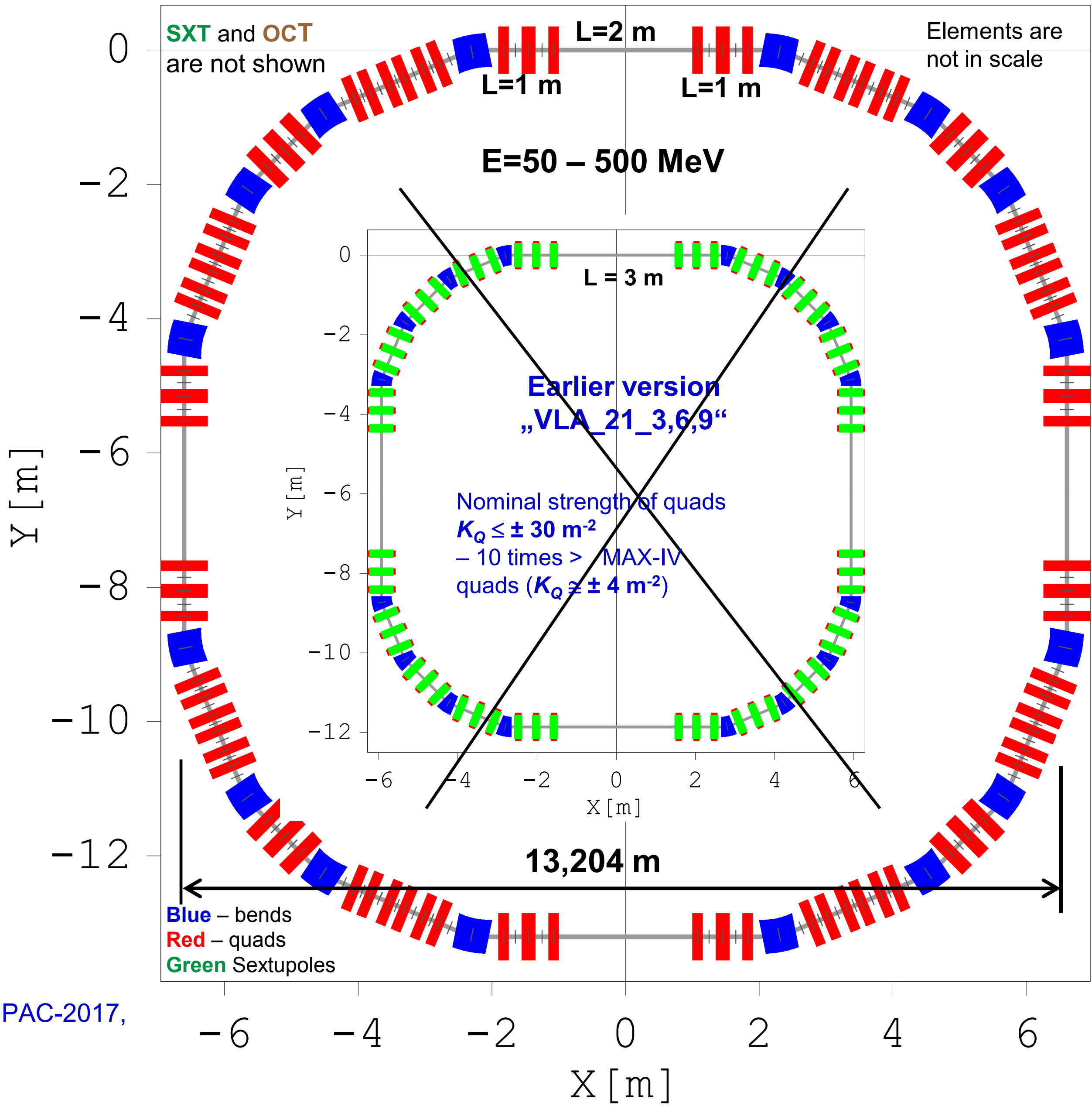
Mom compact. 6.034E-03 (3.6E-03)
Hor.damp.partition Jx 1.397 (0.97)

Energy 50 MeV
Rad.energy/turn [MeV] 0.00
Natural Emittance [nm] 0.18 (0.1)
Appr.vert. Emittance [pm] 0.31
Natural energy spread 4.24E-05
Hor. damping time [s] 24 (20)
Vert. damping time [s] 34 (19)
Long. damping time [s] 21 (9)

Radiation Integrals:

I1	[m]	2.662E-01
I2	[1/m]	4.935E+00
I3	[1/m2]	3.876E+00
I4	[1/m]	-1.96E+00
I5	[1/m]	3.389E-01

A.Papash.Proc. IPAC-2017, IPAC-2018



Parameters of old „21_3,6,9“ lattice are shown in brackets

One Cell. Length L=11.028 m

$E=50 \div 500$ MeV
$B \cdot R=0.167 \div 1{,}67$ T·m
BENDS
TH = 22.5° (22.5°)
L=50 cm (30)
B = 0.13 ÷1,3 T (0.22-2,2)
R =1.273 m (0.764)
$\theta 1=\theta 2=11.25°$ (0)
STR=–3.99 m-2 (0)
GRAD= –(0,67÷6,7) T/m

Quads eff. length
Q1,3 = 15 cm (15)
Q2,7,8 = 20 cm
Q4A-4B 15-15 cm
Q5A-5B 15-15 cm
Q6A-6B 15-15 cm

Quads strength
| Kq | < 16 m^{-2} (30)
| G | < 3 ÷ 30 T/m
MAX-IV Kq < 4 m^{-2}

Chromatic SXT
Integr. Strength Ks·L (m^{-2})
4A-4-4B= +2.2 / 2.2 / 2.2 (+33)
5A-5-5B= –11.3 / 11.3 / 11.3 (-35)
6A-6-6B = +2.1 / 2.1 / 2.1 (+34)
|Ks·L | < 11 m^{-2} (40)
(MAX-IV S < 22 m^{-2})

SXT effective length
15-10-15 cm (15)

Quads strentgh (m^{-2})

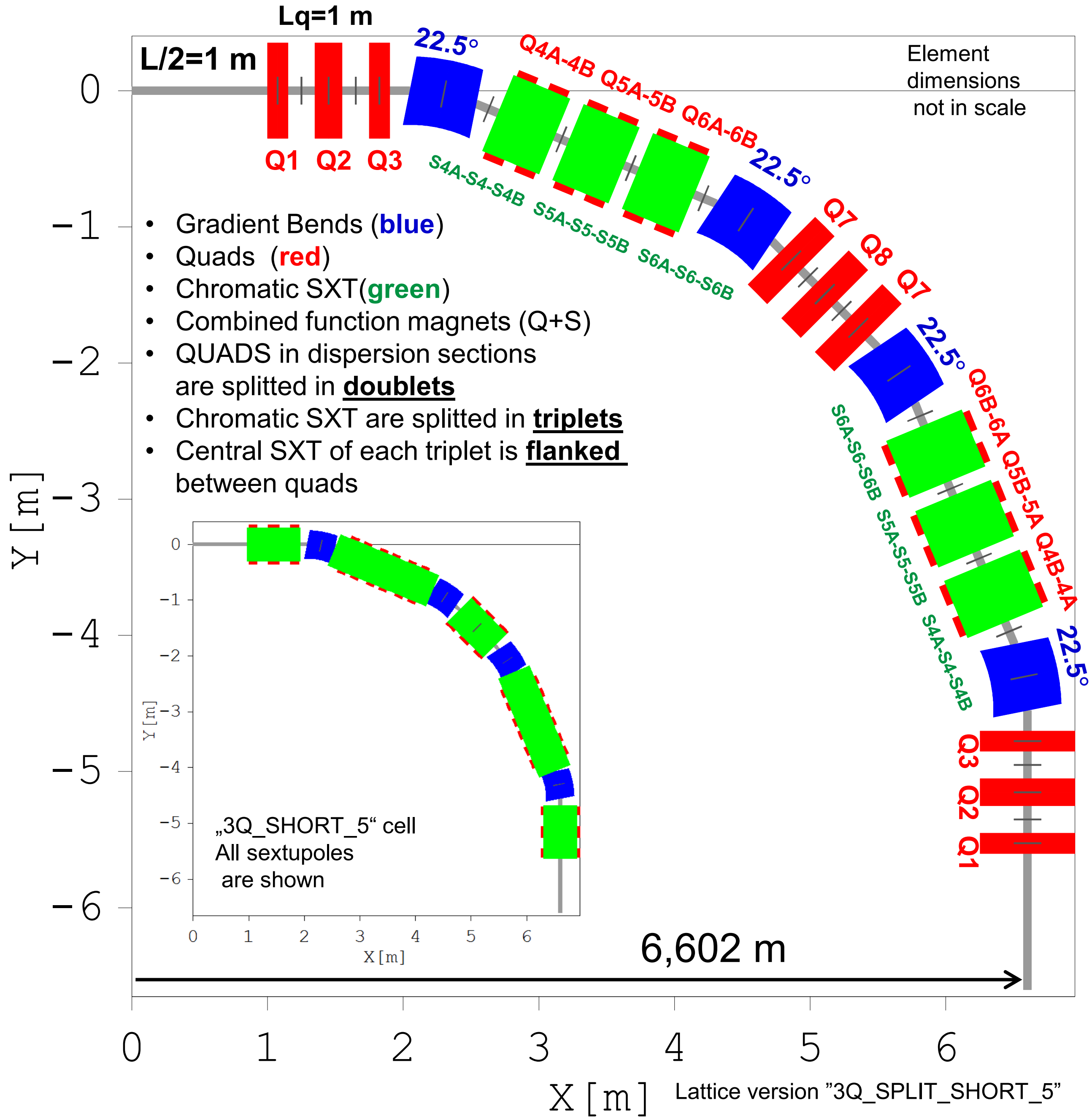
Q1=–2.138 (–5.8)
Q2=+6.736 (+18.2)
Q3=–4.01 (–13.9)

Q4A=+11.7 (+28.7)
Q4B=+11.485

Q5A=–15.305 (–20)
Q5B=–16.435

Q6A=+11.65 (+28.9
Q6B=+11.6 5

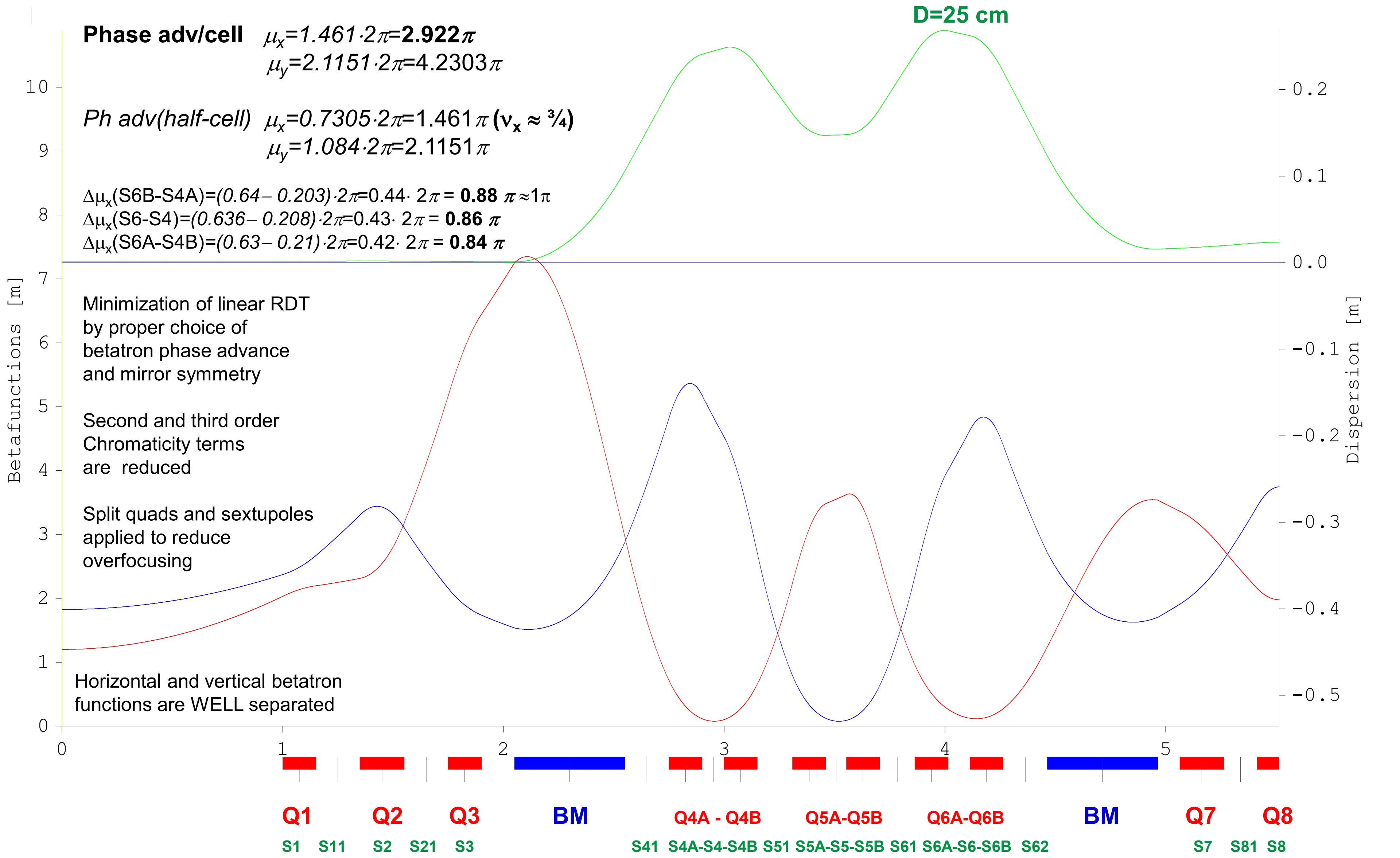
Q7=–2.437 (–12.1)
Q8=+8.460 (+26.2)



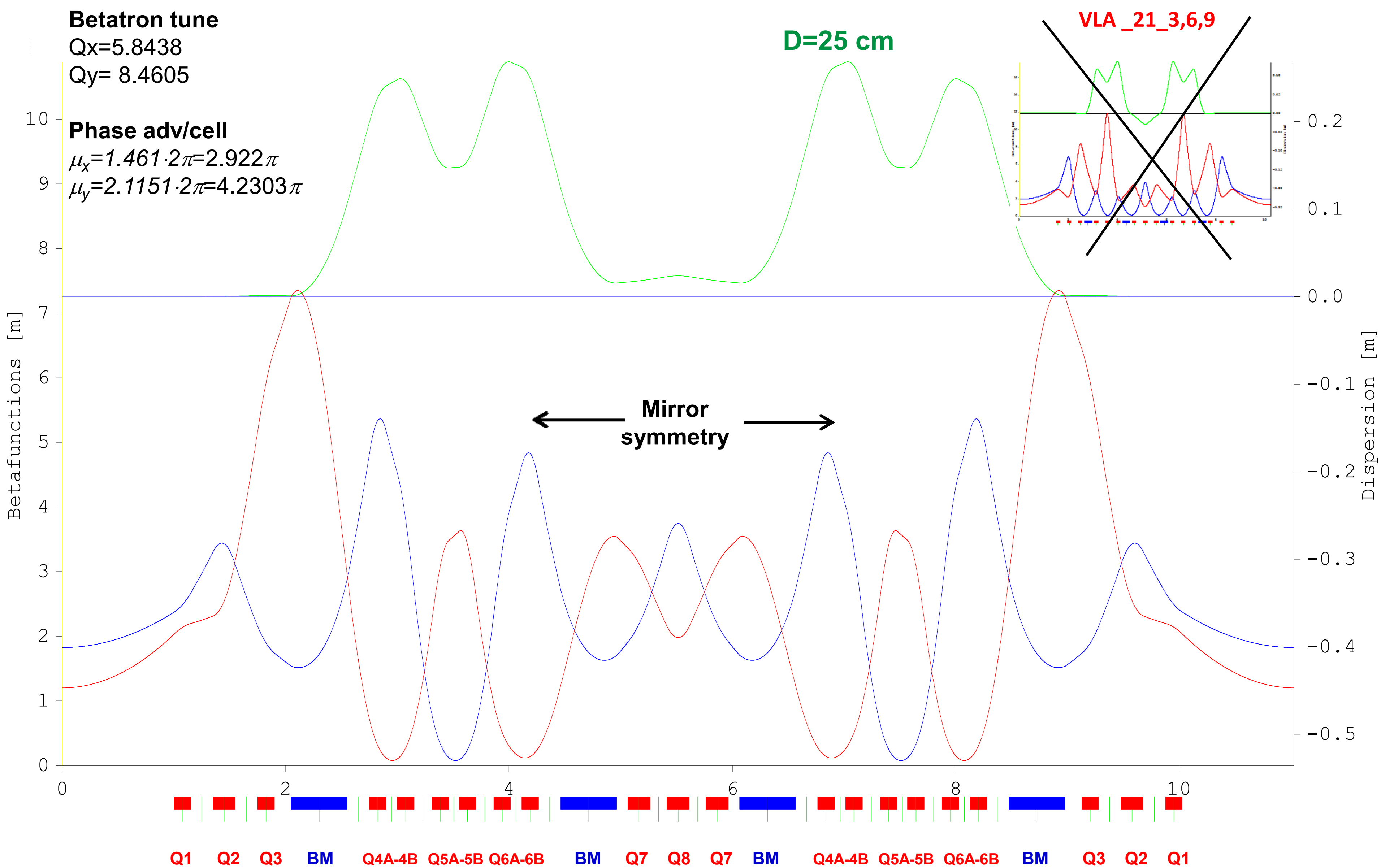
Lattice version ”3Q_SPLIT_SHORT_5”

Lattice "3Q_SPLIT_SHORT_5". half- cell

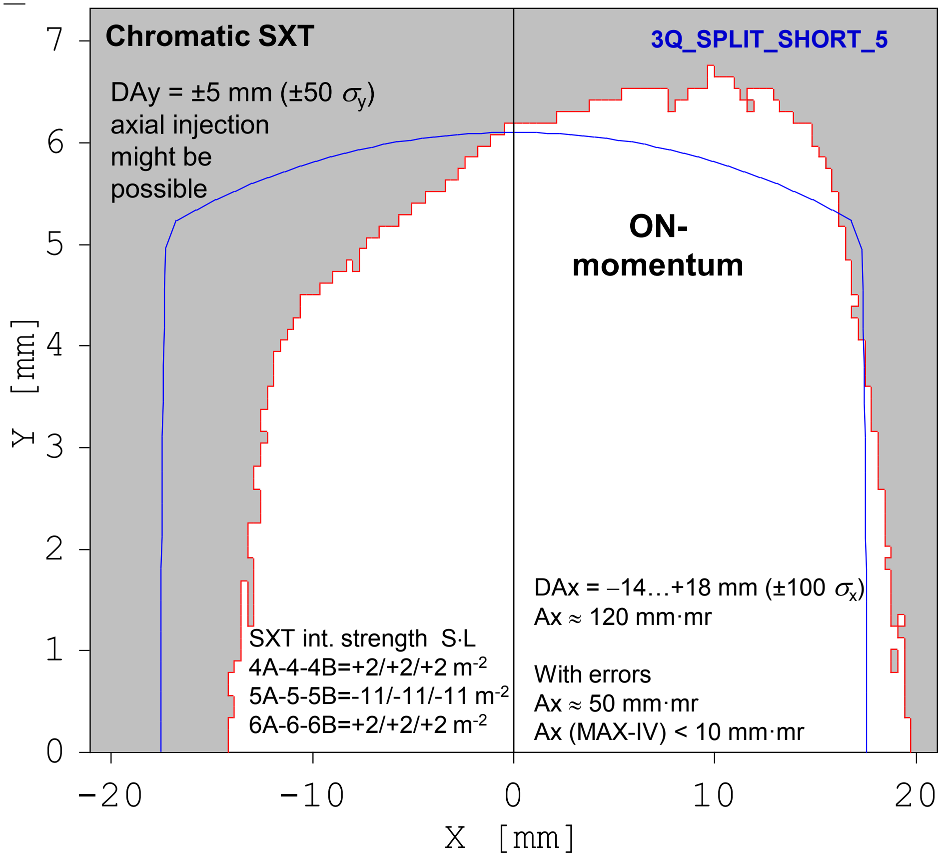
Chromaticity $\xi x = -16$ $\xi y = -21$
Betatron tune **Qx=5.8438** **Qy= 8.4605**

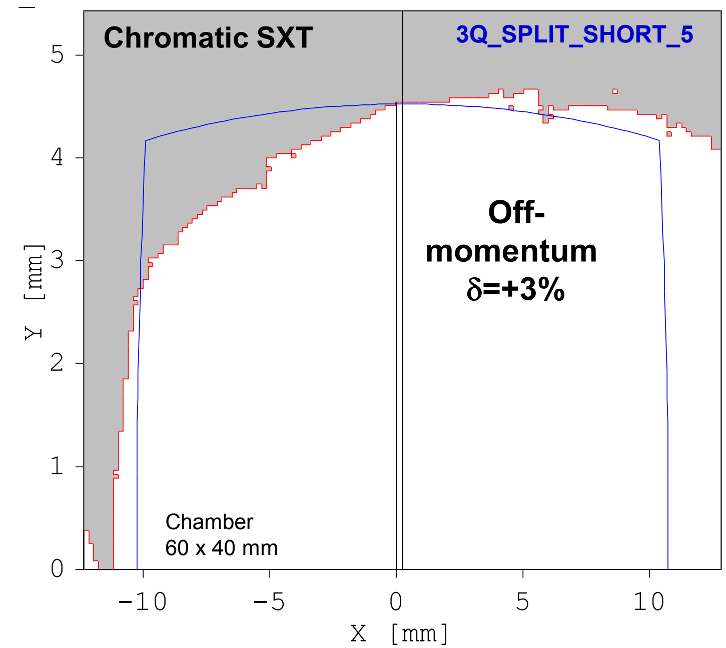


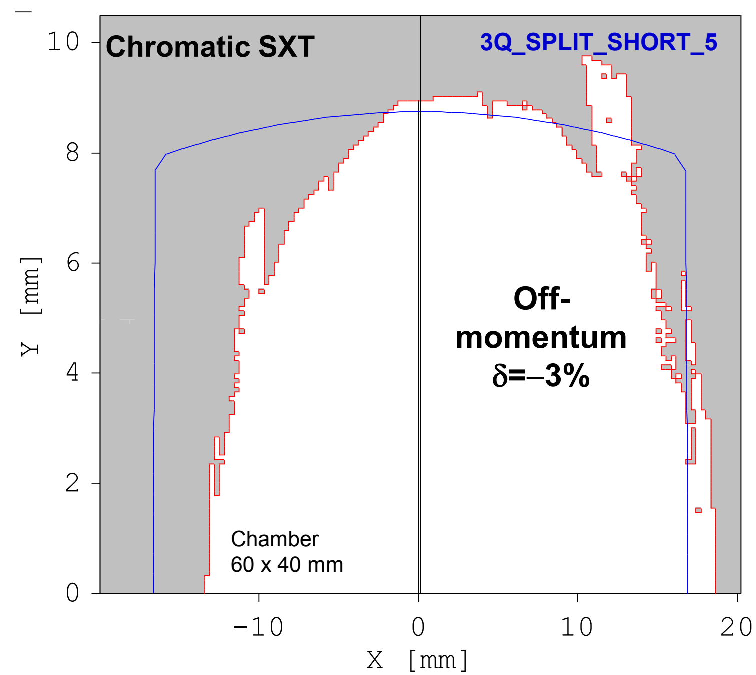
Lattice "3Q_SPLIT_SHORT_5" One cell



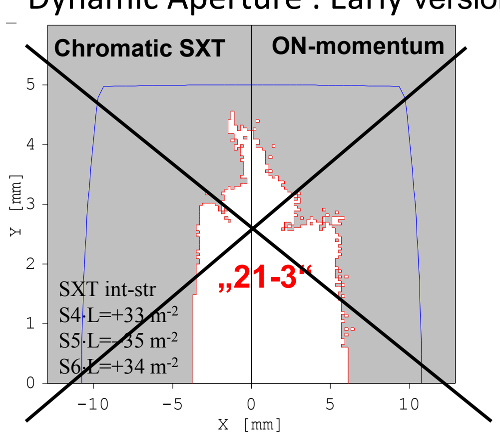
DYNAMIC APERTURE. Lattice "3Q_SPLIT_SHORT_5"

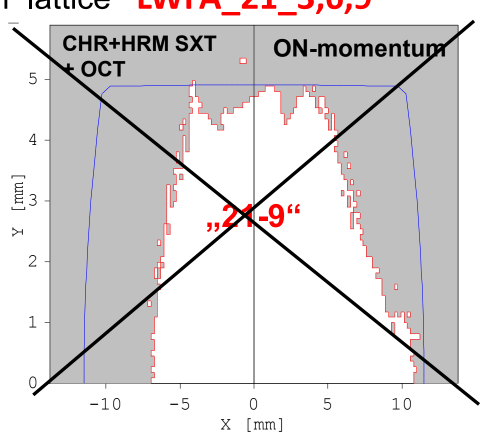






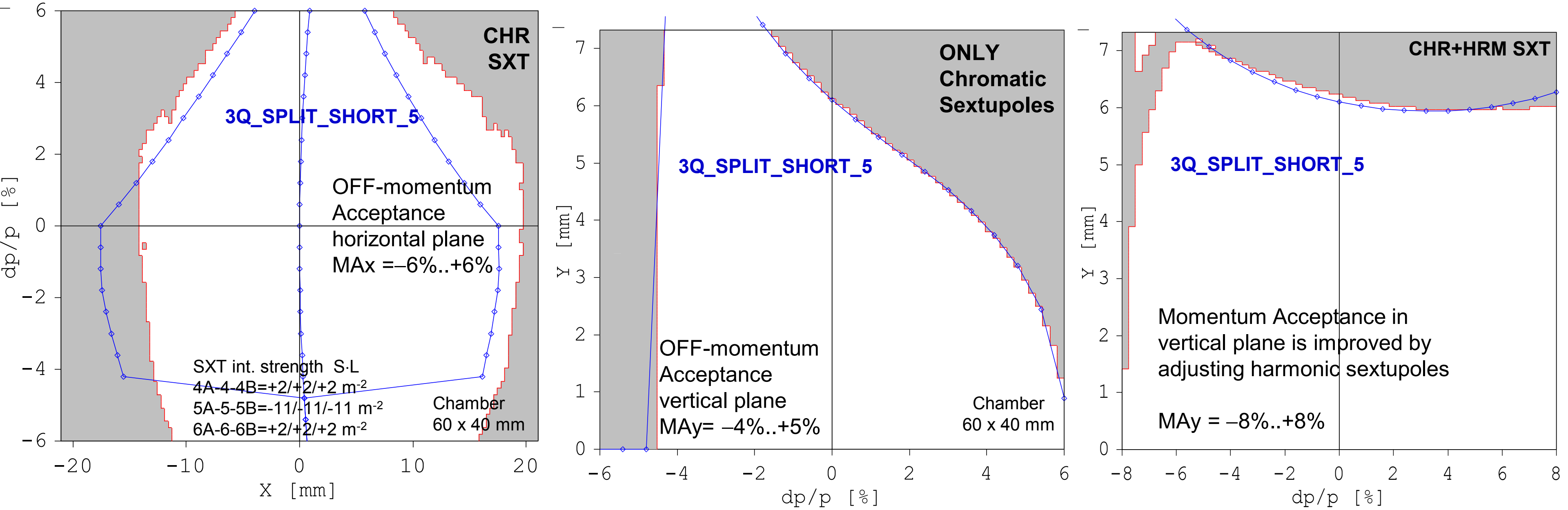
Dynamic Aperture . Early version lattice "LWFA_21_3,6,9"



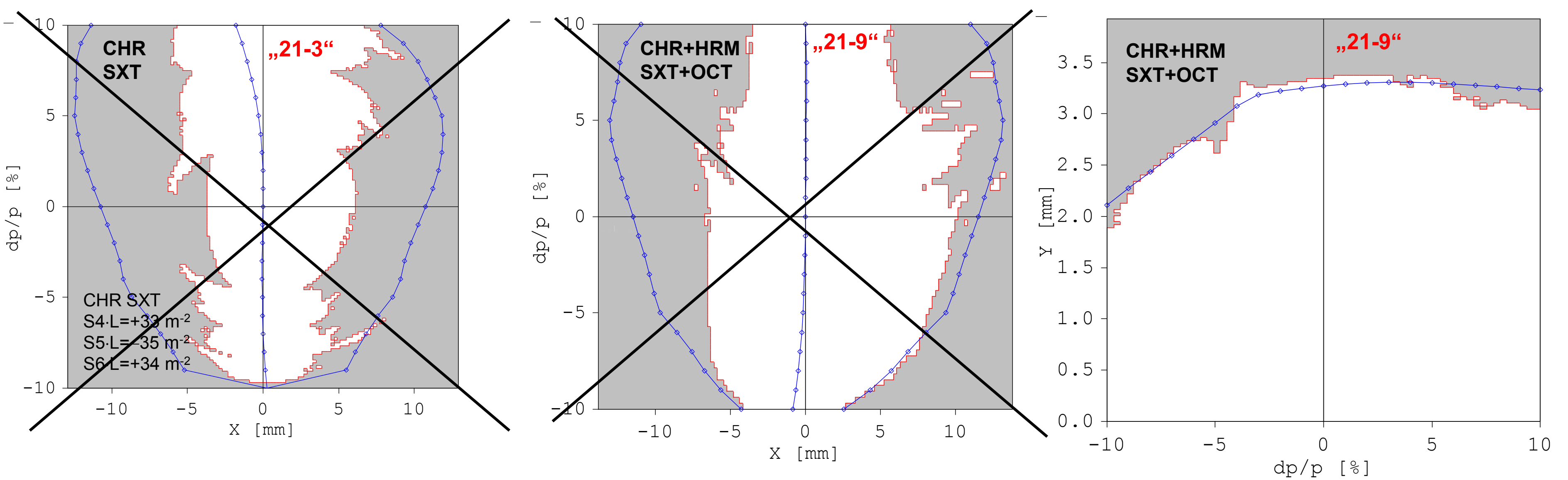


Off-momentum dynamic aperture as function of energy deviation

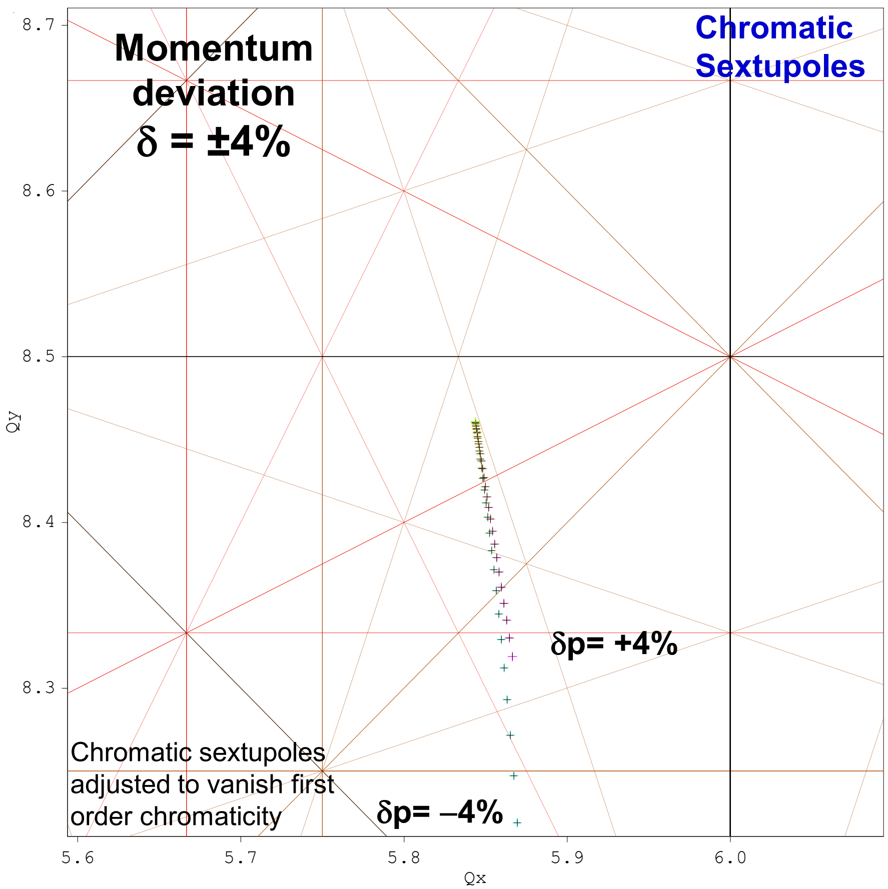
Lattice **"3Q_SPLIT_SHORT_5"**

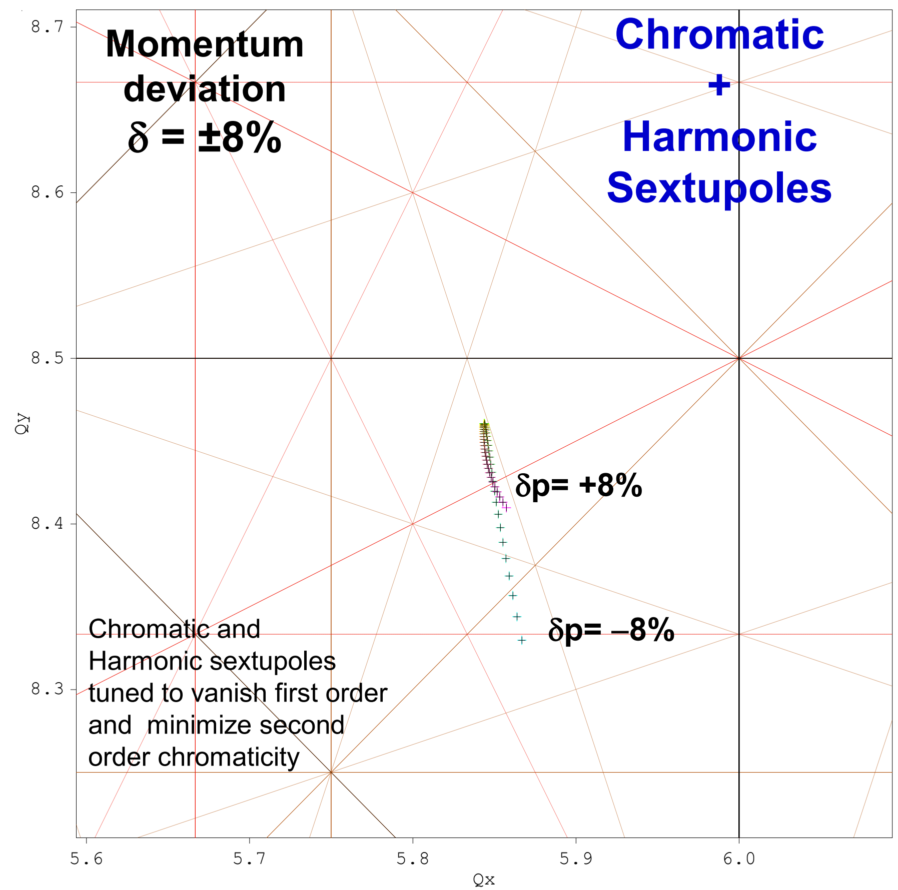


early versions of VLA-cSR lattice. strong settings of quads cause over-focusing and non-linear distortions



Betatron tune diagram. Lattice “3Q_SPLIT_SHORT_5”



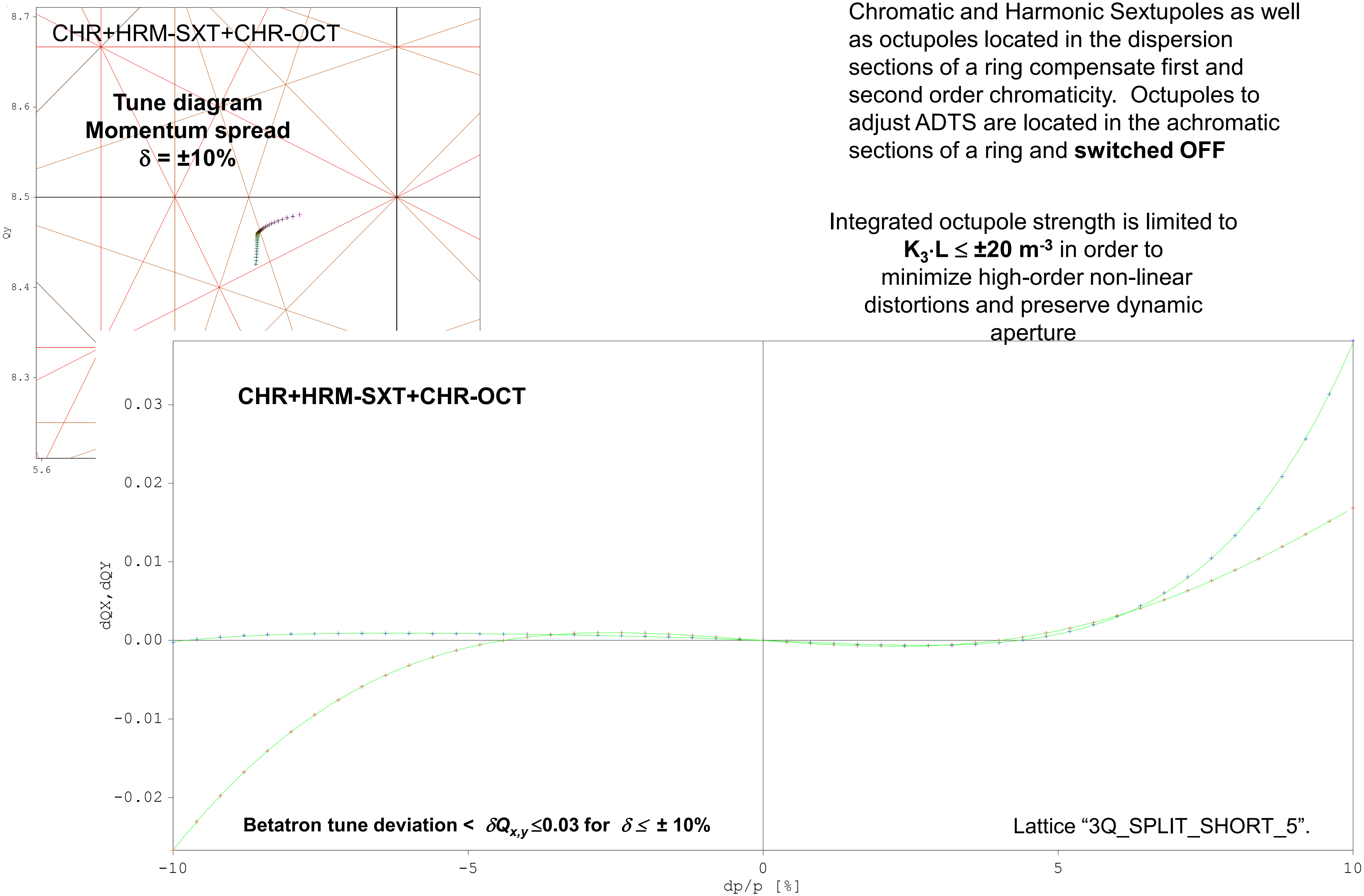


By adjusting chromatic and harmonic sextupole families one can compensate linear as well as second order chromaticity.
Tune deviation for off-momentum particles is reduced and momentum acceptance is improved

Betatron tune deviation for off-momentum particles

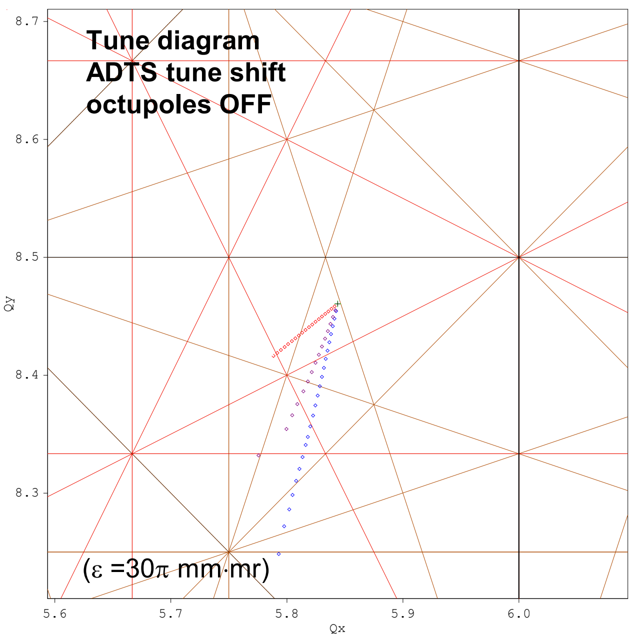
Chromatic and Harmonic Sextupoles as well as octupoles located in the dispersion sections of a ring compensate first and second order chromaticity. Octupoles to adjust ADTS are located in the achromatic sections of a ring and **switched OFF**

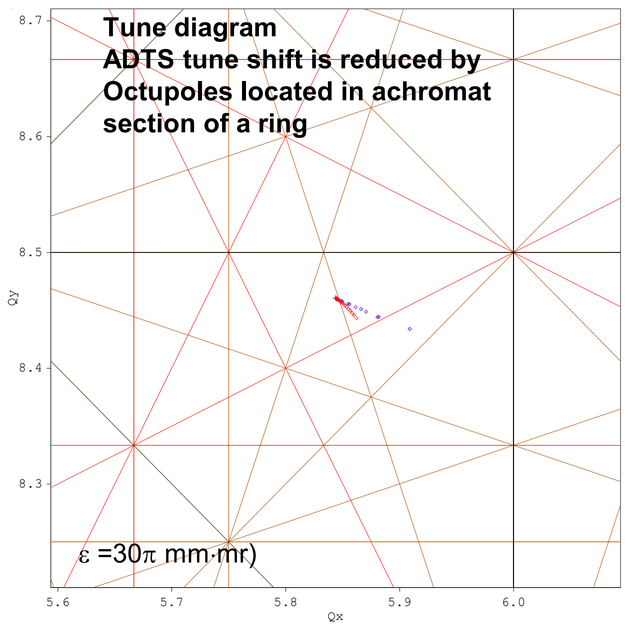
Integrated octupole strength is limited to **$K_3 \cdot L \leq \pm 20\ m^{-3}$** in order to minimize high-order non-linear distortions and preserve dynamic aperture

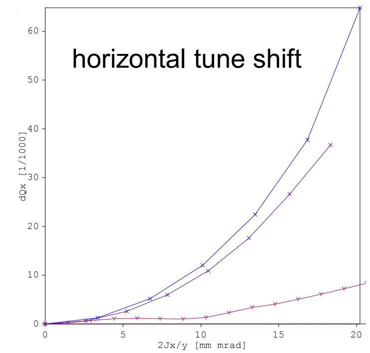


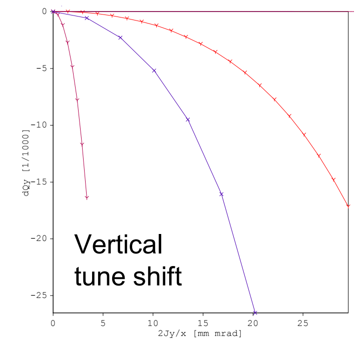
Amplitude dependent tune shift. Lattice “3Q_SPLIT_SHORT_5”.

Octupoles located in achromat section of a ring compensate ADTS



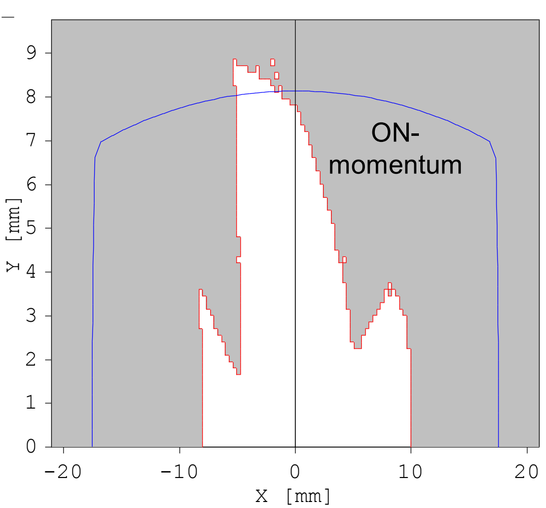


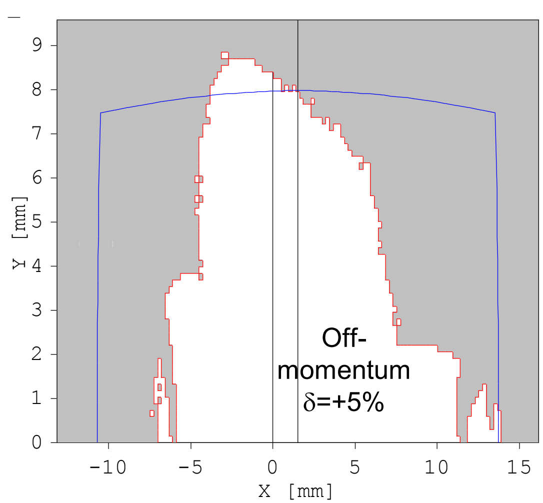


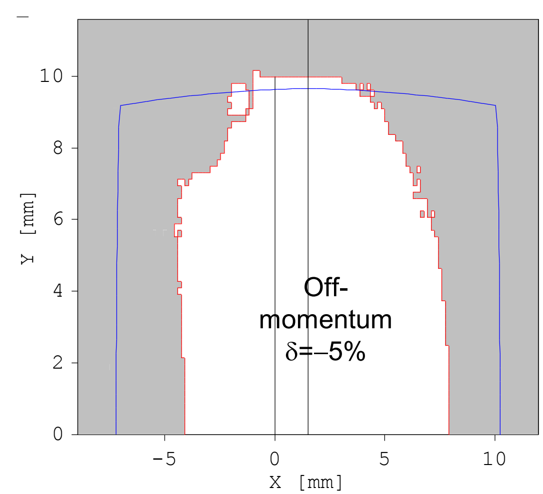


In order to suppress ADTS the integrated octupole strength is increased to **$K_3 \cdot L \approx \pm 250$ m^{-3}**

As a consequence the DA is **shrunk** in horizontal and vertical planes







Early versions of the VLA-cSR lattice suffer from non-linear distortion caused by strong over-focusing

- It was necessary to relax strong settings of early versions (lattice „21.3,6,9“ as an example) and find compromises between lattice parameters
- Quadrupole strength of „21_3,6,9“ lattice **Kq>30 m^{-2}** i.e. ~10 times MORE of MAX-IV quads (**Kq < 4 m^{-2}**)...
- Chromaticity/cell HIGH, dispersion SMALL ***D_{max}*<12 cm** and required SXT strength is HIGH (**S·L≈40 m^{-2}**)
- Dynamic aperture (on-momentum) is SMALL **DAx≈–5..+7 mm** NOT enough for stable circuation

Merit of the „3Q_SPLIT_SHORT_5“ lattice

- Ring dimensions are fitted to existing FLUTE Bunker while main parameters are improved
- Radius of bending magnets is increased, gradient and edge focusing in vertical direction are applied
- Quads are splitted in doublets **QA-QB**, distances between elements are increased
- Quads strength reduced from **Kq=30 m^{-2}** to **Kq<14 m^{-2}** (still few times higher those for MAX-IV)
- The compensation of horizontal chromaticity is done by sets of splitted SXT triplets **SXA–SX–SXB**
- Ring lattice is modelled to satisfy „–I“ condition for mirror symmetry non-interleaved sextupoles
- Max of Dispersion is doubled **D=25** cm, integrated SXT strengh reduced **S·L<20 m-2** (MAX-IV **S·L≈20 m^{-2}**)
- Side effect – decreasing of Geometric Momentum Acceptance to **MA≤ ±7%** (60×30 mm chamber)

Merit of the “3Q_SPLIT_SHORT_5“ lattice

- Chromatic SXT **S4** and **S6** are flanked between quads and located at position of D_{max} and β_{Xmax}
- Phase advance between **S4** and **S6** is close to π and Symmetry conditions are applied

$$\beta_x(S6)=\beta_x(S4) \quad \alpha_x(S6)=-\alpha_x(S4) \quad D(S6)=D(S4) \quad D'(S6)=-D'(S4) \qquad \mu_x(S6-S4)\approx\pi$$

- **S5** is positioned in the middle of dispersion section at MAX of vertical β_y(S5) and MIN β_x(S5)≈5 cm
- Vertical SXT **S5** has little influence on SXT non-linear terms in horizontal plane
- Magnetic rigidity **B·R=1.67 T·m** (500 MeV) and quads gradient is relaxed **∂B/∂R<25 T/m** (MAX-IV≈40 T/m)
- Half-cell betatron phase advance close to **$\mu_x\approx3.5$ ($\nu_x\approx1.75$)** and Resonance Driving Terms (Qx, 2Qx, 3Qx, Qx±2Qy) are reduced due to lattice symmetry and periodicity
- Quadratic (ξ_{2x}), Cubic (ξ_{3x}) CHROMA and ADTS are minimized by lattice geometry and harmonic SXT
- Dynamic aperture **±5 mm** of early lattice versions („**21_3,6,9**“) is OPENED to **-14..+18 mm** (**3Q_SPLIT_SHORT_5)**
- DA for off-momentum particles ($\delta\leq\pm5\%$) is enough for stable circulation of wide momentum spread beam
- Octupoles and decapoles should be added to VLA-cSR lattice to suppress ADTS, quadratic and cubic chromaticities etc. but its strengh must be limited in order to preserve stability of betatgron motion
- Full suppresion of ADTS by octupoles leads to reduction of DA. Full suppression of second order chromaticity by octupoles leads to increase of ADTS and cubic chromaticity terms. As a result the MA is improved but DA is reduced

Outcome of cSTART Feasibiltiy Studies

- Extensive studies of possible geometry and lattice of the very large acceptance compact storage ring operating in the energy range 50 to 500 MeV have been provided

- The main objective of feasibility studies was to create ring model suitable to store the beam after Laser Wake field accelerator with wide momentum spread ($\sigma \sim 1 \div 2\%$) as well as ultra-short electron bunches in a "~fs" range

- More than 40 models of compact ring lattice based on DBA, DBA-FDF, TBA, 5BA cells etc. have been composed, simulated, analyzed and merit of different configurations has been carefully studied

- The DBA-FDF Lattice with relaxed settings and optimized parameters could be accepted as a basis for further Detailed Design studies of the Very Large Acceptance compact Storage Ring

- Proposed VLA-cSR lattice model compromises contradictory conditions

  -- Small circumference of the ring $C \leq 50$ m

  -- Small dispersion $D < 15 \div 25$ cm

  -- Large Dynamic Aperture in the dispersion plane $DAx > 15$ mm

  -- Large Acceptance in both planes $Ax,y > 20$ mm·mr

  -- Wide Momentum Acceptance $MA \sim 5 \div 10\%$

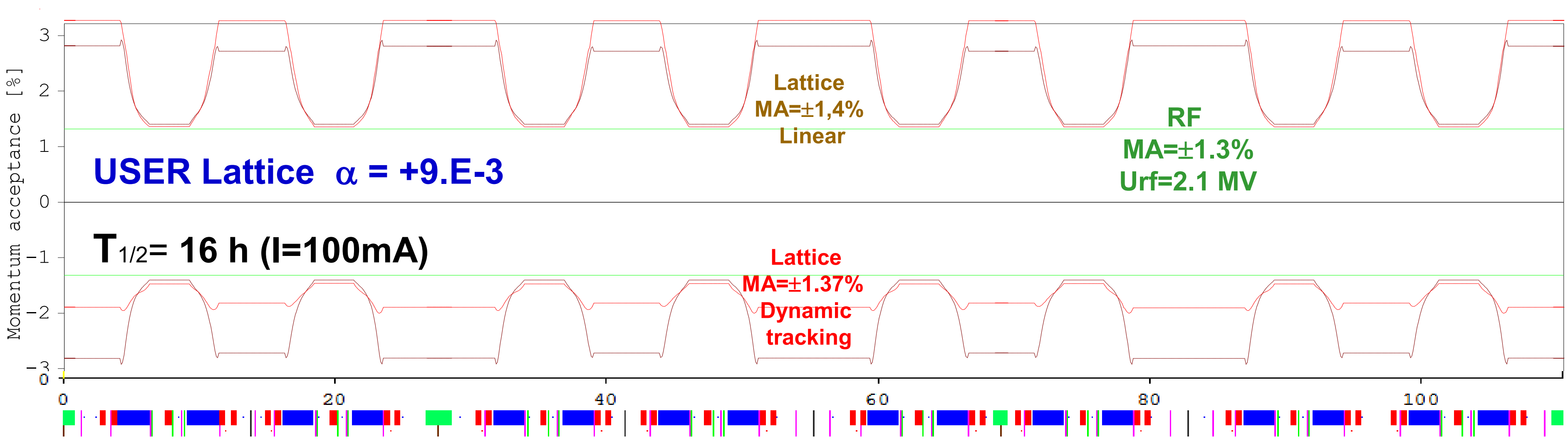
  -- Chromaticity / cell should be limited to $|\xi/\text{cell}| < 5 \div 10$ (NSLS-II sets limits for SXT $|\xi/\text{cell}| < 3$)

- The "–I" condition is provided. The mirror symmetry at position of horizontal chromatic sextupoles is satisfied
- Local maximums of horizontal beta-function and dispersion at position of main chromatic sextupoles help to restrict sextupole strength. Dynamic Aperture is opened significantly from DA= –5..+6 mm to DA= –15.. +20 mm

- Phase advance per cell is adjusted to minimize leading Resonance Driving Terms including high order chromaticity terms. The dynamic aperture for ON- and OFF-momentum particles is enough to store wide energy spread beam

- Harmonic sextupole and octupole families should be used for non-linear studies in particular to operate the ring at negative compaction factor, to manipulate with bunch length and shape etc. but OCT strength must be limited

Momentum Acceptance of KARA ring at 2.5 GeV (OPA simulations)



Proceedings of the 2001 Particle Accelerator Conference, Chicago

UNDERSTANDING THE DYNAMIC MOMENTUM APERTURE OF THE ADVANCED LIGHT SOURCE*

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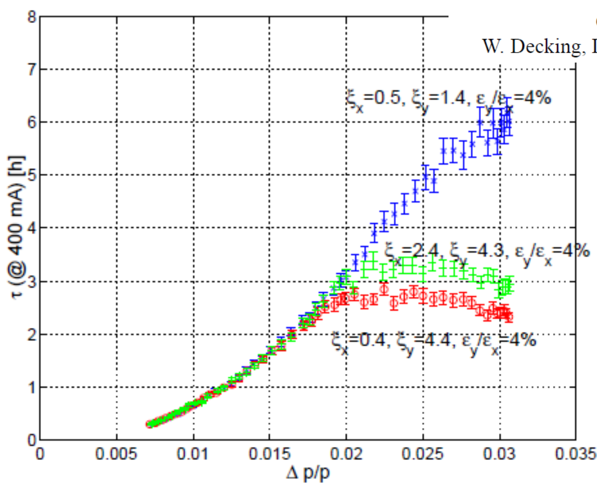


Figure 1: Scan of the RF amplitude and therefore the longitudinal bucket height for three different sets of horizontal and vertical chromaticities (1.5 GeV).

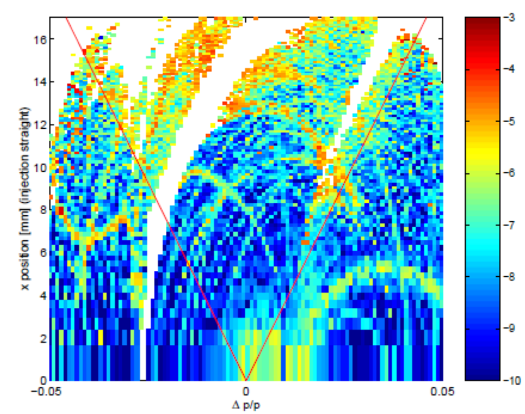
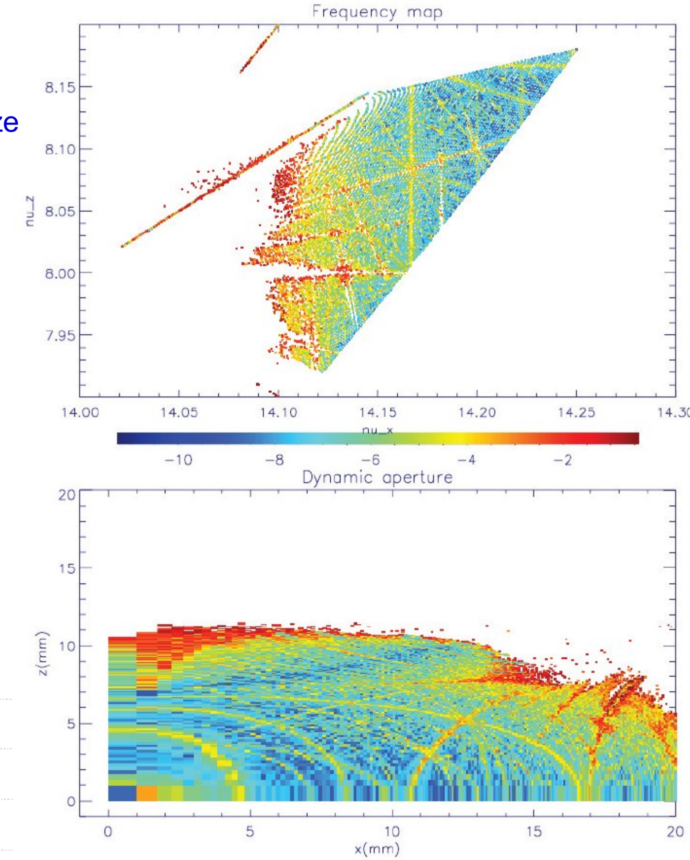


Figure 5: Simulation of a frequency map with the energy offset and the horizontal oscillation amplitudes as the variables spanning the configuration space. The color code indicates the diffusion rate on a logarithmic scale.

J.Laskar. Frequency Map Analysis and Particle Accelerators. Proc.PAC-2003.

A.Nadolski. Methods and Tools to Simulate and Analyze Non-linear Dynamics in Electron Storage rings. Proc. IPAC-2010. San-Sebastian. Spain.

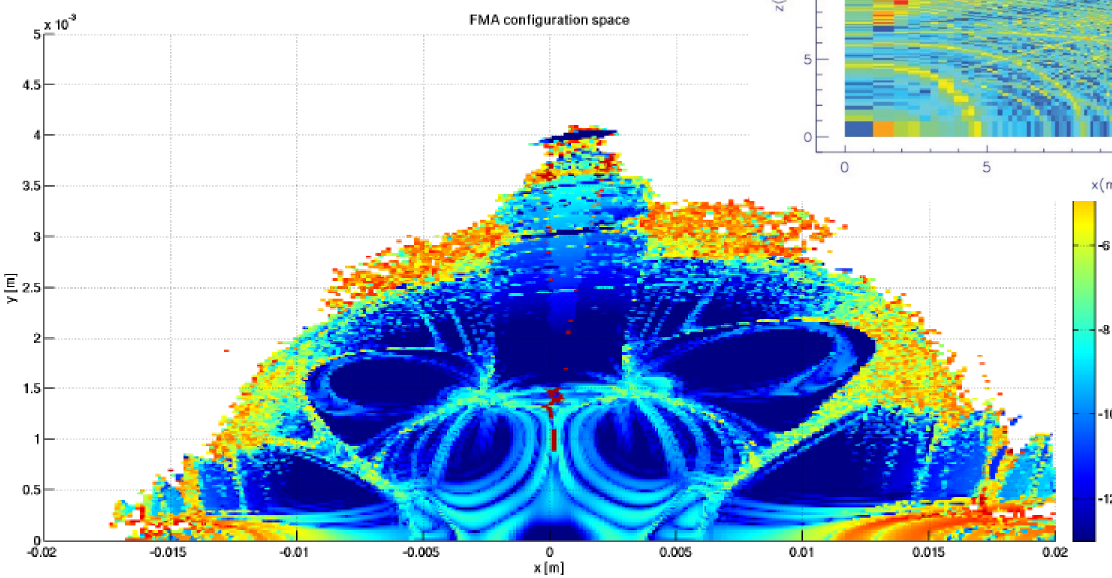
Frequerncy Analys Map of ALS



C. Steier, D. Robin, J. Laskar, L. Nadolski, "Lattice model calibration and frequency map measurements at the ALS," Proc. EPAC2000, Vienna, Austria (2000).

D. Robin, C. Steier, J. Laskar, L. Nadolski, "Global dynamics of the Advanced Light Source revealed through experimental frequency map analysis," Phys. Rev. Lett. 85, 3, pp. 558-561 (2000).

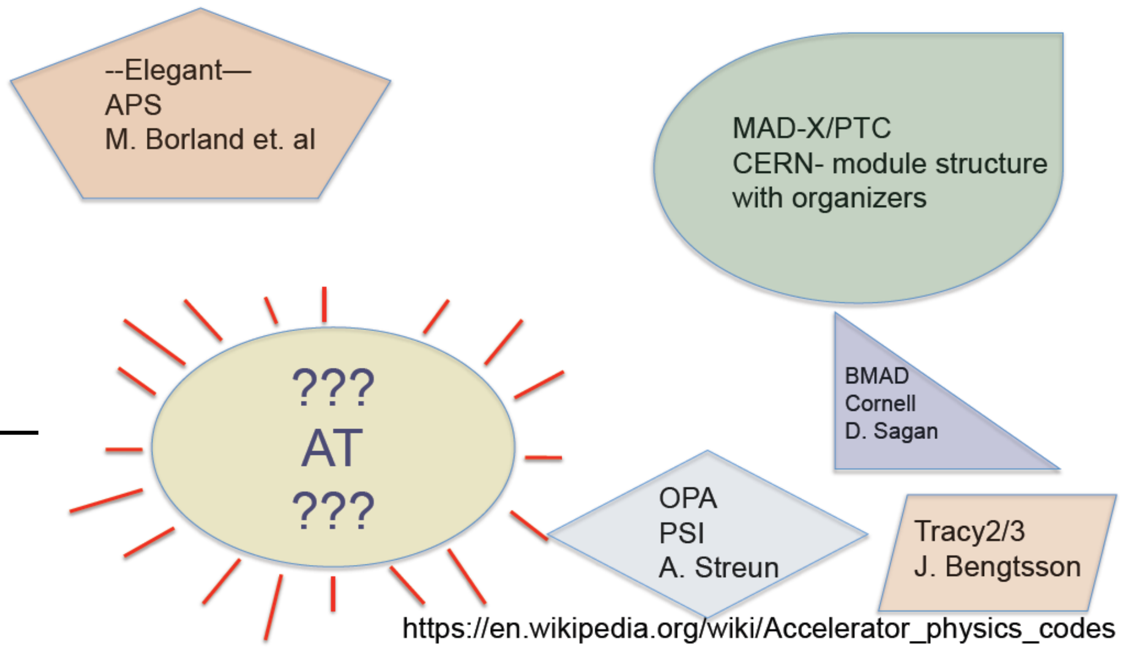
Example of Diffusion map of ESRF



DA/tune diffusion map for EBS, ESRF upgrade

Figure and text are taken from presentation of B.Nash at KIT. July 2017.
„Open problems in electron beam dynamics with applications to Synchrotron radiaton“

Widely used Codes for electron rings



https://en.wikipedia.org/wiki/Accelerator_physics_codes

Electron beam Tracking codes

- So many codes that do the same thing
- Important to know what's in the code you use, but not so useful to keep reinventing the wheel.
- Lattice conversion is a pain.
- Each lab has preferred code: challenge for those switching labs!
- My experience with Tracy and Accelerator Toolbox: can we create a viable multi-laboratory open source collaboration?

CONCLUSION

- It is possible to open on-momentum DA of Diffraction Limited Light Sources to more than ±10 mm even for ultimately low horizontal beam emittance
  - **$\varepsilon \sim 15$ pm·rad**
- Optimization of Dynamic Aperture for ON-momentum particles

might be realized by applying of *"-I"* condition for NON-interleaved pairs of sextupoles located in Dispersion sections of a ring

- MBA cell should be adapted to provide required phase advance
  - $\boldsymbol{\mu_y = (2n \pm 1)\pi}$
- MBA Cell should be designed in such a way that pairs of chromatic sextupole are located at mirror symmetry points of a cell

- Dispersion Bump at position of chromatic sextupole helps to reduce SXT strength

- Optimization of beta-function at position of Sextupole could help to open DA

- Improving of Momentum Acceptance (off-momentum DA) might be achieved by families of harmonic sextupoles located in achromatic sections of a ring

- Octupoles to compensate ADTS should be incorporated in Achromat sections of ring. Octupole strenghs must be limited