Analytic structure of scattering amplitudes

GE, Duarte, Peña, Stadler, 1907.05402 [hep-ph]

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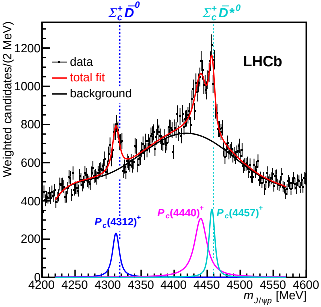
Non-perturbative QFT in Euclidean and Minkowski
Coimbra, Portugal

September 12, 2019

Motivation

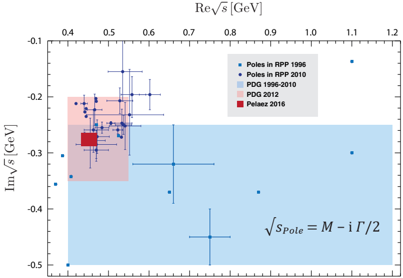
Resonances:

bumps in cross sections ⇔
poles in scattering amplitudes
(complex momentum plane)



LHCb pentaquarks

Aaij, PRL 112 (2019) 222001



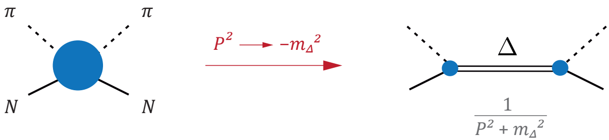
Pelaez, Phys. Rept. 658 (2016) 1

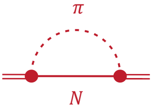
σ/f_0(500): resonance in $\pi\pi$ scattering

- **PDG 2010:** "$f_0(600)$"
  $\sqrt{s} \sim$ (400...1200) – i (250...500) MeV
- **PDG 2012:** "$f_0(500)$"
  $\sqrt{s} \sim$ (400...550) – i (200...350) MeV

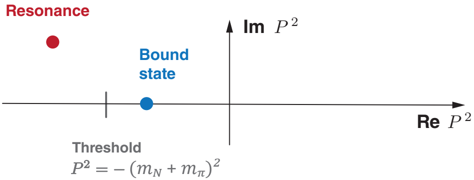
Motivation

Resonances:





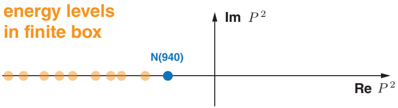
internal $N\pi$ poles produce width



Motivation

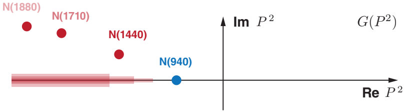
Lattice QCD:

$$\langle \dots \rangle = \int \mathcal{D}[\psi, \bar{\psi}, A] \, e^{-S[\psi, \bar{\psi}, A]} \, (\dots)$$



- **Finite volume:**
  bound states & scattering states

vary volume,
Luescher method

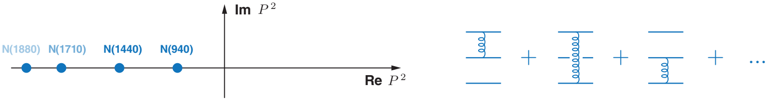


- **Infinite volume:**
  Bound states, resonances, branch cuts

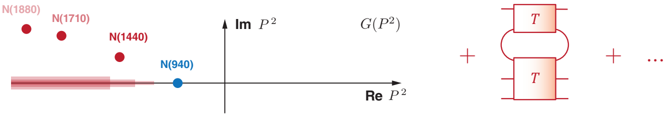
Motivation

In terms of quarks and gluons?

Bound states:



Resonances by **meson-baryon interactions:**



Motivation

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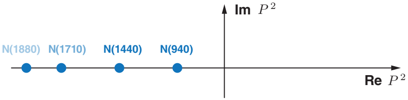
Bound states:



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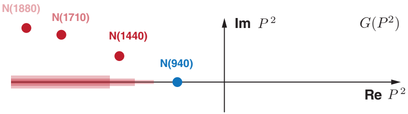
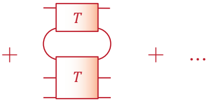
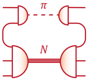




Both **bound states** and **resonances**
must be generated from quark-gluon structure!

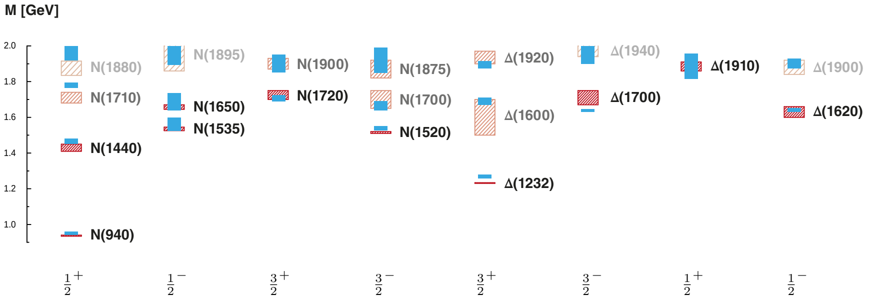


Analogue for $\rho \to \pi\pi$:
Williams, 1804.11161 [hep-ph],
Miramontes, Sanchis-Alepuz,
1906.06227 [hep-ph]

Motivation

Baryon excitation spectrum:

Review: GE, Sanchis-Alepuz, Williams, Alkofer, Fischer, PPNP 91 (2016), 1606.09602



- These are still **bound states**

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Baryon excitation spectrum:

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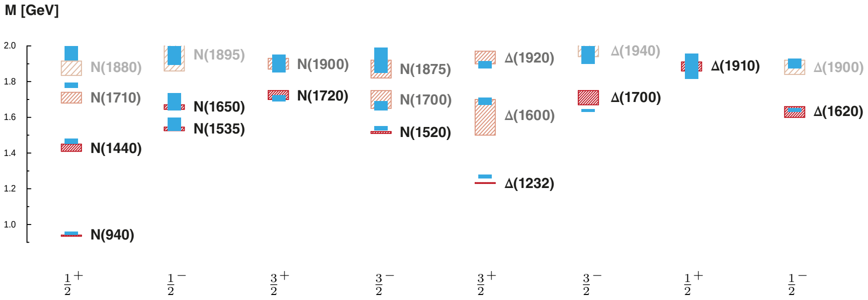
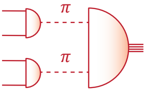


- These are still **bound states**

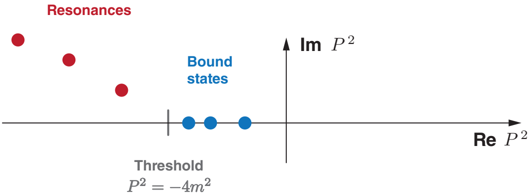
- **Tetraquarks are resonances:** internal poles emerge dynamically!

  $\sigma, \kappa, a_0/f_0$: GE, Fischer, Heupel, PLB 753 (2016)

  $X(3872)$: Wallbott, GE, Fischer, PRD 100 (2019)



Questions



- Instead of extracting resonance information from spacelike data ($P^2 > 0$), can we calculate them directly in **complex plane?**
- On the **second Riemann sheet?**
- From **four-dimensional, Lorentz-invariant** integral equations?
- Related: **Euclidean** vs. **Minkowski space** – what's the deal?

Euclidean vs. Minkowski

- *"We live in Minkowski space and not Euclidean space!"* ⚡

  Choice of **metric** cannot affect physics: $P_M^\mu = \begin{pmatrix} P_0 \\ \boldsymbol{P} \end{pmatrix} \Leftrightarrow P_E^\mu = \begin{pmatrix} \boldsymbol{P} \\ iP_0 \end{pmatrix}$

Euclidean vs. Minkowski

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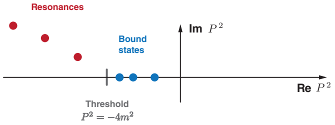
- **Spacelike** (“Euclidean”) vs. **timelike** (“Minkowski”)? ⚡



What about $P^2 \in \mathbb{C}$?

What if phase space is multi-dimensional?

Euclidean vs. Minkowski

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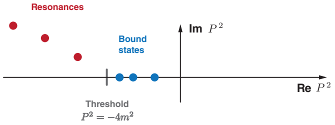


What about $P^2 \in \mathbb{C}$?

What if phase space is multi-dimensional?

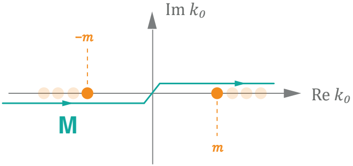
- It's about the **integration path**... but

E = M

≠ **E'** ... "naive Euclidean"

"We need XY in Minkowski space" ϟ
"We calculate XY directly in Minkowski space" ϟ
"The Euclidean calculation is wrong" ϟ

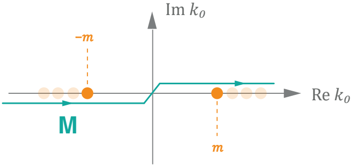
Textbook example



$$i\int d^4k\, \frac{1}{k^2-m^2+i\epsilon}\,\cdots = i\int d^3k \int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0\, \frac{1}{k_0^2-\omega^2}\,\cdots$$

- Do k_0 integration first, pick up $\boldsymbol{k}$-dependent residues, integrate over $\boldsymbol{k}$

Textbook example



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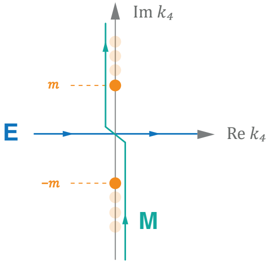
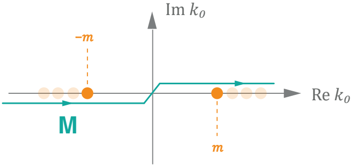


Euclidean: $k_4 = ik_0$, but $d^4k_E = -id^4k_M$

$$\int d^3k\int_{-\infty}^{\infty} dk_4\,\frac{1}{k_4^2+\omega^2}\,\cdots$$

Textbook example



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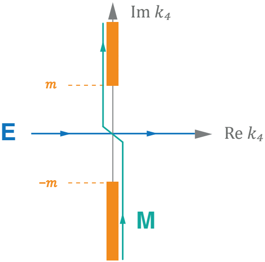

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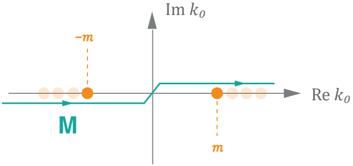
$$\int d^3k\int_{-\infty}^{\infty} dk_4\,\frac{1}{k_4^2+\omega^2}\;\cdots$$

- Now exchange $d^3k \leftrightarrow dk_4$ integration:

$$\int_{-\infty}^{\infty} dk_4\int d^3k\,\frac{1}{k_4^2+\omega^2}\;\cdots$$

has **cuts** instead of poles
→ avoid cuts in k_4 integration

Textbook example



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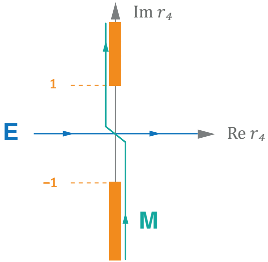


Euclidean:

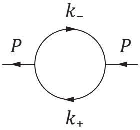
- Make everything dimensionless: $r^\mu = k_E^\mu / m$
- For manifest Lorentz invariance: k_E^2, $d\Omega$ instead of k_4, d^3k:

$$\int_{-\infty}^{\infty} dk_E^2 \int d\Omega\, \frac{1}{k_E^2+m^2}\,\cdots \qquad\qquad k_E^2 = -k^2$$

→ avoid cuts in k_E^2 integration

Two poles

Consider two-point function (current correlator, self energy, vacuum polarization, ...)



$$\int d^4k \; \frac{1}{k_+^2 + m^2} \, \frac{1}{k_-^2 + m^2}$$

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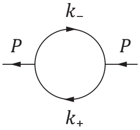

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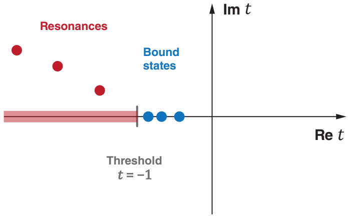
Define $P^2 = 4m^2 t$:



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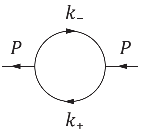

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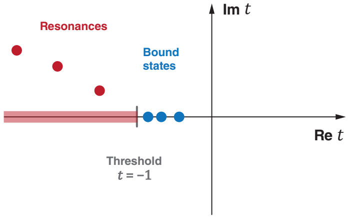
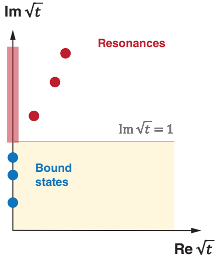


Simpler in $\sqrt{t}$:



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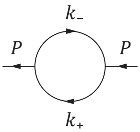

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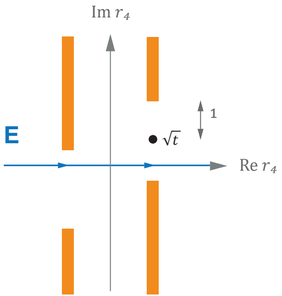
Then:



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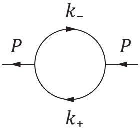

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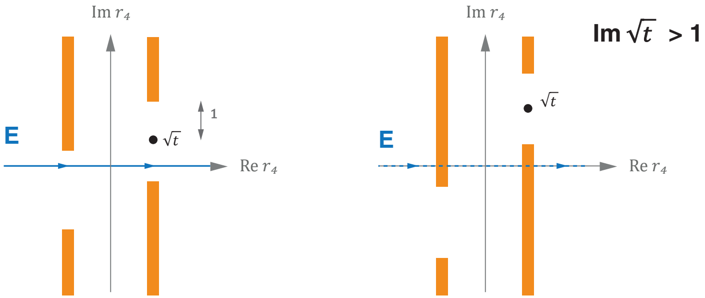

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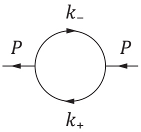

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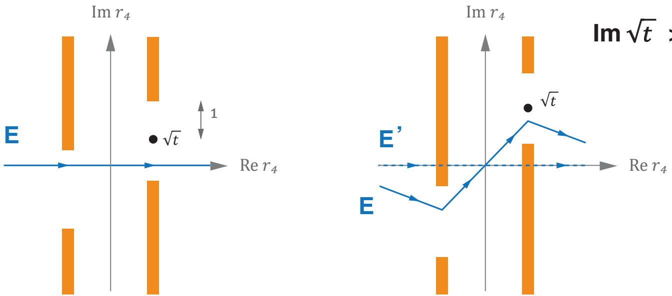

Im $\sqrt{t}$ > 1

Two poles

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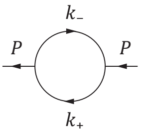

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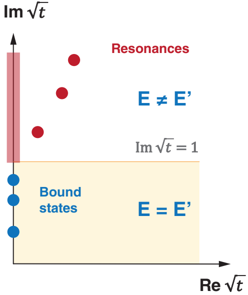

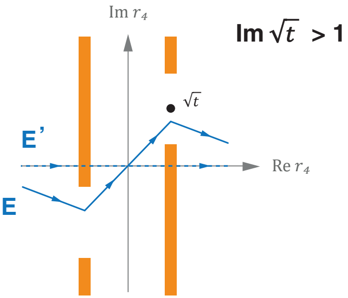

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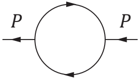

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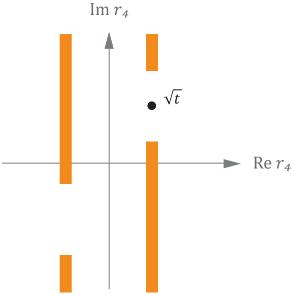

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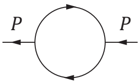

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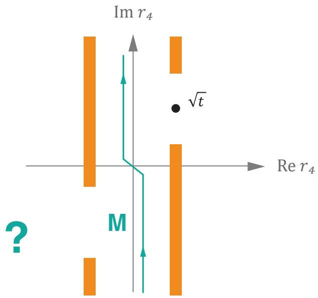

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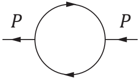
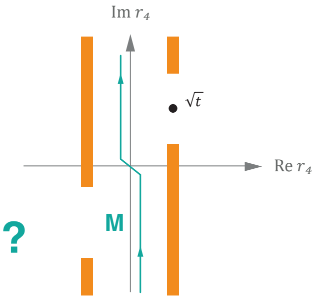


$$\int d^4k \, \frac{1}{k_+^2 + m^2} \, \frac{1}{k_-^2 + m^2}$$

Where does the *iϵ* come from?

$$\sum_{n=0}^{\infty} e^{-iE_n T} |n\rangle\langle n|\Omega\rangle \xrightarrow{T\to\infty(1-i\epsilon)} e^{-iE_0 T}|0\rangle\langle 0|\Omega\rangle$$

$$\int_{-\infty(1+i\epsilon)}^{\infty(1+i\epsilon)} dk_0 \quad \Leftrightarrow \quad \int_{-\infty(i-\epsilon)}^{\infty(i-\epsilon)} dr_4$$



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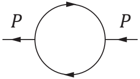

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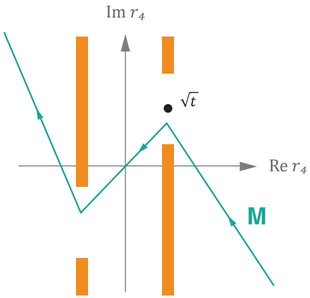
$$\int_{-\infty(i-\epsilon)}^{\infty(i-\epsilon)} dr_4$$



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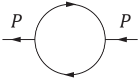
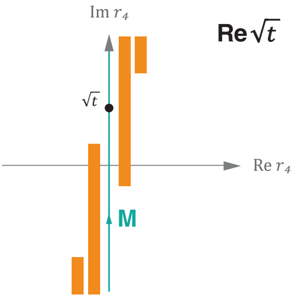


$$\int d^4k \, \frac{1}{k_+^2 + m^2} \, \frac{1}{k_-^2 + m^2}$$

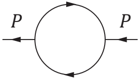
$$\int_{-\infty(i-\epsilon)}^{\infty(i-\epsilon)} dr_4 \ldots$$



Re $\sqrt{t}$ = 0

Two poles

Consider two-point function (current correlator, self energy, vacuum polarization, ...)



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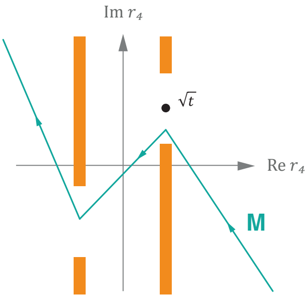

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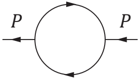

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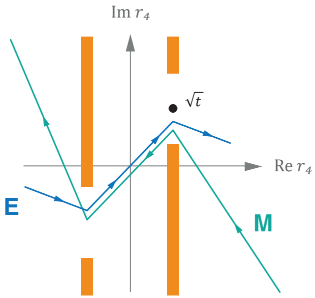
$$\int\limits_{-\infty(i-\epsilon)}^{\infty(i-\epsilon)} dr_4 \ldots$$



So:

E = M

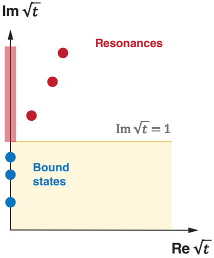
$\int d^3k \int_{-\infty}^{\infty} dk_4$ ... close contours analytically, pick up **residues**

$\int_{-\infty}^{\infty} dk_4 \int d^3k$ ... avoid cuts by numerical **contour deformation**

Suggestions for better wording:

"We need XY ~~in Minkowski space"~~
"We calculate XY ~~directly in Minkowski space"~~
... in the full kinematical domain
... above threshold
... using residue calculus

The naive Euclidean calculation ~~is~~ would be wrong
in certain kinematical regions (if anyone actually did that)



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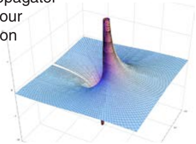
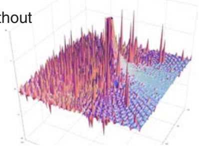
Quark propagator **with** contour deformation



... and without



Contour deformations

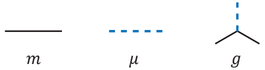
2-point functions:

- Fermion propagator in QED3
  Maris, PRD 52 (1995)
- Quark propagator in QCD
  GE, PhD thesis (2009)
- Gluon and ghost propagators in QCD
  Strauss, Fischer, Kellermann, PRL 109 (2012)
- Glueball correlator in YM
  Windisch, Alkofer, Haase, Liebmann, CPC 184 (2013),
  Windisch, Huber, Alkofer, PRD 87 (2013)
- Finite-T spectral functions from FRG
  Pawlowski, Strodthoff, Wink, PRD 98 (2018)

3-point functions:

- Rare pion decay $\pi^0 \to e^+e^-$
  Weil, GE, Fischer, Williams, PRD 96 (2017)
- Rho-meson decay
  Williams, 1804.11161
- Quark-photon vertex
  Miramontes, Sanchis-Alepuz, 1906.06227

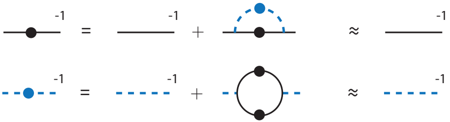
Scalar system



2 parameters:

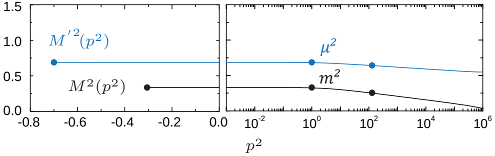
$$c = \frac{g^2}{(4\pi m)^2}\,, \qquad \beta = \frac{\mu}{m}$$

Dressed propagators do not change much:



Tree-level propagators ok – at least for small coupling

Ahlig, Alkofer, Ann. Phys. 275 (1999)

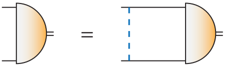


$$D(p^2) = \frac{1}{Z}\,\frac{1}{p^2 + M^2(p^2)}$$

$$D'(p^2) = \frac{1}{Z'}\,\frac{1}{p^2 + M'^2(p^2)}$$

Bound states & resonances

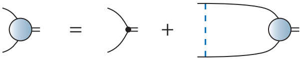
- **Homogeneous BSE:** $\psi = KG_0\,\psi$



→ **BS amplitude:** eigenvalue spectrum of KG_0 for given J^{PC} channel

Wick 1954, Cutkosky 1954, Nakanishi 1969, ...

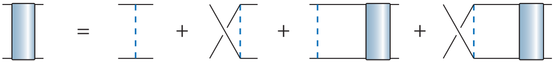
- **Inhomogeneous BSE:** $\Gamma = \Gamma_0 + KG_0\,\Gamma$



→ **Vertex:** bound-state and resonance poles for given J^{PC} channel

$$\Gamma = \frac{\Gamma_0}{1 - KG_0}$$

- **Scattering equation:** $T = K + KG_0\,T$

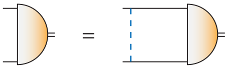


→ **Scattering amplitude,** all singularities

$$T = \frac{K}{1 - KG_0}$$

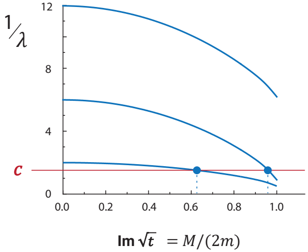
Bound states & resonances

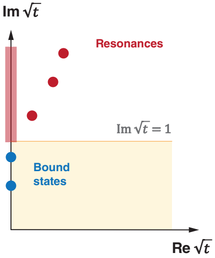
- **Homogeneous BSE:**



$$\Rightarrow \quad \psi(t) = c\, K G_0(t)\, \psi(t)$$

$$\psi(X,Z,t) = c \int dx \int dz\;\; K(X,x,Z,z,t)\;\; G_0(x,z,t)\;\; \psi(x,z,t) \qquad \Rightarrow \quad \frac{1}{\lambda(t)} \overset{!}{=} c$$





Bound states & resonances

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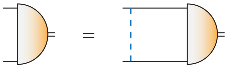

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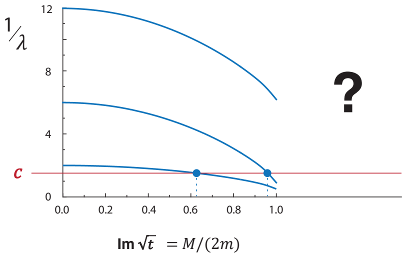
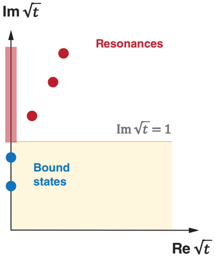
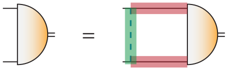




Contour deformation

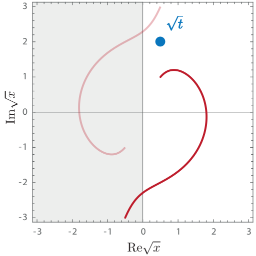
- **Homogeneous BSE:**



$$\psi(X,Z,t) = \int_0^\infty dx \int_{-1}^{1} dz \; K(X,x,Z,z) \; \underbrace{G_0(x,z,t)}_{\frac{1}{(x+t+1)^2 - 4\,x\,t\,z^2}} \; \psi(x,z,t)$$

→ cuts from G_0 in complex x plane for given t

→ cuts from K in complex x plane for given X



Contour deformation

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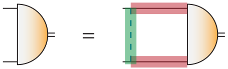

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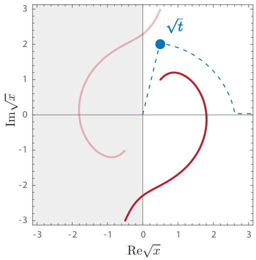
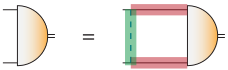
→ cuts from G_0 in complex x plane for given t
→ cuts from K in complex x plane for given X



- Find path in x that avoids G_0 cuts
- Paths in X and x must match →
  each point on path creates another cut from K

Contour deformation

- **Homogeneous BSE:**



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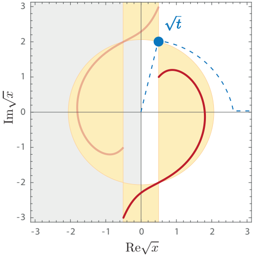
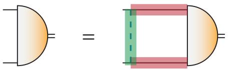
→ cuts from G_0 in complex x plane for given t
→ cuts from K in complex x plane for given X



- Find path in x that avoids G_0 cuts
- Paths in X and x must match →
  each point on path creates another cut from K
- All cuts in yellow area
- $\mathrm{Re}\sqrt{x}$ and $\mathrm{Abs}\sqrt{x}$ must never decrease
- Can cover **entire complex t plane!**

Contour deformation

- **Homogeneous BSE:**



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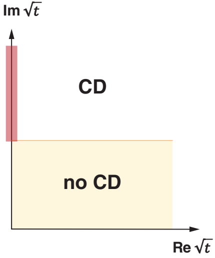
→ cuts from G_0 in complex x plane for given t

→ cuts from K in complex x plane for given X



- Find path in x that avoids G_0 cuts
- Paths in X and x must match → each point on path creates another cut from K
- All cuts in yellow area
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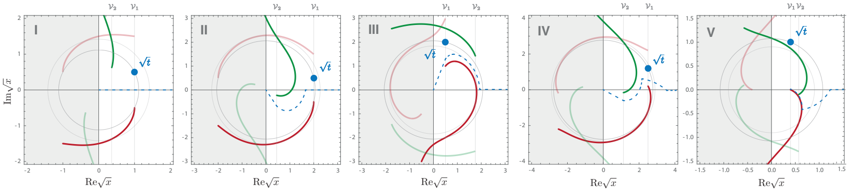
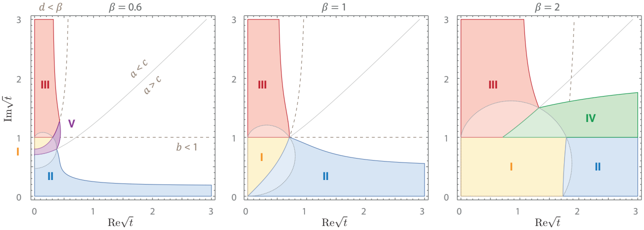
Contour deformation

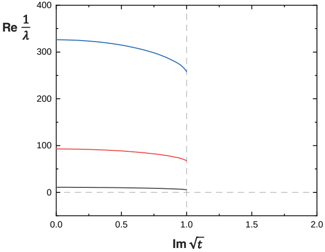
For **onshell scattering amplitude** more complicated:

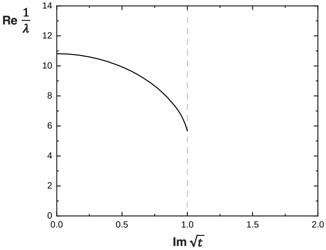


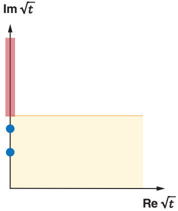
Can still cover **parts** of complex t plane:



BSE Eigenvalues







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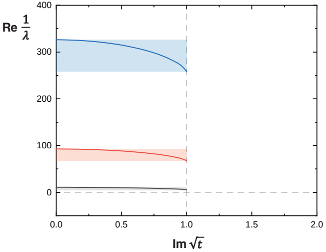
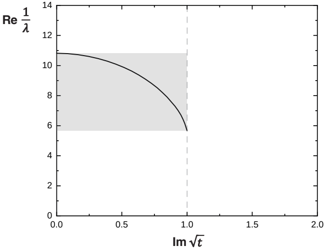
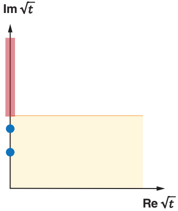
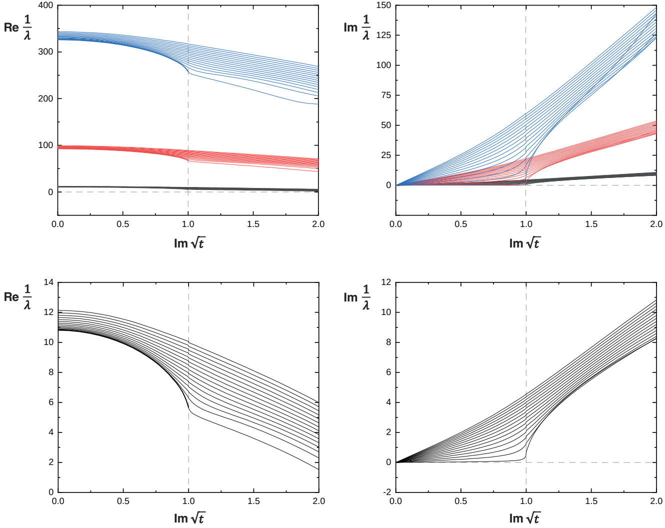
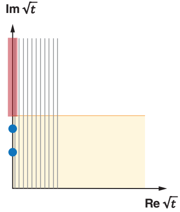






BSE Eigenvalues





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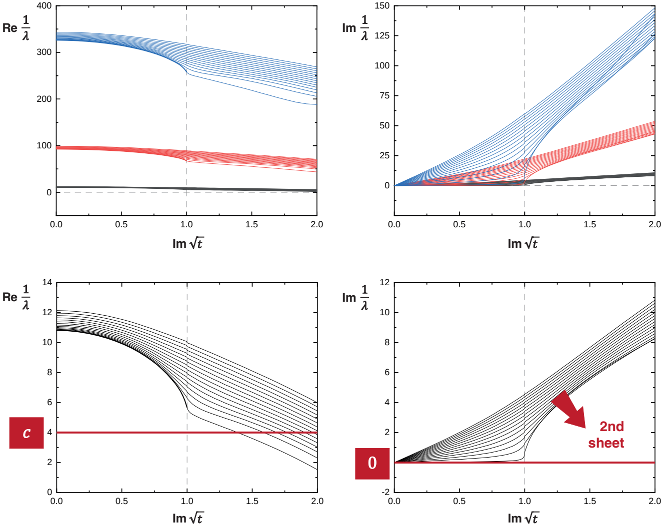
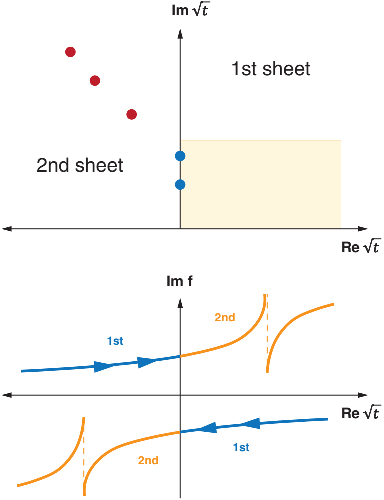


$$\frac{1}{\lambda(t)} \stackrel{!}{=} c + 0 \cdot i$$

still valid for **complex poles:**

can detect resonances from **homogeneous BSE**

How to access 2nd sheet?

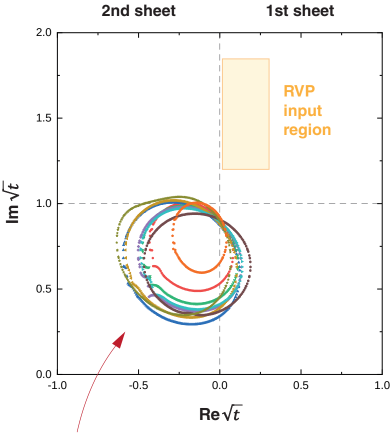


RVP: Resonances via Padé / Schlessinger point method / Continued fraction

Schlessinger, Phys. Rev. 167 (1968)
Tripolt, Haritan, Wambach, Moiseyev, PLB 774 (2017)

$$f(z) = \cfrac{c_1}{1 + \cfrac{c_2\,(z-z_1)}{1 + \cfrac{c_3\,(z-z_2)}{1 + \cfrac{c_4\,(z-z_3)}{\dots}}}}$$

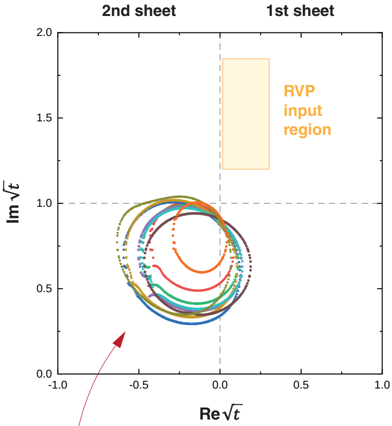
Pole trajectories



Pole trajectories:
Zeros of **Im** $1/\lambda$

- **No resonances above threshold**
- But RVP sensitive to # input points, also doesn't handle cuts well

Pole trajectories



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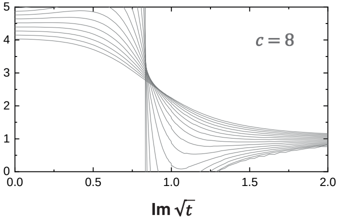

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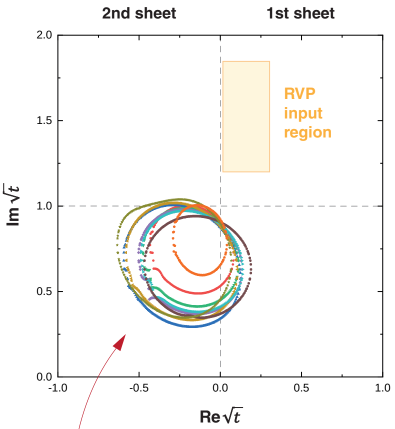

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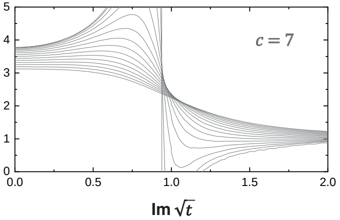

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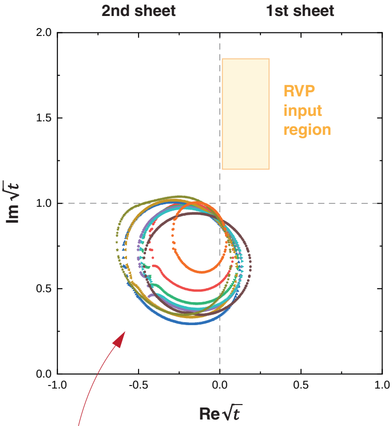

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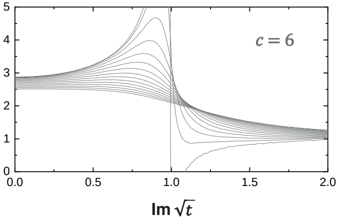
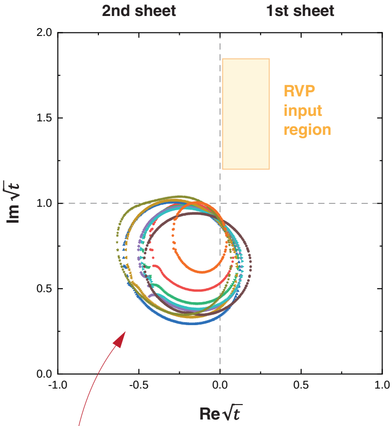


Pole trajectories



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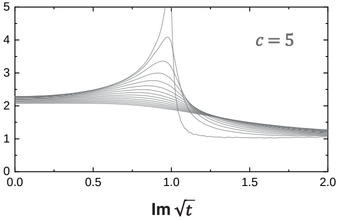

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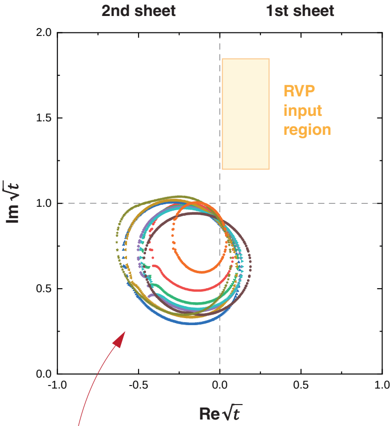

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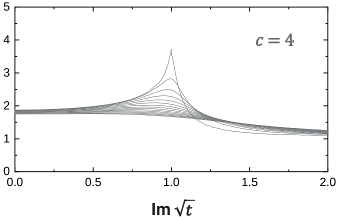

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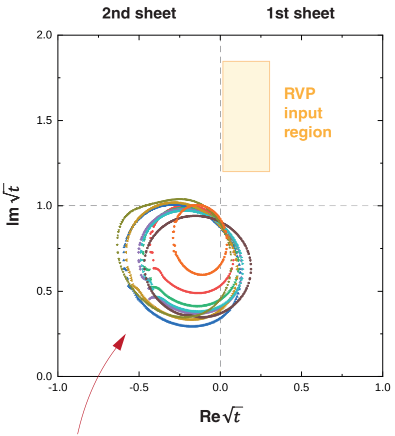

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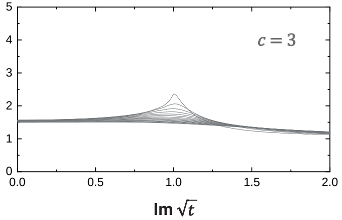
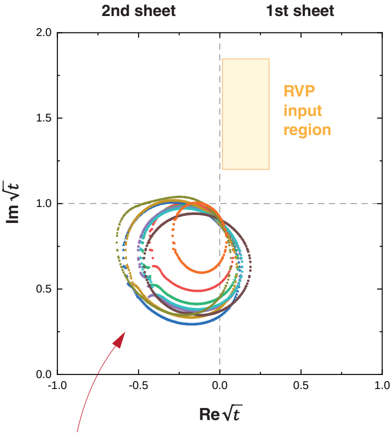
- But RVP sensitive to # input points, also doesn't handle cuts well
- **Vertex from inhomogeneous BSE:** only threshold cusp, no resonance bump



Pole trajectories



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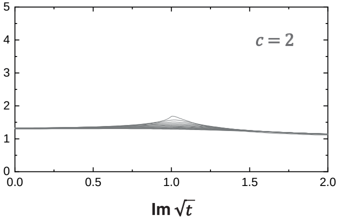
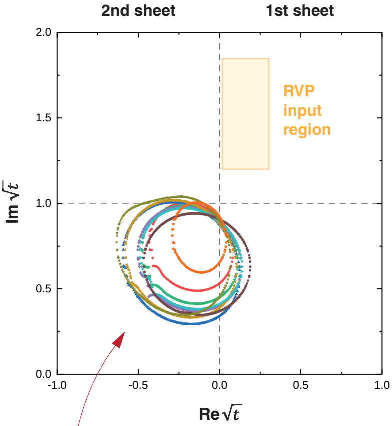


Pole trajectories



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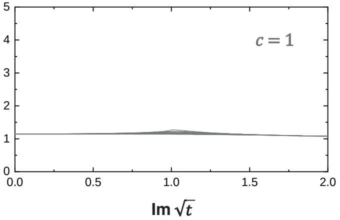
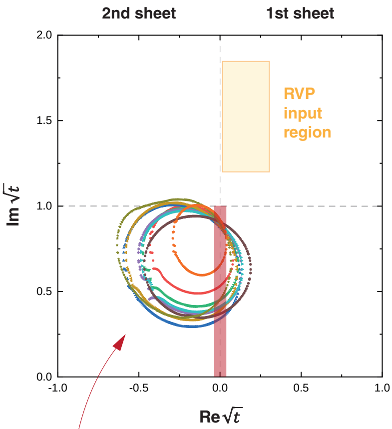


Pole trajectories



Pole trajectories:
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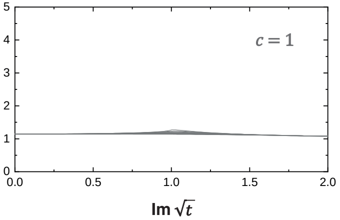
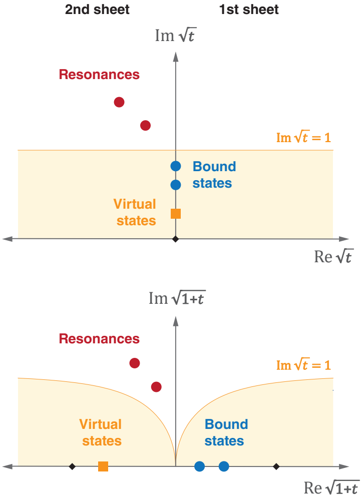


- **Virtual bound states?**
  Glöckle, "The QM Few-Body Problem", 1983
  Hanhart, Pelaez, Rios, PLB 739 (2014)

Poles on 2nd sheet

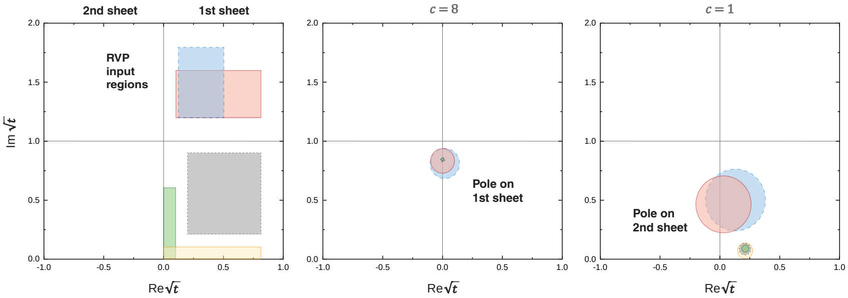


No cut in $\sqrt{1+t}$ plane

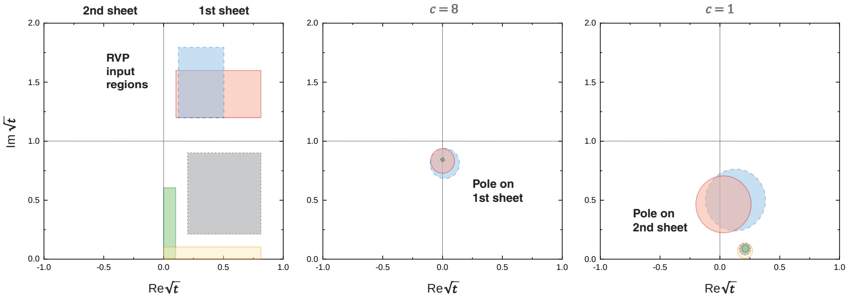
Hanhart, Pelaez, Rios, PLB 739 (2014)

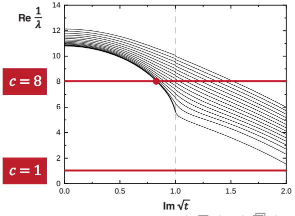
→ can analytically continue eigenvalues of homogeneous BSE!

Poles on 2nd sheet

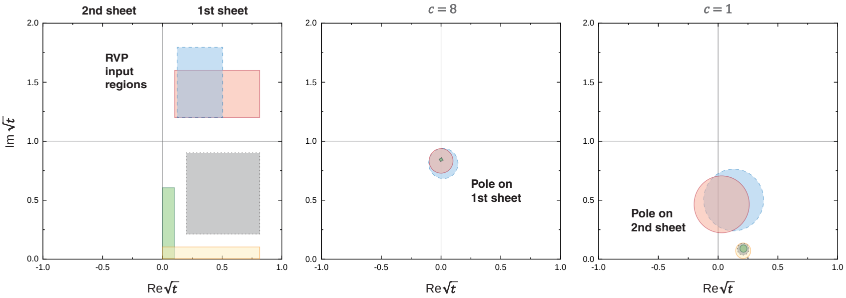


Poles on 2nd sheet

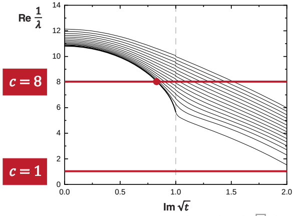




Poles on 2nd sheet



- RVP accurately reproduces **bound-state pole** on 1st sheet
- For small couplings, RVP points to **virtual states** (poles on axis of 2nd sheet)



Two-body unitarity

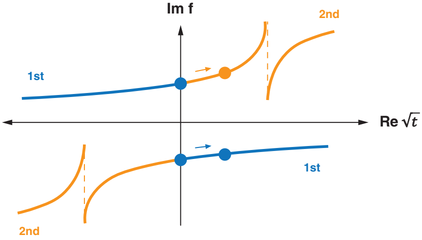
Follows from scattering equation:

$$T = K + K G_0 T \quad \Rightarrow \quad T^{-1} = K^{-1} - G_0$$

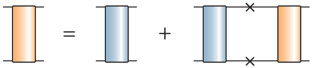
$$\Rightarrow \quad T_+^{-1} - T_-^{-1} = (K_+^{-1} - K_-^{-1}) - (G_{0+} - G_{0-})$$

$$\Rightarrow \quad T_+ - T_- = T_+ (G_{0+} - G_{0-}) T_- + (\dots)$$

e.g.: Gribov Lectures on Theoretical Physics, Cambridge 2008



If $T_\pm = T(t \pm i\epsilon)$:



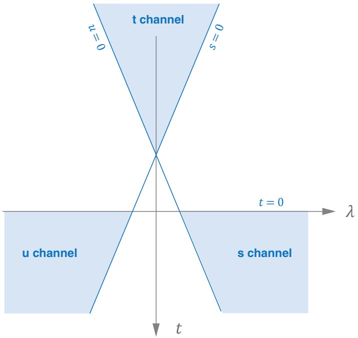
With **partial-wave decomposition:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

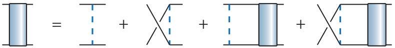
→ But this requires **scattering amplitude**

Scattering amplitude

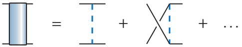
Depends on two variables: t and crossing variable $\lambda = \dfrac{s-u}{4m^2}$



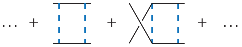
- Bound states, resonances and t-channel cuts at fixed $t \rightarrow$ determined by scattering equation



- Exchange-particle poles from K at fixed $s=\mu^2$ and $u=\mu^2$ (no poles in $T-K$)

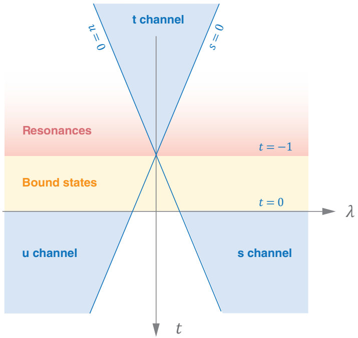


- Perturbative cuts for $s > 4\mu^2$ and $u > 4\mu^2$



Scattering amplitude

Depends on two variables: t and crossing variable $\lambda = \frac{s-u}{4m^2}$



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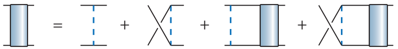

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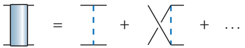

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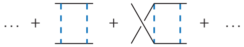
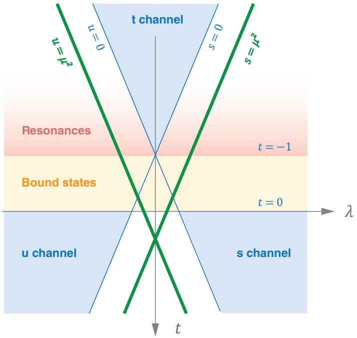


Scattering amplitude

Depends on two variables: t and crossing variable $\lambda = \dfrac{s-u}{4m^2}$



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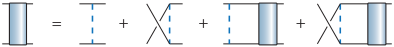

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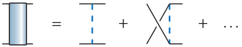

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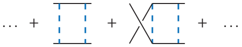
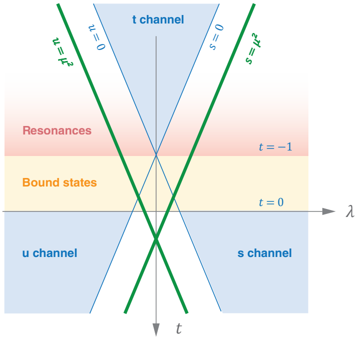
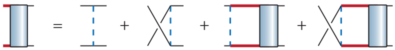


Scattering amplitude

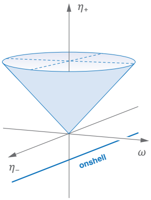
Depends on two variables: t and crossing variable $\lambda = \frac{s-u}{4m^2}$



- To obtain onshell scattering amplitude, must first solve **half-offshell** scattering equation



- Kinematics same as in **Compton scattering**
  GE, Ramalho, PRD 98 (2018)



Scattering amplitude

Partial-wave expansion:

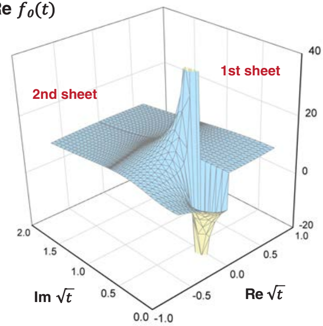
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

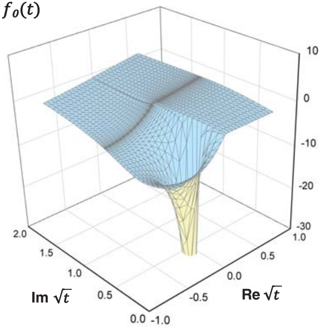
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 1$

Re $f_0(t)$



Im $f_0(t)$



Scattering amplitude

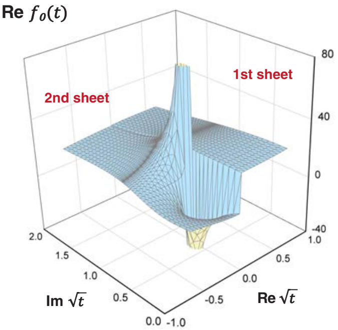
Partial-wave expansion:

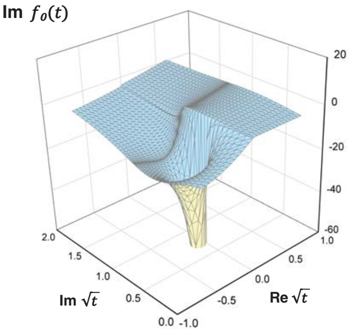
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 2$





Scattering amplitude

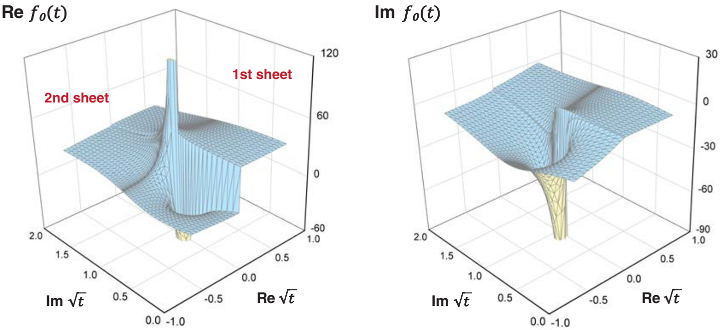
Partial-wave expansion:

$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 3$



Scattering amplitude

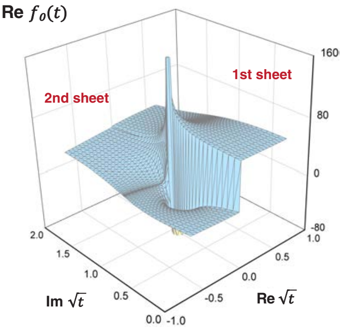
Partial-wave expansion:

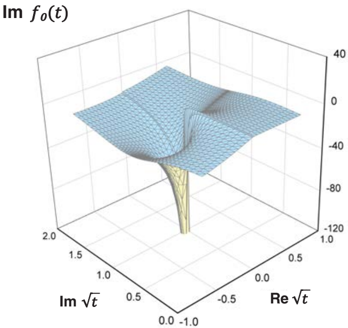
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 4$





Scattering amplitude

Partial-wave expansion:

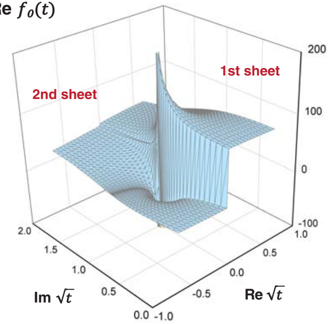
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

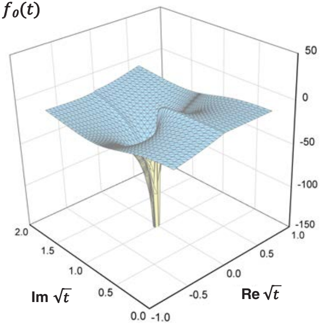
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 5$

Re $f_0(t)$



Im $f_0(t)$



Scattering amplitude

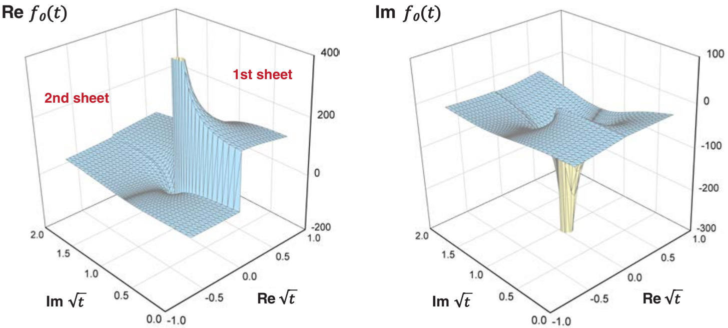
Partial-wave expansion:

$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 7$



Scattering amplitude

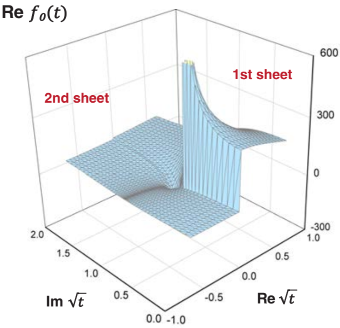
Partial-wave expansion:

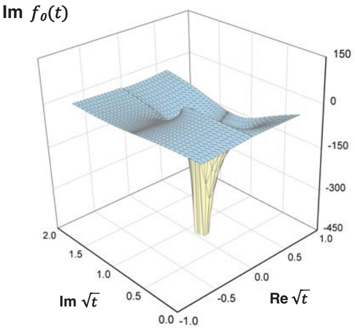
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 8$





Scattering amplitude

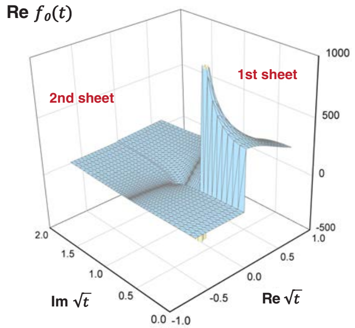
Partial-wave expansion:

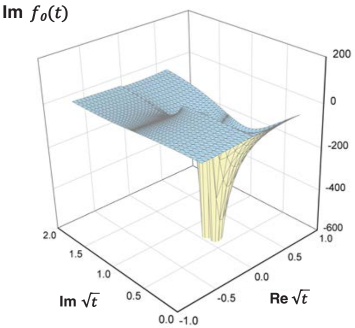
$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

Amplitude on **2nd sheet:**

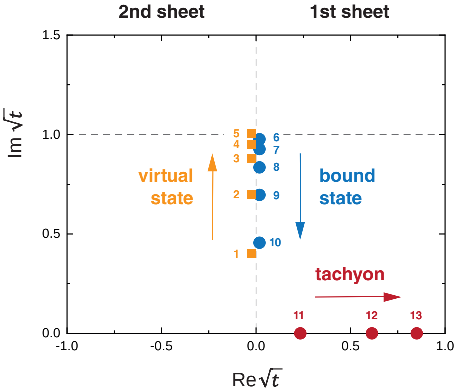
$$f_l(t)_{II} = \frac{f_l(t)_I}{1 - 2i\,\tau(t)\, f_l(t)_I}$$

$c = 9$





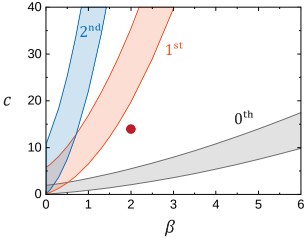
Pole trajectories



- Scalar model doesn't have resonances but only **virtual bound states**
- Need full **scattering equation** to find them (2-body unitarity)
- For **nearby resonances,** (in-)homogeneous BSE + RVP probably sufficient

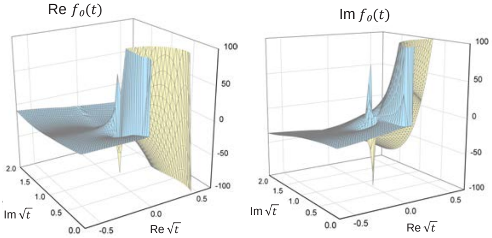
Pole trajectories

Analogous for other values of β
(i.e., exchange-particle masses)



Inside each band a state is bound

At fixed β, when increasing coupling:
virtual states → bound states → tachyons



Here for $\beta = 2,\ c = 12$:

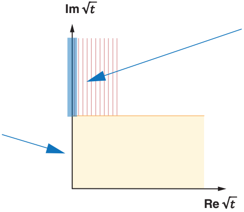
- Ground state has become tachyonic, 1st excited state is not yet bound
- Large structure is exchange particle pole at fixed s (or u), cf. Mandelstam plane

Benchmarks

- **Binding energies**

$c = 1, \beta = 0.5$

$\mathrm{Im}\sqrt{t}$	π/λ_0 this work	π/λ_0 [1, 2]	π/λ_0 [3]
0.999	1.18(3)	1.211	1.216
0.995	1.43(1)	1.440	1.440
0.99	1.623	1.624	1.623
0.95	2.498	2.498	2.498
0.90	3.251	3.251	3.251
0.80	4.416	4.416	4.416
0.75	4.901	4.901	4.901
0.6	6.094	6.096	6.094
0.4	7.205	7.206	7.204
0.2	7.849	7.850	7.849
0	8.061	8.062	8.061

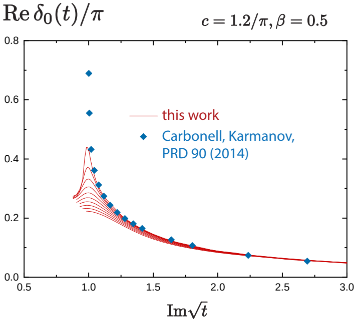


[1, 2] Kusaka, Simpson, Williams, PRD 56 (1997)
Karmanov, Carbonell, EPJ A 28 (2006)

[3] Frederico, Salmè, Viviani, PRD 89 (2014)

- **Phase shifts**

$$f_l(t) = \frac{1}{2i\,\tau(t)}\left[e^{2i\,\delta_l(t)} - 1\right]$$



Summary & Outlook

- **Contour deformations:**
  Toolbox for treating resonances with integral equations

- Can be taken over without changes to NN, $N\pi$ scattering, etc. → amplitude analyses

- QCD with functional methods: must still implement **resonance mechanism,**
  tetraquarks are good starting point (it's automatic)

- **Homogeneous BSE** is good enough to extract **pole positions on 2nd sheet**
  (nearby resonances, otherwise at least ballpark estimates)

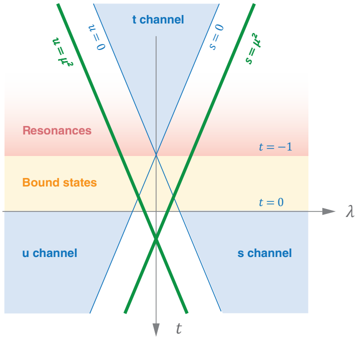
- Generally applicable for **circumventing singularities** (e.g. from quark propagator)
  → highly excited states, timelike FFs, FFs at large Q^2, PDFs, GPDs, TMDs, . . .

- Scalar system: template for resonances in **Higgs sector.**
  Model considered here doesn't have resonances but **virtual bound states**

Backup slides

Scattering amplitude

Depends on two variables: t and crossing variable $\lambda = \dfrac{s-u}{4m^2}$



- λ dependence "boring": exchange poles and cuts, $T-K$ almost flat (cuts only)
- Also $T(\lambda)$ is flat if exchange poles far away ⇒ **partial-wave expansion** converges rapidly

$$T(t,\lambda) = \sum_{l=0}^{\infty} (2l+1)\, f_l(t)\, P_l(\cos\Theta) \approx f_0(t)$$

- Extract **phase shifts**, make Argand plots, etc.

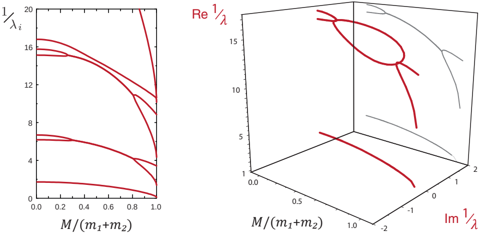
$$f_l(t) = \frac{1}{2i\,\tau(t)}\left[e^{2i\,\delta_l(t)} - 1\right]$$

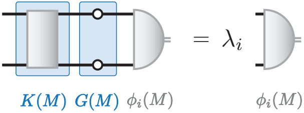
Complex eigenvalues?

Excited states: some EVs are complex conjugate?

Typical for **unequal-mass** systems, already in Wick-Cutkosky model
Wick 1954, Cutkosky 1954

Connection with "**anomalous**" **states?**
Ahlig, Alkofer, Ann. Phys. 275 (1999)





K and G are Hermitian (even for unequal masses!) but KG is not

If $G = G^\dagger$ and $G > 0$:
Cholesky decomposition $G = L^\dagger L$

$$K\, L^\dagger L\, \phi_i = \lambda_i\, \phi_i$$
$$(LKL^\dagger)\,(L\phi_i) = \lambda_i\,(L\phi_i)$$

⇒ Hermitian problem with same EVs!

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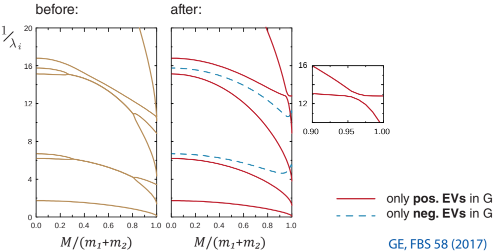
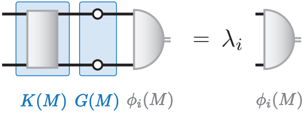


GE, FBS 58 (2017)



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⇒ Hermitian problem with same EVs!

⇒ all EVs strictly **real**
⇒ level repulsion
⇒ "anomalous states" removed?

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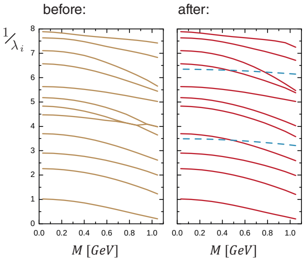
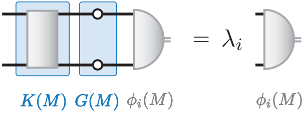


Eigenvalue spectrum for pion channel
GE, FBS 58 (2017)

only **pos. EVs** in G
only **neg. EVs** in G



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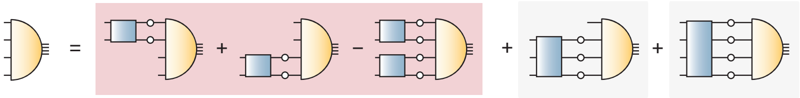
⇒ "anomalous states" removed?

Four-body equation



Two-body interactions
... plus permutations:

$(qq)(\bar{q}\bar{q}), \;(q\bar{q})(q\bar{q}), \;(q\bar{q})(q\bar{q})$

(12) (34) (23) (14) (13) (24)

3-body
(+ permutations)

4-body

Bethe-Salpeter amplitude:

$$\Gamma(p,q,k,P) = \sum_i f_i\left(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}\right) \tau_i(p,q,k,P) \quad \otimes \quad \text{Color} \quad \otimes \quad \text{Flavor}$$

9 Lorentz invariants:

$$p^2, \quad q^2, \quad k^2$$

$$\omega_1 = q\cdot k \qquad \eta_1 = p\cdot P$$
$$\omega_2 = p\cdot k \qquad \eta_2 = q\cdot P$$
$$\omega_3 = p\cdot q \qquad \eta_3 = k\cdot P$$

$$P^2 = -M^2$$

256 Dirac-Lorentz tensors

2 Color tensors:

$3\otimes\bar{3}, \quad 6\otimes\bar{6}$ or
$1\otimes 1, \quad 8\otimes 8$
(Fierz-equivalent)

Structure of the amplitude

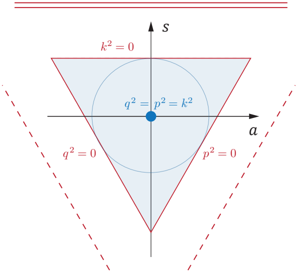
Lorentz invariants can be grouped into
multiplets of the permutation group S4:
GE, Fischer, Heupel, PRD 92 (2015)

- **Singlet:** symmetric variable, carries overall scale:

  $\mathcal{S}_0 = \frac{1}{4}(p^2 + q^2 + k^2)$

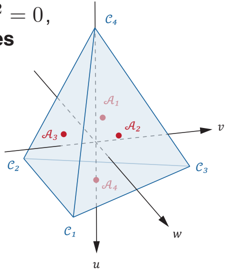
- **Doublet:** $\mathcal{D}_0 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix} \sqrt{3}(q^2 - p^2) \\ p^2 + q^2 - 2k^2 \end{bmatrix}$

  Mandelstam triangle,
  outside: **meson and diquark poles!**



- **Triplet:** $\mathcal{T}_0 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix} 2(\omega_1 + \omega_2 + \omega_3) \\ \sqrt{2}(\omega_1 + \omega_2 - 2\omega_3) \\ \sqrt{6}(\omega_2 - \omega_1) \end{bmatrix}$

  tetrahedron bounded by $p_i^2 = 0$,
  outside: **quark singularities**



- Second **triplet:**
  3dim. sphere

  $\mathcal{T}_1 = \frac{1}{4\mathcal{S}_0}\begin{bmatrix} 2(\eta_1 + \eta_2 + \eta_3) \\ \sqrt{2}(\eta_1 + \eta_2 - 2\eta_3) \\ \sqrt{6}(\eta_2 - \eta_1) \end{bmatrix}$

Tetraquark mass