# Continuum Limit of Gluon and Ghost Propagators in Minimal Landau Gauge

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# Color Confinement

Millennium Prize Problems by the Clay Mathematics Institute (US\$1,000,000): Yang-Mills Existence and Mass Gap: Prove that, for any compact simple gauge group $G$, a non-trivial quantum Yang-Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.

Lattice simulations can solve QCD exactly (in discretized space-time), allowing quantitative predictions for the physics of hadrons. But they can also help reveal the principles behind a central phenomenon of QCD: color confinement. In fact, we can try to understand the QCD vacuum (the "battle for nonperturbative QCD"*) by using inputs from lattice simulations and numerically testing approximations introduced in analytic approaches (Dyson-Schwinger equations, Bethe-Salpeter equations, Pomeron dynamics, QCD-inspired models, etc).

* *The QCD vacuum, hadrons and the superdense matter*, Edward V. Shuryak

# Possible Pathways to Confinement

- Green's functions carry all information of a QFT's physical and mathematical structure.
- Gluon propagator (two-point function) as the most basic quantity of QCD.
- Confinement given by behavior at large distances (small momenta) $\Rightarrow$ nonperturbative study of IR gluon propagator. Proposal by Mandelstam (1979) linking linear potential to infrared behavior of gluon propagator as $1/p^4$.
- Gribov-Zwanziger confinement scenario based on suppressed gluon propagator and enhanced ghost propagator in the infrared.

# Gauge-Related Lattice Features

- Gauge action written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements.
- Under gauge transformations $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^\dagger(x+\mu)$, where $g \in SU(N_c)$ $\Rightarrow$ closed loops are gauge-invariant.
- Integration volume is finite: no need for gauge-fixing.
- When gauge fixing, procedure is incorporated in the simulation, no need to consider Faddeev-Popov matrix.
- Get FP matrix without considering ghost fields explicitly.
- Lattice momenta given by $\hat{p}_\mu = 2\,\sin\left(\pi\, n_\mu / N\right)$ with $n_\mu = 0, 1, \ldots, N/2$ $\Leftrightarrow$ $p_{min} \sim 2\pi/(a\,N) = 2\pi/L$, $p_{max} = 4/a$ in physical units.

# Lattice Landau Gauge

In the continuum: $\partial_\mu A_\mu(x) = 0$. On the lattice the (minimal) Landau gauge is imposed by minimizing the functional

$$S[U; g] = -\sum_{x,\mu} Tr\, U_\mu^g(x) \,,$$

where $g(x) \in SU(N)$ and $U_\mu^g(x) = g(x)\, U_\mu(x)\, g^\dagger(x + ae_\mu)$ is the lattice gauge transformation. By considering the relations $U_\mu(x) = e^{iag_0 A_\mu(x)}$ and $g(x) = e^{i\tau\theta(x)}$, we can expand $S[U; g]$ (for small $\tau$):

$$S[U; g] = S[U; 1\!\!1] + \tau\, S^{'}[U; 1\!\!1](b, x)\, \theta^b(x)$$

$$+ \frac{\tau^2}{2}\, \theta^b(x)\, S^{''}[U; 1\!\!1](b, x; c, y)\, \theta^c(y) + \ldots$$

where $S^{''}[U; 1\!\!1](b, x; c, y) = \mathcal{M}(b, x; c, y)[A]$ is a lattice discretization of the Faddeev-Popov operator $-D \cdot \partial$ with $A_\mu(x) = \left[U_\mu(x) - U_\mu^\dagger(x)\right]_{\text{traceless}} / (2i)$.

# Constraining the Functional Integral

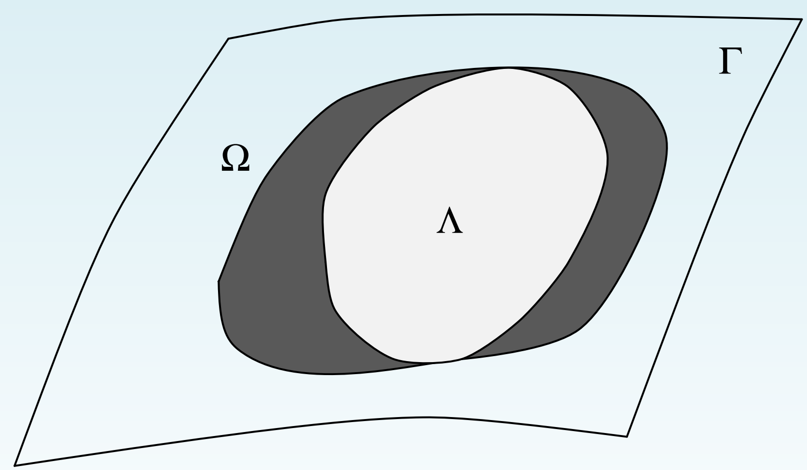
At a stationary point $S^{'}[U; 1\!\!1](b, x) = 0$, one obtains

$$\sum_{\mu} A_{\mu}^{b}(x) - A_{\mu}^{b}(x - a\, e_{\mu}) = 0 ,$$

which is a discretized version of the (continuum) Landau gauge condition. At a local minimum one also has $\mathcal{M}(b, x; c, y)[A] \geq 0$ . This defines the first Gribov region (V.N. Gribov, 1978)

$$\Omega \equiv \{ U : \partial \cdot A = 0, \mathcal{M} \geq 0 \} \equiv \text{ all local minima of } S[U; \omega] .$$

All gauge orbits intersect $\Omega$ (G. Dell'Antonio & D. Zwanziger, 1991) but the gauge fixing is not unique (Gribov copies).
Absolute minima of $S[U; \omega]$ define the fundamental modular region $\Lambda$, free of Gribov copies in its interior. (Finding the absolute minimum is a spin-glass problem.)



# Numerical Simulations

When we are interested in gauge-dependent quantities we consider the following steps:

1. Choose an initial configuration $\mathcal{C}_0 = U_\mu(x) \in$ SU($N_c$)
2. Thermalize the initial configuration (heat-bath, etc.) $\mathcal{C}_0 \to \mathcal{C}_1$
3. Fix the gauge for the configuration $\mathcal{C}_i$ with $i = 1, 2, \ldots$
4. Evaluate (gauge-dependent) quantities using the configuration $\mathcal{C}_i$
5. Produce a new (independent) configuration $\mathcal{C}_i \to \mathcal{C}_{i+1}$
6. Go back to step 3

We do not need to simulate anti-commuting variables or to evaluate the determinant of the Faddeev-Popov matrix!

# Gluon and Ghost Propagators

As a consequence of the restriction of the measure to the region $\Omega$:

- In minimal Landau gauge the gluon propagator

$$D^{ab}_{\mu\nu}(p) \,=\, \sum_x e^{-2i\pi k\cdot x} \langle A^a_\mu(x)\, A^b_\nu(0)\rangle \,=\, \delta^{ab}\, \left( g_{\mu\nu} \,-\, \frac{p_\mu\, p_\nu}{p^2} \right) D(p^2)$$

  is suppressed in the IR limit, i.e. $D(0)$ is finite (and nonzero) and reflection positivity is violated. This result may be viewed as an indication of gluon confinement (the propagator presents poles with complex-conjugate masses).

- Infinite volume favors configurations on the first Gribov horizon, where $\lambda_{min}$ of $\mathcal{M}$ goes to zero. In turn, the ghost propagator

$$G(p) \,=\, \frac{1}{N_c^2 - 1} \sum_{x,\,y,\,a} \frac{e^{-2\pi i\, k\cdot(x-y)}}{V} \,\langle\, \mathcal{M}^{-1}(a, x; a, y)\,\rangle\,,$$

  is IR enhanced at intermediate momenta, but it is free-like in the IR limit.

# Fits of the Propagators (I)

In Phys. Rev. D85 (A.C. et al., 2012) we have shown that the 4d SU(2) gluon propagator $D(p^2)$ can be well fitted using the function (Gribov-Stingl propagator, RGZ propagator)

$$f_1(p^2) \,=\, C\,\frac{p^2+s}{p^4+u^2\,p^2+t^2}\;,$$

implying complex-conjugate poles

$$f_2(p^2) \,=\, \frac{\alpha_+}{p^2+\omega_+^2}\,+\,\frac{\alpha_-}{p^2+\omega_-^2}\;,$$

with $\alpha_\pm = a \pm ib$ and $\omega_\pm^2 = v \pm iw$. Similar results in Annals Phys. 397 (D.Dudal et al., 2018) for 4d SU(3).

In Phys. Rev. D93 (A.C. et al., 2016) we have shown that the 4d SU(2) ghost propagator $G(p^2)$ can be well fitted using the function

$$F_3(p^2) \,=\, \frac{z}{p^2}\,\frac{t+p^2/s^2+\log\left(1+p^2/s^2\right)}{1+p^2/s^2}\;,$$

which has $1/p^2$ leading IR and UV behaviors.

# Fits of the Propagators (II)

For the ghost propagator $G(p^2)$ we have also considered the 1-loop expression, evaluated in Phys. Rev. D85 (A.C. et al., 2012) using the $f_2(p^2)$ fit of the gluon propagator $D(p^2)$, i.e.

$$G(p^2) \; = \; \frac{1}{p^2} \, \frac{1}{1 - \sigma(p^2)}$$

with

$$\sigma(p^2) \; = \; \frac{g^2 N_c}{32\pi^2 R^2} \left[ -p^2 t_1(p^2) \, + \, R^2 t_2(p^2) \, + \, p^{-2} t_3(p^2) \, - \, p^{-4} t_4(p^2) \right] .$$

Here, $R = \sqrt{v^2 + w^2}$ and $t_1(p^2), t_2(p^2), t_3(p^2)$ and $t_4(p^2)$ are written in terms of $p^2$ and the pole parameters $a, b, v, w$ (and an arbitrary momentum scale $\mu$).

In this case the only fitting parameter is $g^2$.

Note that, in the limit $p \to 0$ one finds $\sigma(p^2)/g^2 \; \to \; c_1 + \left[ -c_2 + c_3 \, \log(p^2/R) \right] \, p^2$, with $c_1, c_2, c_3 > 0$. Thus, for a critical value $g_c^2$ one can obtain $\sigma(0) = 1$, yielding a ghost propagator with a $1/p^4$ singularity in the IR limit (JHEP 0806, Ph.Boucaud et al., 2008).

# Previous Simulations

In Phys. Rev. D85 and Phys. Rev. D93 (A.C. et al., 2012 and 2016) we considered simulations with lattice sides $N = 48, 56, 64, 80, 96$ and $128$ at $\beta = 2.2$. In this case the lattice spacing is approximately 0.210 fermi, so that the smallest nonzero momentum is about 46 MeV and the largest physical lattice volume $V = N^4$ is about (27 fermi)$^4$.

We want to check if the results obtained are confirmed when considering smaller lattice spacings (continuum limit).

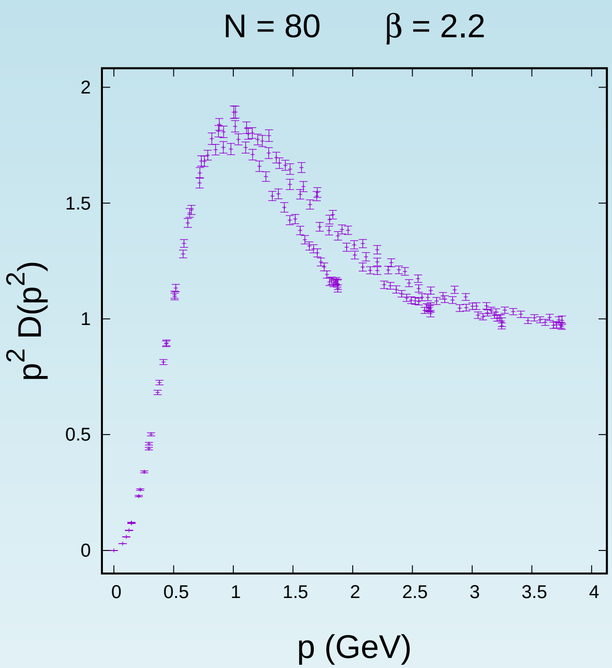
Let us recall that in Phys. Rev. D90 (A.C. et al., 2012), when studying the so-called Bose-ghost propagator (related to the issue of BRST symmetry breaking in minimal Landau gauge) we have found a change in the pole structure when the lattice spacing is decreased.

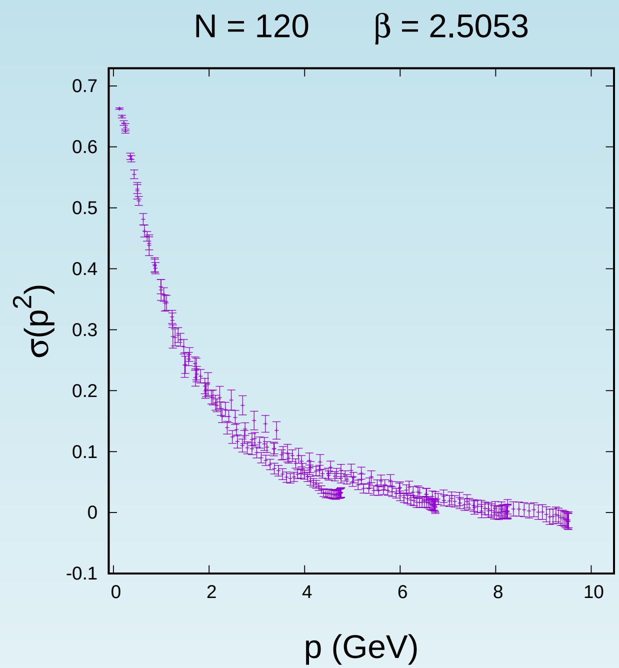
# New Simulations

| $V = N^4$ | $\beta$ | $L\,(fermi)$ | $p_{min}$ (MeV) | # conf | $\mu$ (GeV) |
|---|---|---|---|---|---|
| $48^4$ | $2.2$ | 10.097 | 122.71 | 500 | 3.0 |
| $72^4$ | $2.3494$ | 10.097 | 122.76 | 250 | 5.0 |
| $96^4$ | $2.4367$ | 10.097 | 122.77 | 100 | 7.0 |
| $120^4$ | $2.5053$ | 10.097 | 122.78 | 100 | 9.0 |
| $80^4$ | $2.2$ | 16.828 | 73.66 | 600 | 3.5 |
| $128^4$ | $2.3688$ | 16.832 | 73.65 | 496 | 5.5 |
| $160^4$ | $2.4366$ | 16.833 | 73.65 | 400 | 7.0 |
| $192^4$ | $2.4927$ | 16.827 | 73.68 | 292/295 | 8.5 |

Eight sets of parameters $(N, \beta)$ for two constant physical lattice sizes $L = Na = 10.097$ and $16.83$ (runs done with the `Blue Gene/P` and `Blue Gene/Q` supercomputers at Rice University). I will show a preliminary analysis for these data.

# Breaking of Rotational Invariance (I)




Gluon dressing function $p^2 D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using unimproved momenta $p^2 = 4 \sum_\mu \sin^2(\pi\, n_\mu / N)$.

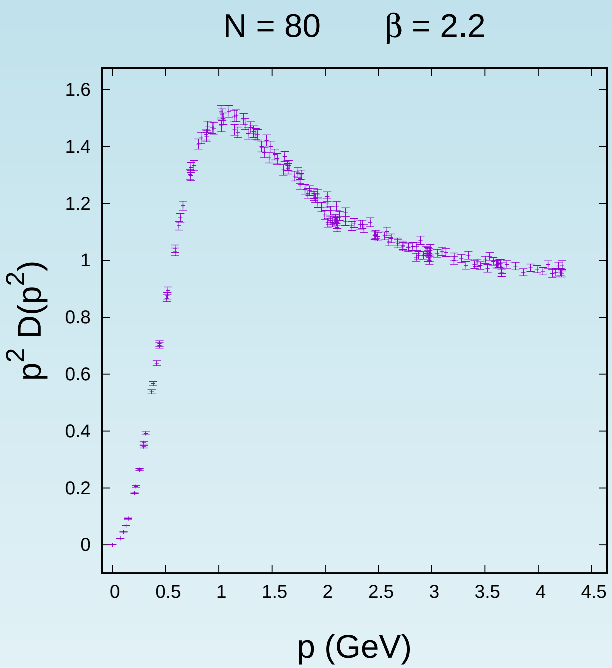
Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using unimproved momenta $p^2 = 4 \sum_\mu \sin^2(\pi\, n_\mu / N)$.

# Breaking of Rotational Invariance (II)

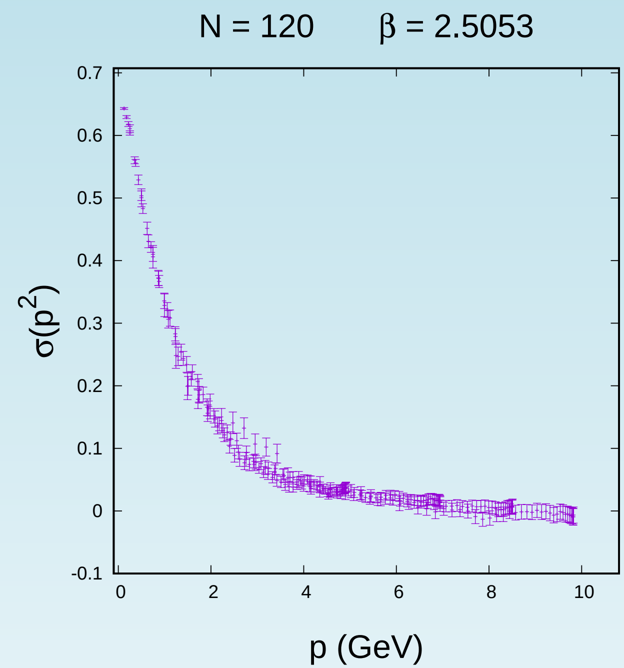
| | gluon propagator | | | ghost propagator | | |
|---|---|---|---|---|---|---|
| $N^4$ | $r_4$ | $r_6$ | $< \chi^2 >$ | $r_4$ | $r_6$ | $< \chi^2 >$ |
| $48^4$ | 0.054 | 0.000 | 2.62 | 0.016 | — | 1.30 |
| $72^4$ | 0.084 | -0.006 | 2.46 | 0.017 | — | 2.41 |
| $96^4$ | 0.107 | -0.015 | 2.35 | 0.014 | — | 0.48 |
| $120^4$ | 0.073 | -0.005 | 2.39 | 0.016 | — | 3.02 |
| $80^4$ | 0.091 | -0.006 | 2.70 | 0.021 | — | 1.70 |
| $128^4$ | 0.059 | -0.002 | 1.96 | 0.016 | — | 2.85 |
| $160^4$ | 0.070 | -0.006 | 2.67 | 0.019 | — | 2.95 |
| $192^4$ | 0.073 | -0.006 | 2.01 | 0.008 | — | 2.29 |

For each lattice volume $V = N^4$ we show the parameters $r_4$ and $r_6$ used to define improved momenta $p^2 = \sum_\mu \hat{p}_\mu^2 + r_4\,\hat{p}_\mu^4 + r_6\,\hat{p}_\mu^6$ with $\hat{p}_\mu = 2\,\sin\left(\pi\, n_\mu / N\right)$, for the gluon and ghost propagators.

# Breaking of Rotational Invariance (III)

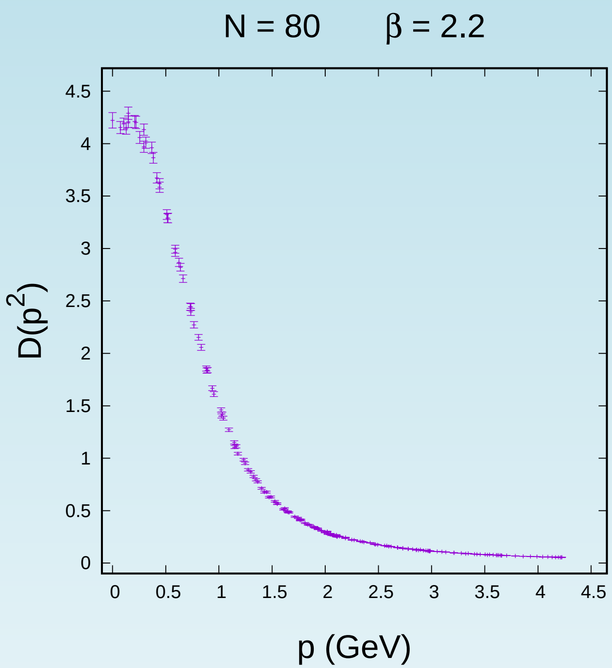


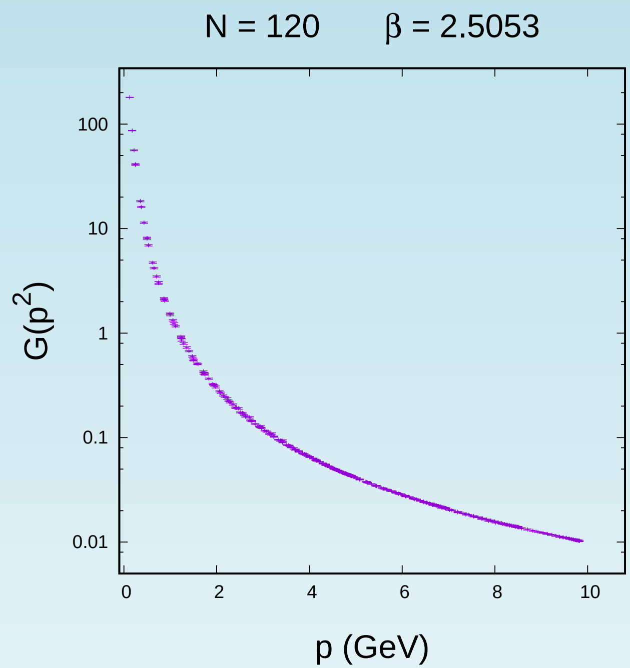

Gluon dressing function $p^2 D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta.




Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta.

# Breaking of Rotational Invariance (IV)




Gluon propagator $D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta.

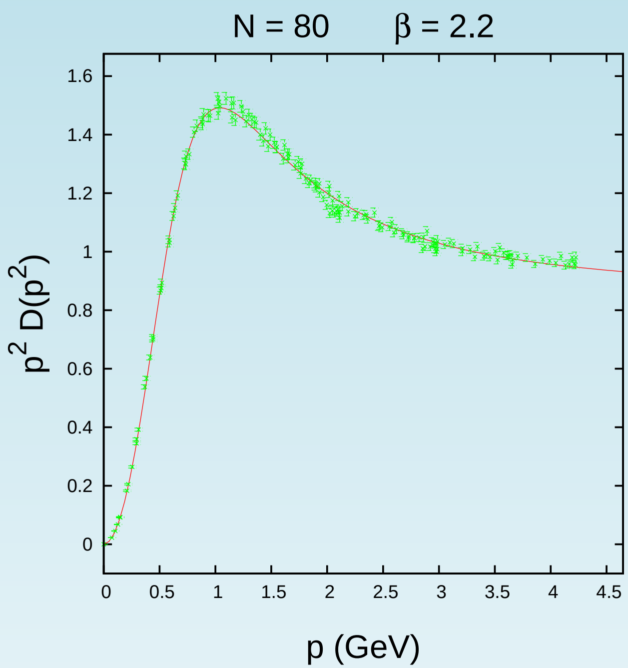
Ghost propagator $G(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta.

# Fits for the Gluon Propagator (I)

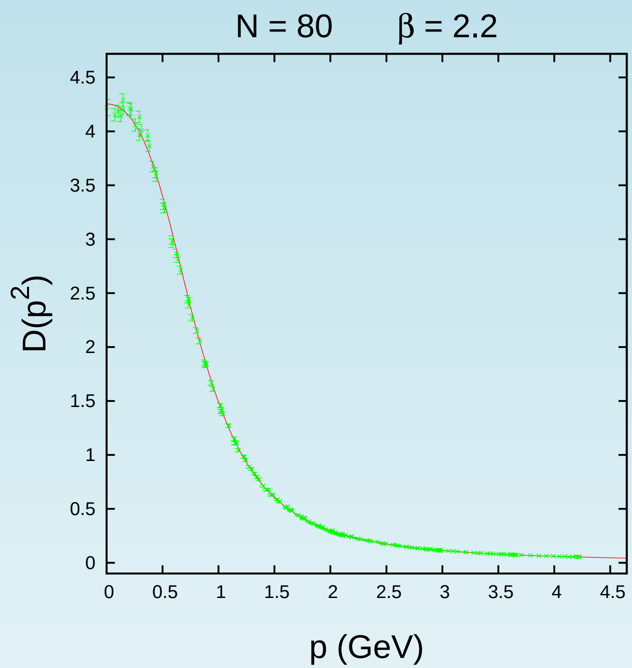
| $N^4$ | $C$ | $u$(GeV) | $t$(GeV$^2$) | $s$(GeV$^2$) | $\chi^2$/d.o.f. |
|---|---|---|---|---|---|
| $48^4$ | 0.853 (0.007) | 0.671 (0.027) | 0.674 (0.012) | 2.388 (0.109) | 2.08 |
| $72^4$ | 0.883 (0.005) | 0.680 (0.028) | 0.616 (0.009) | 2.844 (0.110) | 1.73 |
| $96^4$ | 0.902 (0.006) | 0.826 (0.038) | 0.673 (0.012) | 3.945 (0.178) | 1.47 |
| $120^4$ | 1.014 (0.004) | 0.766 (0.034) | 0.646 (0.010) | 3.749 (0.140) | 1.30 |
| $80^4$ | 0.856 (0.004) | 0.709 (0.017) | 0.707 (0.008) | 2.485 (0.073) | 1.70 |
| $128^4$ | 0.894 (0.003) | 0.712 (0.017) | 0.658 (0.005) | 3.224 (0.070) | 2.08 |
| $160^4$ | 0.917 (0.002) | 0.757 (0.015) | 0.676 (0.005) | 3.693 (0.064) | 1.48 |
| $192^4$ | 0.953 (0.002) | 0.785 (0.015) | 0.641 (0.004) | 3.758 (0.064) | 1.28 |

Fits of the gluon-propagator data, for different lattice volumes $V = N^4$ and $\beta$ couplings, using the fitting function $f_1(p^2)$ and improved momenta. The whole range of momenta was considered for the fits. Errors shown in parentheses correspond to one standard deviation. Note: the renormalization condition $D(\mu^2) = 1/\mu^2$ affects only the coefficient $C$.

# Fits for the Gluon Propagator (II)




Gluon dressing function $p^2 D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta, and the corresponding fit.




Gluon propagator $D(p^2)$ (in minimal Landau gauge, $N = 80$ and $\beta = 2.2$) for four different sets of momenta, using improved momenta, and the corresponding fit.

# Poles and Gluon Mass

| $N^4$ | $v$(GeV$^2$) | $w$(GeV$^2$) | $m_g$(GeV) | $\Gamma_g$(GeV) |
|---|---|---|---|---|
| $48^4$ | 0.225 (0.018) | 0.636 (0.014) | 0.475 (0.019) | 1.341 (0.061) |
| $72^4$ | 0.231 (0.019) | 0.571 (0.012) | 0.481 (0.020) | 1.188 (0.055) |
| $96^4$ | 0.341 (0.031) | 0.581 (0.023) | 0.584 (0.027) | 0.995 (0.060) |
| $120^4$ | 0.293 (0.026) | 0.576 (0.017) | 0.541 (0.024) | 1.064 (0.056) |
| $80^4$ | 0.252 (0.012) | 0.660 (0.010) | 0.502 (0.012) | 1.316 (0.038) |
| $128^4$ | 0.254 (0.012) | 0.607 (0.008) | 0.504 (0.012) | 1.206 (0.032) |
| $160^4$ | 0.287 (0.011) | 0.613 (0.007) | 0.536 (0.010) | 1.144 (0.026) |
| $192^4$ | 0.308 (0.012) | 0.562 (0.008) | 0.555 (0.010) | 1.013 (0.024) |

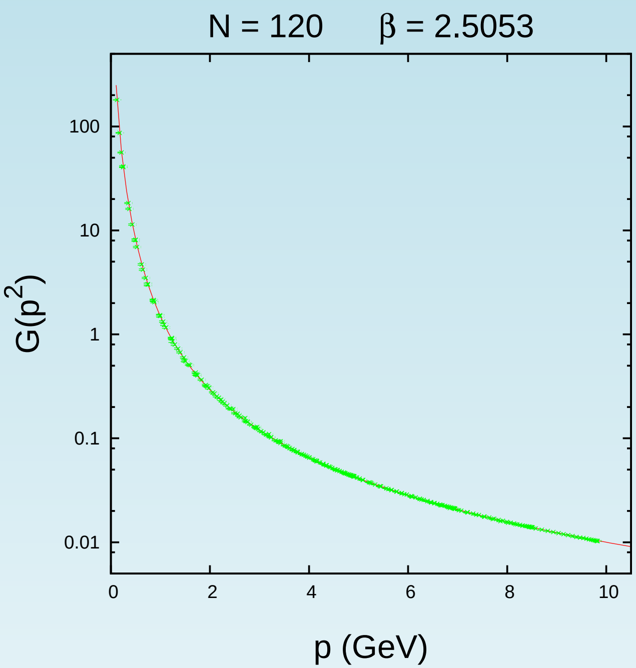
Estimates of the parameters of the function $f_2(p^2)$ from fits (see previous table) to the equivalent form $f_1(p^2)$. All poles are complex-conjugate pairs. We also show the gluon mass $m_g = \sqrt{v}$ and its width $\Gamma_g = w/m_g$, both in GeV. Errors shown in parentheses correspond to one standard deviation. Note: the renormalization condition $D(\mu^2) = 1/\mu^2$ affects only the coefficients $a$ and $b$.

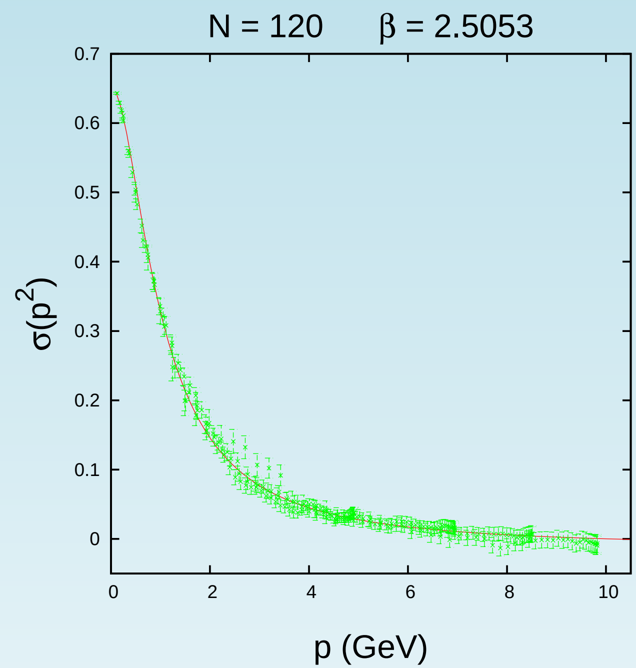
# $F_3(p^2)$ Fits for the Ghost Propagator (I)

| $N^4$ | $z$ | $t$ | $s$(GeV) | $\chi^2$/d.o.f. |
|---|---|---|---|---|
| $48^4$ | 0.900 (0.003) | 3.310 (0.012) | 0.372 (0.003) | 0.95 |
| $72^4$ | 0.969 (0.001) | 3.028 (0.007) | 0.383 (0.003) | 0.68 |
| $96^4$ | 0.995 (0.001) | 2.869 (0.005) | 0.359 (0.002) | 0.33 |
| $120^4$ | 0.988 (0.0008) | 2.907 (0.008) | 0.382 (0.003) | 0.62 |
| $80^4$ | 0.921 (0.002) | 3.321 (0.009) | 0.389 (0.003) | 2.17 |
| $128^4$ | 0.975 (0.0008) | 2.987 (0.003) | 0.379 (0.002) | 1.51 |
| $160^4$ | 0.983 (0.0005) | 2.847 (0.002) | 0.377 (0.001) | 1.12 |
| $192^4$ | 0.999 (0.0009) | 2.974 (0.005) | 0.414 (0.003) | 4.31 |

Fits of the ghost-propagator data, for different lattice volumes $V = N^4$ and $\beta$ couplings, using the fitting function $F_3(p^2)$ and improved momenta. The whole range of momenta was considered for the fits. Errors shown in parentheses correspond to one standard deviation. Note: the renormalization condition $G(\mu^2) = 1/\mu^2$ affects only the coefficient $z$.

# $F_3(p^2)$ Fits for the Ghost Propagator (II)




Ghost propagator $G(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta, and the corresponding fit.

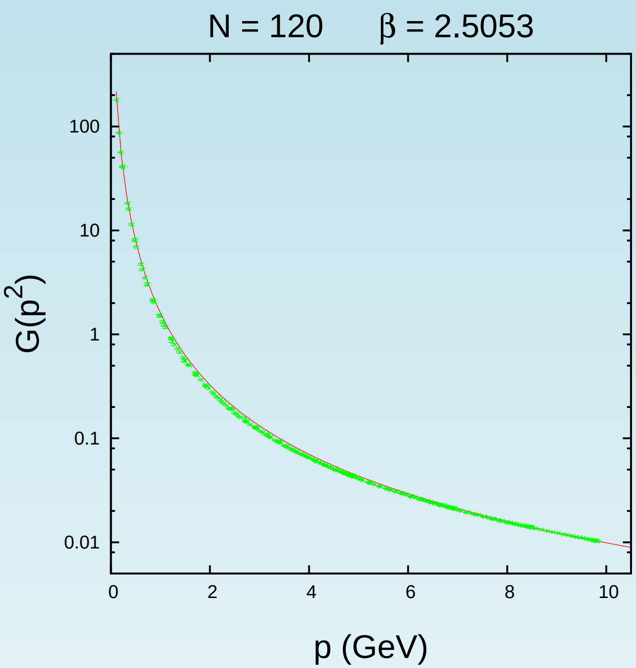
Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta, and the corresponding fit.

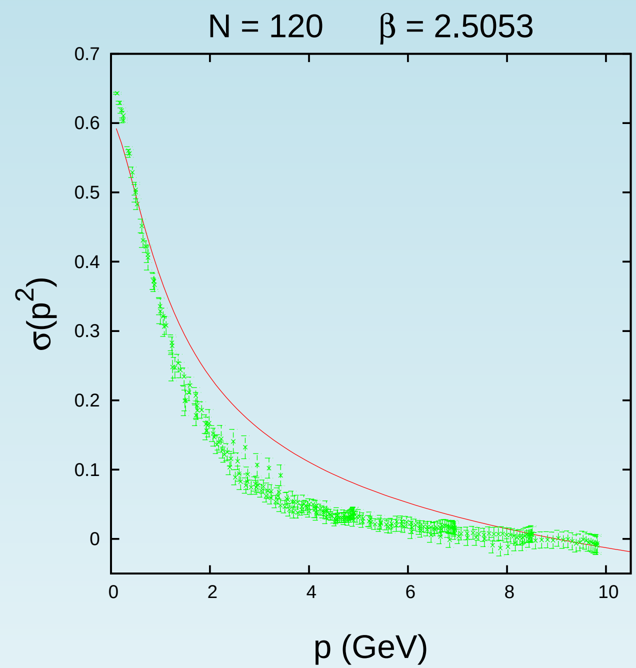
# 1-Loop Fits for the Ghost Propagator (I)

| $N^4$ | $g^2$ | $\chi^2$/d.o.f. | $g_c^2$ |
|---|---|---|---|
| $48^4$ | 11.59 (0.04) | 24.0 | 16.14 |
| $72^4$ | 8.22 (0.03) | 15.6 | 11.84 |
| $96^4$ | 6.22 (0.03) | 15.2 | 9.33 |
| $120^4$ | 5.53 (0.06) | 39.6 | 8.74 |
| $80^4$ | 11.27 (0.02) | 44.6 | 15.52 |
| $128^4$ | 7.76 (0.02) | 73.1 | 11.22 |
| $160^4$ | 6.72 (0.03) | 117.7 | 10.07 |
| $192^4$ | 6.05 (0.02) | 68.4 | 8.79 |

Fits of the ghost-propagator data, for different lattice volumes $V = N^4$ and $\beta$ couplings, using the 1-loop expression, with $g^2$ as the only free parameter, and improved momenta. The whole range of momenta was considered for the fits. Errors shown in parentheses correspond to one standard deviation. We also show the critical value $g_c^2$, which sets $\sigma(0) = 1$. Note: the renormalization condition $D(\mu^2) = 1/\mu^2$ affects (multiplicatively) both $g^2$ and $g_c^2$.

# 1-Loop Fits for the Ghost Propagator (II)




Ghost propagator $G(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta, and the corresponding fit.

Gribov ghost form factor $\sigma(p^2)$ (in minimal Landau gauge, $N = 120$ and $\beta = 2.5053$) for four different sets of momenta, using improved momenta, and the corresponding fit.

# Conclusions

The features observed for gluon and ghost propagators, in minimal Landau gauge, using relatively coarse lattices are confirmed in the continuum limit:

- the gluon propagator is finite (and nonzero) at zero momentum, and it is characterized by complex-conjugate poles;
- if we interpret these poles as describing an unstable particle, its mass and decay width seem to have a nice continuum limit;
- the fit proposed for the ghost propagator works very well, also for smaller lattice spacings;
- the value for $g_c^2$ does not get closer to the fitted value $g^2$ in the continuum limit.

THANKS!