Space Charge

JAI Graduate Course

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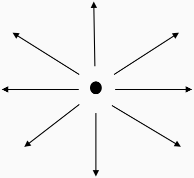
Introduction

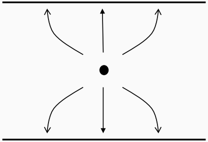
Space Charge

The basic idea behind space charge is very simple. We impose electromagnetic fields on a beam of particles, but we must also take into account the EM fields *produced by the beam itself*.

These fields can consist of:



1. Direct self fields
2. Image self fields
3. Wakefields
   - not discussed in this lecture



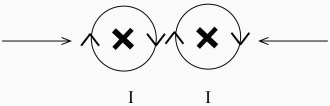
Space Charge

Consider two point charges, q, spaced a distance r apart. They experience a repulsive Coulomb force,

$$F_{elec.} = \frac{q^2}{4\pi\epsilon_0 r^2} \tag{1}$$

In an accelerator the particles are moving with some velocity, v. This is equivalent to a current carrying wire with $I = qv$. Recall that between two current carrying wires, there is in fact an *attractive* force,

$$F_{mag.} = \frac{\mu_0 I^2}{4\pi r^2} = \frac{\mu_0 q^2 v^2}{4\pi r^2} = \frac{v^2}{c^2} F_{elec.} \tag{2}$$



Space Charge

Combining Eqns. [1] and [2], the overall force is repulsive

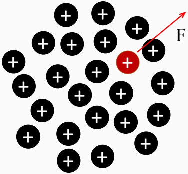
$$F_{total} = (1 - v^2/c^2)F_{elec.} \tag{3}$$

This cancels to almost zero in the case $v \approx c$, i.e. for electrons travelling near to the speed of light. For hadron (proton or ion) machines, often $\beta = v/c \approx 0.5$ so the space charge repulsion becomes significant.

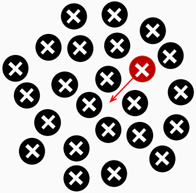
Of course, this is only for two charges. In reality we have a full beam with some intensity.

Space Charge

If we take a cross-section through the beam:



Repulsive Coulomb force



Attractive magnetic force

Note that the force on a test particle at the *centre* of the beam is *zero* and the force increases nearer the beam edge.

Aside: What does 'space charge' mean?

There are two 'regimes' to describe the net effects of Coulomb interactions in a system with many particles.

- Collisional regime: dominated by particle-on-particle collisions and described by *single particle effects*.
- Space Charge regime: dominated by the self fields of the distribution of particles themselves, which varies over distances which are larger than the average particle separation and described by *collective effects*.

Aside: What does 'space charge' mean?

To tell which regime we're in, it is useful to consider the *Debye length* λ_D.

In a beam moving at relativistic velocity, but assuming the transverse motion is non-relativistic,

$$\lambda_D = \sqrt{\frac{\epsilon_0 \gamma^2 k_B T}{q^2 n}} \tag{4}$$

k_B is the Boltzmann constant, T is temperature, thus $k_B T$ is the average kinetic energy of the particles, and n is the particle density N/V.

If the $\lambda_D << a$ (beam radius), collective effects due to self fields play an important role and we can use smooth functions of the charge and field distributions. For most beams of practical interest[1], collisional forces are small and can be neglected.

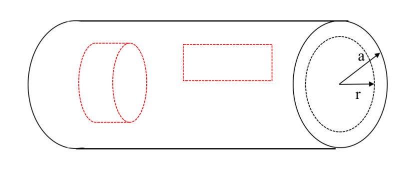
[1] See M. Reiser, Chapter 4 for more discussion on this. Note that intrabeam scattering in high energy storage rings is an exception where collisional forces play a key role.

Space Charge Forces

Unbunched Uniform Beam

Consider a beam as a continuous cylinder of charge, length l, beam radius a, charge density

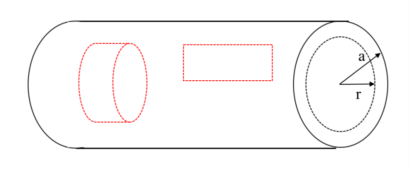
$$\rho(r) = qn(r) = \frac{I_{beam}}{\pi a^2 v} \tag{5}$$



The electric field is radial and inside the beam is given by Gauss' Flux theorem:

$$\int \epsilon_0 E.dS = \int \rho dV \tag{6}$$

Unbunched Uniform Beam



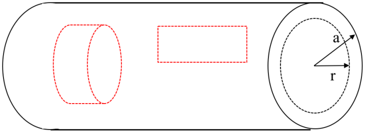
The electric field:

$$2\pi r l \epsilon_0 E_r = \begin{cases} \rho \pi r^2 l, & \text{if } r \leq a \\ \rho \pi a^2 l, & \text{if } r > a \end{cases}$$

Therefore:

$$E_r = \begin{cases} \frac{I_{beam}}{2\pi\epsilon_0\beta c}\frac{r}{a^2}, & \text{if } r \leq a \\ \frac{I_{beam}}{2\pi\epsilon_0\beta c}\frac{1}{r}, & \text{if } r > a \end{cases}$$

Unbunched Uniform Beam



The magnetic field is angular, $\vec{B} = B_\phi$ from Ampére's law:

$$\int B.dl = \mu_0 \times \{\text{current flowing through a loop}\} \tag{7}$$

$$2lB_\rho = \mu_0 Jlr \tag{8}$$

Where $J = \frac{I_{beam}}{\pi a^2}$.

$$\therefore B_\phi = \frac{\mu_0 I_{beam} r}{2\pi a^2} \text{for } r \leq a \tag{9}$$

Unbunched Uniform Beam

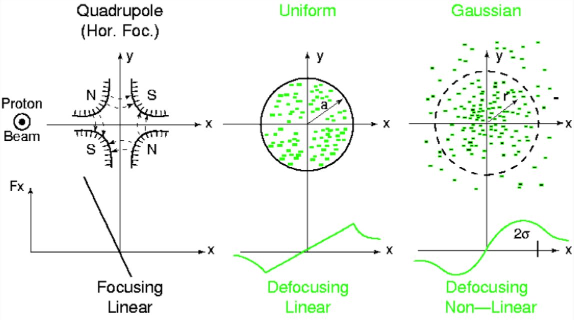
The force experienced by a test particle in the beam is given (as always...) by the Lorentz force. Taking the E and B fields from previous slides,

$$F_r = e(E_r - v_s B_\phi) \tag{10}$$

$$F_r = \frac{eI_{beam}}{2\pi\epsilon_0\beta c}(1-\beta^2)\frac{r}{a^2} = \frac{eI_{beam}}{2\pi\epsilon_0\beta c^2}\frac{1}{\gamma^2}\frac{r}{a^2} \tag{11}$$

Unbunched Uniform Beam

$$F_x = \frac{eI_{beam}}{2\pi\epsilon_0\beta c^2}\frac{1}{\gamma^2}\frac{x}{a^2}, \quad F_y = \frac{eI_{beam}}{2\pi\epsilon_0\beta c^2}\frac{1}{\gamma^2}\frac{y}{a^2} \tag{12}$$



In the x and y directions for a circular beam, uniform charge density this gives a linear force in x, y, decreasing with energy. Note this is a defocusing lens in BOTH planes.

Unbunched Gaussian Beam

If instead we assume the bunch has a transverse Gaussian profile (a bit more realistic):

$$n(r) = A\exp(-\frac{r^2}{2\sigma^2}) \tag{13}$$

Where $A = N/2\pi\sigma^2$ and N is particles per unit length. Working this through gives us the space charge force as:

$$F(r) = \frac{Nq^2}{2\pi\epsilon_0\gamma^2}\frac{1}{r}\left(1 - \exp(-\frac{r^2}{2\sigma^2})\right) \tag{14}$$

Space Charge in Transport Line

Hill's equation

In a FODO transport line, we know the motion is described by Hill's equation, where we can add a perturbation term from the force due to space charge:

$$x'' + (k(s) + k_{SC}(s))x = 0 \tag{15}$$

k_{SC} is derived by expressing x'' in terms of transverse acceleration d^2x/dt^2 and thus of the force F_x

$$x'' = \frac{2r_0 I_{beam}}{ea^2\beta^3\gamma^3 c} x \tag{16}$$

Where the classical particle radius $r_0 = e^2/(4\pi\epsilon_0 m_0 c^2) = 1.54 \times 10^{-18}$ for protons. Which yields the new Hill's equation:

$$x'' + \left(k(s) - \frac{2r_0 I_{beam}}{ea^2\beta^3\gamma^3 c}\right) x = 0 \tag{17}$$

Incoherent tune shift

Space charge leads to defocusing in both planes, so we would expect that there will be a shift in betatron tune, ΔQ. If we take the simplest case of an unbunched beam, with uniform circular cross section, we find by calculating the (effective) gradient errors around the ring:

$$\Delta Q_x = \frac{1}{4\pi}\int_0^{2\pi r} k_{SC}\beta_x(s)ds \tag{18}$$

Using k_{SC} from before:

$$\Delta Q = -\frac{1}{4\pi}\int_0^{2\pi R} \frac{2r_0 I_b}{e\beta^3\gamma^3 c}\frac{\beta_x(s)}{a^2}ds = -\frac{r_0 R I_b}{e\beta^3\gamma^3 c}\left\langle \frac{\beta_x(s)}{a^2(s)} \right\rangle. \tag{19}$$

If we use that $\left\langle \frac{\beta_x(s)}{a^2(s)} \right\rangle = \frac{1}{\epsilon_0}$, the 100% emittance, and replace $I = Ne\beta c/(2\pi R)$, we get for the *direct space charge tune shift*:

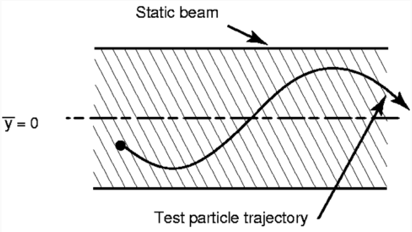
$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\epsilon_{x,y}\beta^2\gamma^3} \tag{20}$$

Incoherent tune shift

Things to note about the tune shift:

$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\epsilon_{x,y}\beta^2\gamma^3} \tag{21}$$

- 'Direct' space charge, unbunched beam in a synchrotron
- Vanishes for $\gamma >> 1$
- Important for low-energy machines
- Independent of machine size $2\pi R$ for a given N
- Incoherent motion - particle moves within the beam.



Incoherent tune shift

$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\epsilon_{x,y}\beta^2\gamma^3} \tag{22}$$

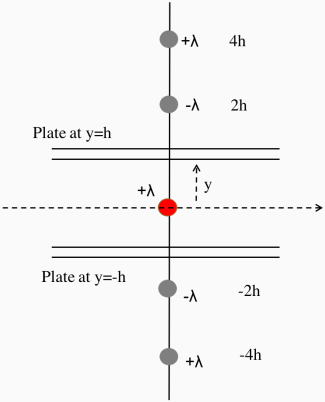
Taking the values for the CERN PS Booster, (assume unbunched), calculate the tune shift:

- $N = 1 \times 10^{13}$ protons
- $\epsilon_{x,y} = 80, 27\mu$rad m
- $E = 50$ MeV, i.e. $\gamma = 1.053, \beta = 0.314$

Image Effects

Parallel Conducting Plates

Perfectly conducting plate parallel to beam pipe, produces an infinite system of images.



Field created by a line charge at distance d is

$$E_y = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{d} \tag{23}$$

From first pair of images:

$$E_{1y} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h-y} - \frac{1}{2h+y}\right) \tag{24}$$

$$E_{2y} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{4h-y} - \frac{1}{4h+y}\right) \tag{25}$$

Parallel Conducting Plates

$$E_{iny} = \frac{(1)^{n+1}\lambda}{2\pi\epsilon_0}\left(\frac{1}{2nh-y} - \frac{1}{2nh+y}\right) = (1)^{n+1}\frac{\lambda}{4\pi\epsilon_0}\frac{y}{n^2h^2} \tag{26}$$

$$E_{iy} = \sum_{n=1}^{\infty} = \frac{\lambda}{4\pi\epsilon_0 h^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}y = \frac{\lambda}{4\pi\epsilon_0 h^2}\frac{\pi^2}{12}y \tag{27}$$

$$\therefore F_y^i = \frac{q\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}y, F_x^i = -\frac{q\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48}x \tag{28}$$

- The vertical image field vanishes at $y = 0$
- Field is linear in y, vertically defocusing
- Field is large if vacuum chamber is small

Incoherent Tune Shift

The total incoherent tune shift for a round beam between parallel conducting walls:

$$\Delta Q_x = -\frac{2 r_0 I_b R \langle \beta_x \rangle}{q c \beta^3 \gamma} (\underbrace{\frac{1}{2 \langle a^2 \rangle \gamma^2}}_{\text{direct}} - \underbrace{\frac{\pi^2}{48 h^2}}_{\text{image}}) \tag{29}$$

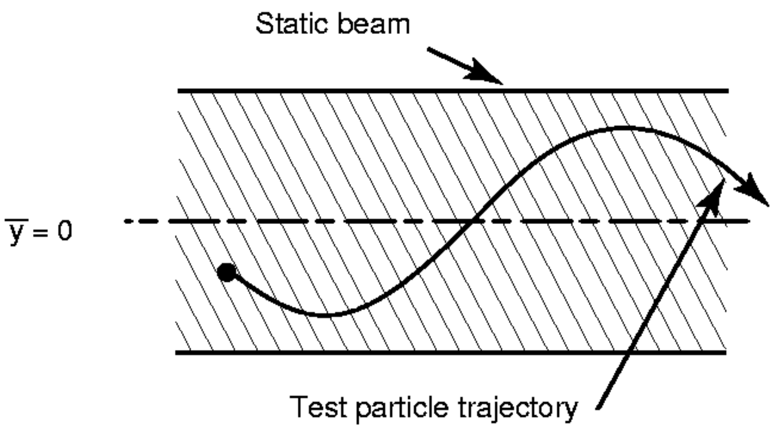
$$\Delta Q_y = -\frac{2 r_0 I_b R \langle \beta_y \rangle}{q c \beta^3 \gamma} (\underbrace{\frac{1}{2 \langle a^2 \rangle \gamma^2}}_{\text{direct}} + \underbrace{\frac{\pi^2}{48 h^2}}_{\text{image}}) \tag{30}$$

- Image effects $\propto 1/\gamma$
- They do not vanish for large γ so not negligible for electron machines
- Electrical image effects are normally focusing in horizontal, defocusing in vertical plane
- Note there are also image effects from the ferromagnetic boundary

Incoherent vs Coherent Effects

Incoherent and Coherent Motion

Incoherent motion

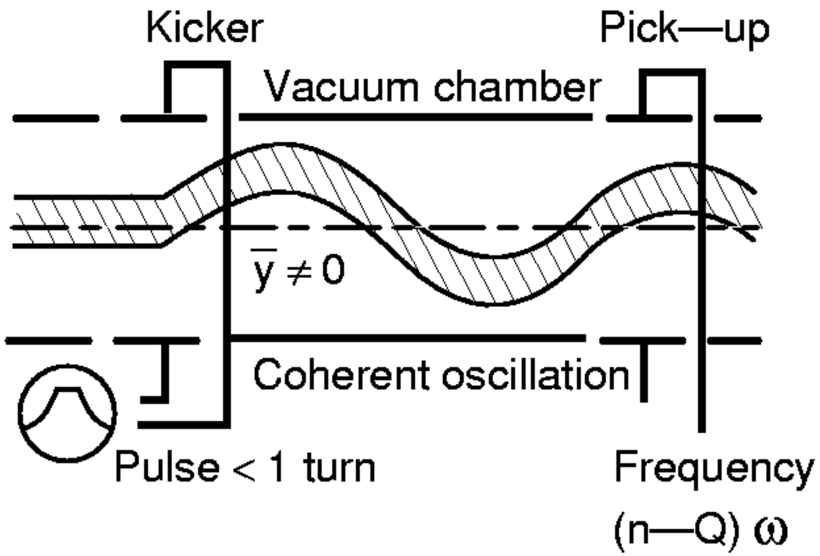


Test particle in a beam whose **centre of mass does not move**

The **beam environment does not "see"** any motion

Each particle features its **individual amplitude and phase**

Coherent motion

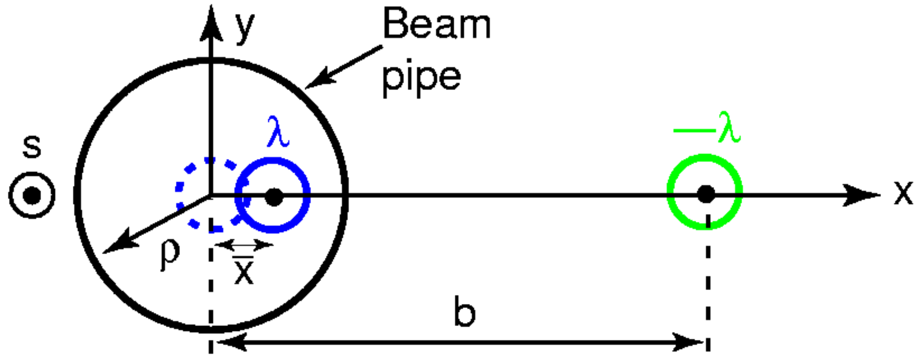


The **centre of mass moves** doing betatron oscillation as a whole

The **beam environment** (e.g. a position monitor **"sees"** the **"coherent** motion")

On top of the coherent motion, each particles has still **its individual** one

Coherent Tune Shift, Round Beam Pipe



$\bar{x}$..hor. beam position (centre of mass)
a…beam radius
ρ…beam pipe radius (ρ » a)

$b\bar{x} = \rho^2$ (mirror charge on a circle)

$$E_{ix}(\bar{x}) = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{b-\bar{x}} \approx \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{b} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho^2}\bar{x}$$

$$F_{ix}(\bar{x}) = \frac{e\lambda}{2\pi\varepsilon_0}\frac{1}{\rho^2}\bar{x}$$

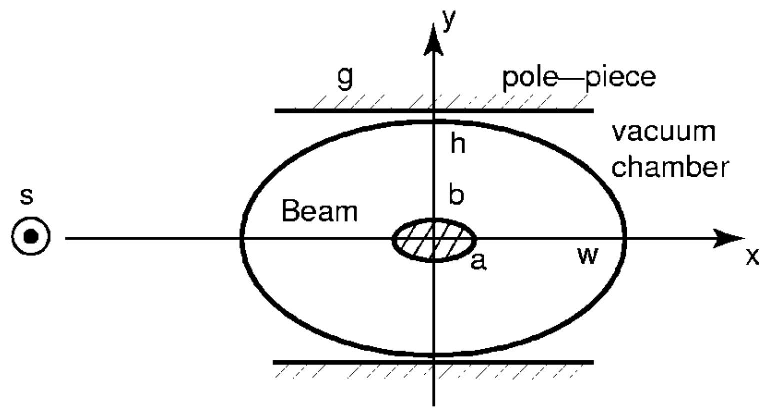
- ❑ same in vertical plane (y) due to symmetry
- ❑ force linear in $\bar{x}$
- ❑ force positive hence defocusing in both planes

$$\Delta Q_{x,y\,coh} = -\frac{r_0 R\langle\beta_{x,y}\rangle I}{ec\beta^3\gamma\rho^2} = -\frac{r_0\langle\beta_{x,y}\rangle}{2\pi\beta^2}\frac{N}{\gamma\rho^2}$$

Coherent tune shift, round pipe

- ❑ negative (defocusing) both planes
- ❑ only weak dependence on γ
- ❑ ΔQ_{coh} always negative

The “Laslett”* Coefficients



$$\Delta Q_{y,inc} = -\frac{N r_0 \langle \beta_y \rangle}{\beta^2 \gamma \pi} \left(\frac{\varepsilon_0^y}{b^2 \gamma^2} + \frac{\varepsilon_1^y}{h^2} + \beta^2 \frac{\varepsilon_2^y}{g^2} \right)$$

direct electr. image magnet. image

$$\Delta Q_{y,coh} = -\frac{N r_0 \langle \beta_y \rangle}{\beta^2 \gamma \pi} \left(\frac{\xi_1^y}{h^2} + \beta^2 \frac{\xi_2^y}{g^2} \right)$$

Laslett coefficients	Circular $(a = b,\ w = h)$	Elliptical (e.g. $w = 2h$)	Parallel plates $(h/w = 0)$
ε_0^x	$1/2$	$\frac{b^2}{a(a+b)}$	
ε_0^y	$1/2$	$\frac{b}{a+b}$	
ε_1^x	0	-0.172	-0.206
ε_1^y	0	0.172	0.206
ξ_1^x	$1/2$	0.083	0
ξ_1^y	$1/2$	0.55	$0.617(\pi^2/16)$
ε_2^x	$-0.411(-\pi^2/24)$	-0.411	-0.411
ε_2^y	$0.411(\pi^2/24)$	0.411	0.411
ξ_2^x	0	0	0
ξ_2^y	$0.617(\pi^2/16)$	0.617	0.617

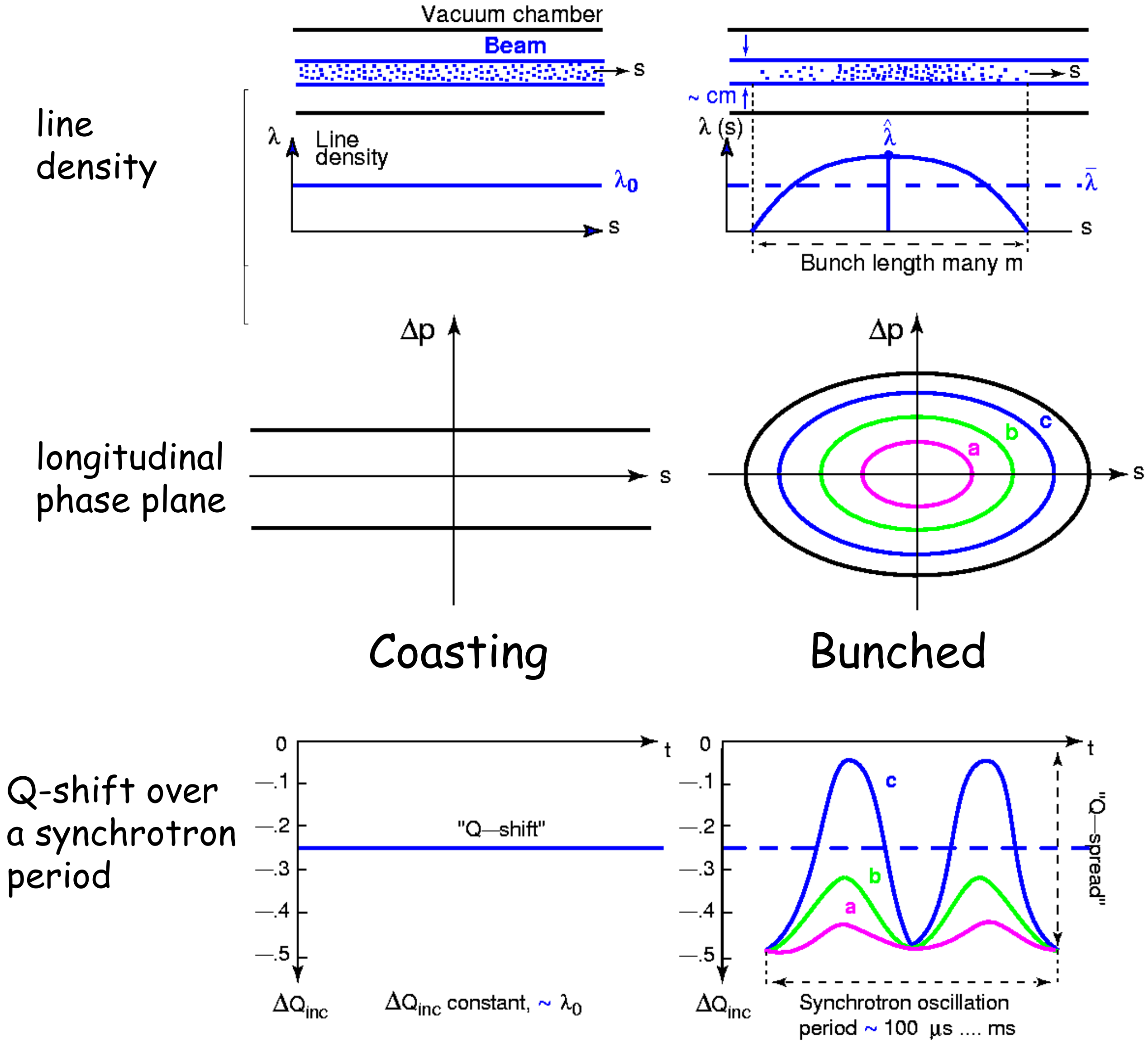
$\pi^2/4$

8

Uniform, elliptical beam in an elliptical beam pipe.
Similar formulae for ΔQ_x
In general, $\Delta Q_y > \Delta Q_x$

*L.J. Laslett, 1963

Bunched Beam in a Synchrotron



What's different with bunched beams?

- Q-shift much larger in bunch centre than in tails
- Q-shift changes periodically with ω_s
- peak Q-shift much larger than for unbunched beam with same N (number of particles in the ring)
- Q-shift ⇨ Q-spread over the bunch

Incoherent ΔQ: A Practical Formula

$$\langle \beta \rangle = \frac{R}{Q}$$

$$\Delta Q_y = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_y G_y}{B_f}\left\langle \frac{\beta_y}{b(a+b)} \right\rangle \qquad \left\langle \frac{\beta_y}{b(a+b)} \right\rangle = \left\langle \frac{\beta_y}{b^2\left(1+\frac{a}{b}\right)} \right\rangle \approx \frac{1}{E_y\left(1+\sqrt{\frac{E_x Q_y}{E_y Q_x}}\right)}$$

$$\boxed{\Delta Q_{x,y} = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_{x,y} G_{x,y}}{B_f}\frac{1}{E_{x,y}\left(1+\sqrt{\frac{E_{y,x} Q_{x,y}}{E_{x,y} Q_{y,x}}}\right)}}$$

q/A...... charge/mass number of ions (1 for protons, e.g. 6/16 for $_{16}O^{6+}$)

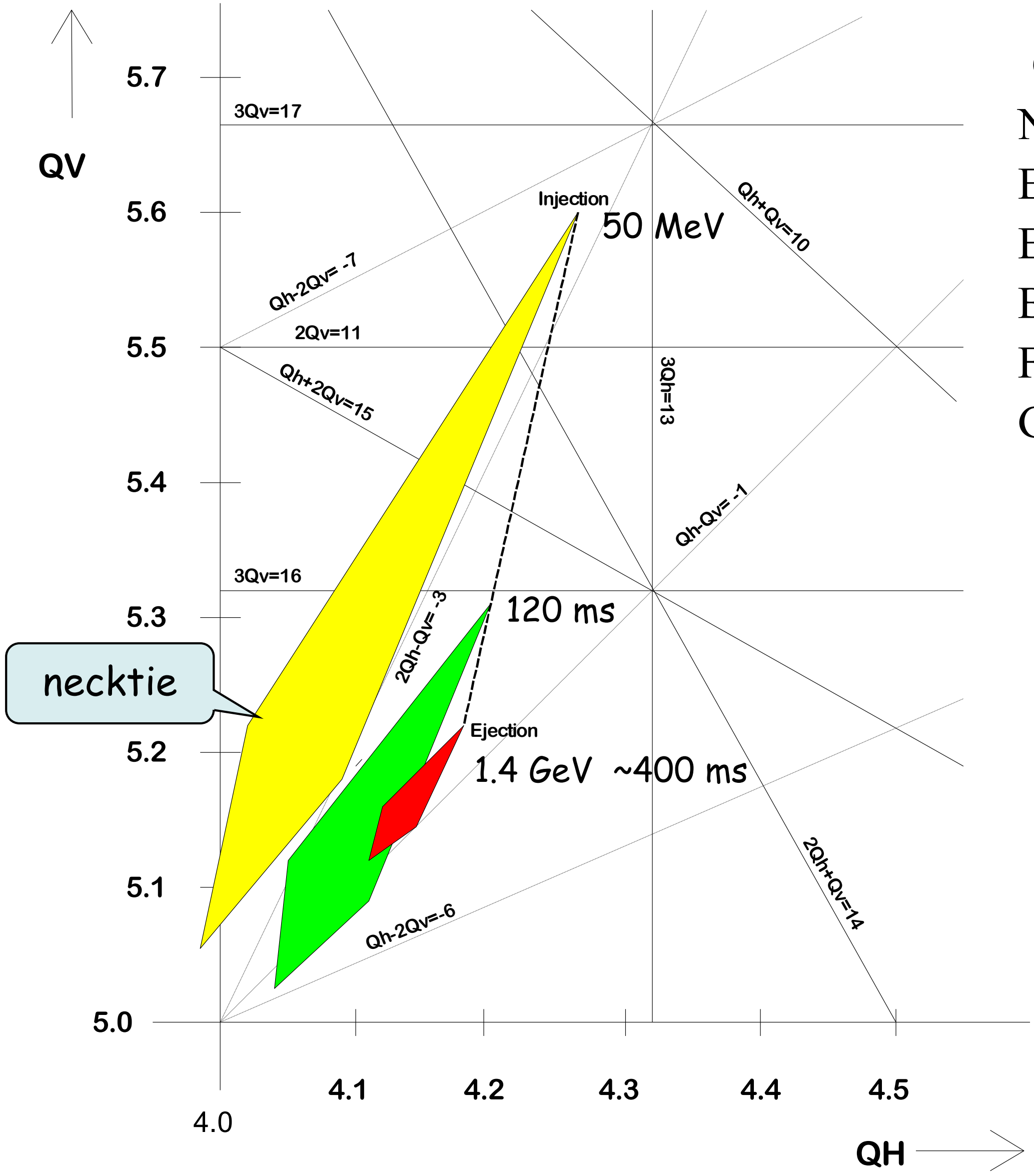
$F_{x,y}$......"Form factor" derived from Laslett's image coefficients $\varepsilon_1^x, \varepsilon_1^y, \varepsilon_2^x, \varepsilon_2^y$ ($F \approx 1$ if dominated by direct space charge)

$G_{x,y}$......Form factor depending on particle distribution in x,y. In general, $1 < G \le 2$
Uniform $G=1$ ($E_{x,y}$ 100% emittance)
Gaussian $G=2$ ($E_{x,y}$ 95% emittance)

B_f...... "Bunching Factor": average/peak line density $B_f = \frac{\bar{\lambda}}{\hat{\lambda}} = \frac{\bar{I}}{\hat{I}}$

Examples

A Space-Charge Limited Accelerator



CERN PS Booster Synchrotron

N = 10^{13} protons

$E_x^* = 80$ μrad m $[4\ \beta\gamma\ \sigma_x^2/\beta_x]$ hor. emittance

$E_y^* = 27$ μrad m vertical emittance

$B_f = 0.58$

$F_{x,y} = 1$

$G_x/G_y = 1.3/1.5$

- Direct space charge tune spread **~0.55 at injection**, covering 2nd and 3rd order stop-bands
- **"necktie"-shaped tune spread shrinks rapidly** due to the $1/\beta^2\gamma^3$ dependence
- Enables the working point to be moved **rapidly** to an area clear of strong stop-bands

How to Remove the Space-Charge Limit?

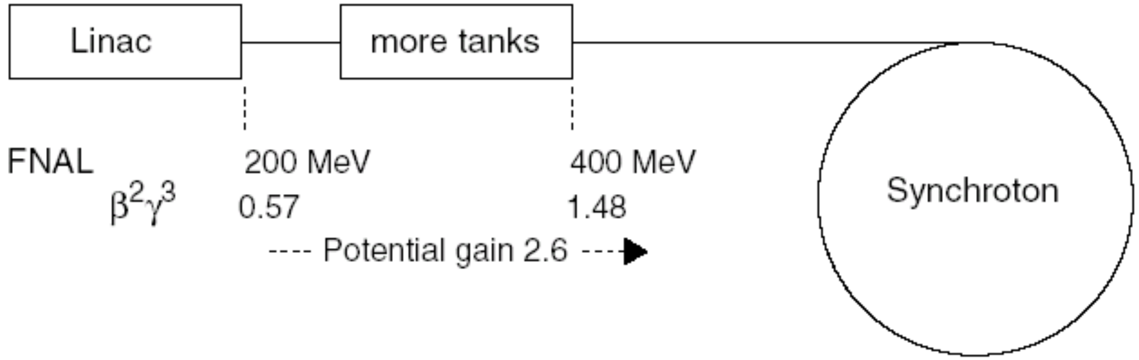
Direct space charge

$$\Delta Q_y \approx \frac{N}{E_y \beta^2 \gamma^3} \frac{\hat{I}}{\bar{I}}$$

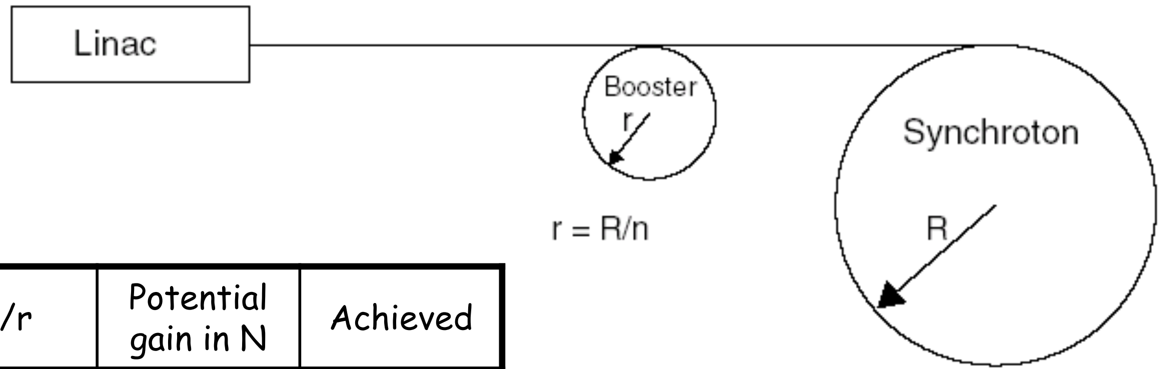
Problem: A **large proton synchrotron is limited in N** because ΔQ_y reaches 0.3 … 0.5 when filling the (vertical) acceptance.

Solution: **Increase N by raising the injection energy and thus $\beta^2\gamma^3$** while keeping to the same ΔQ. Ways to do this:

Make a **longer** (higher-energy) **Linac** (by adding tanks as has been done in Fermilab)



Add a **small "Booster" synchrotron** of radius r = R/n with n the number of batches (BNL) or rings (CERN)



	Linac (MeV)	Booster (GeV)	n=R/r	Potential gain in N	Achieved
CERN PS	50	1	4(rings)	59	~15
BNL AGS	200	1.5	4(batches)	26	~8

High Intensity Proton Beam in a FODO Line

Transverse phase planes
rad vs. m

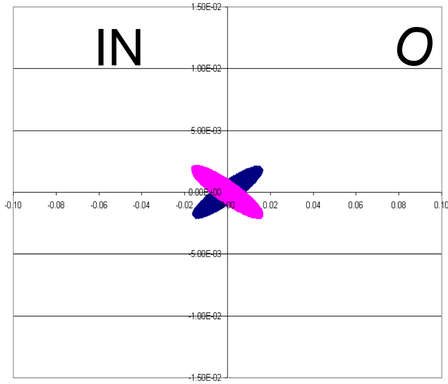
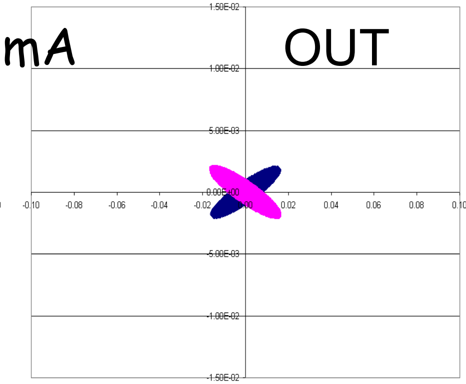
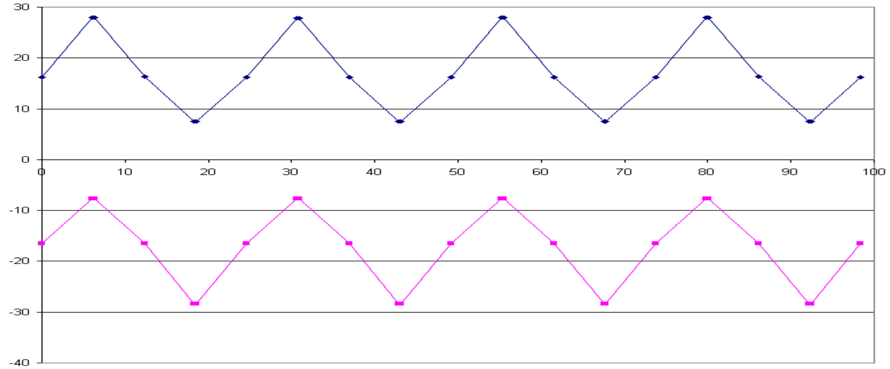
horizontal
vertical

Transverse envelopes
mm vs. m

IN

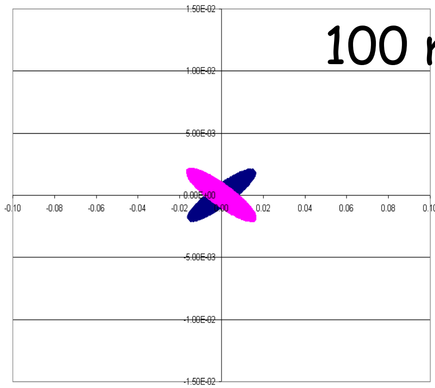
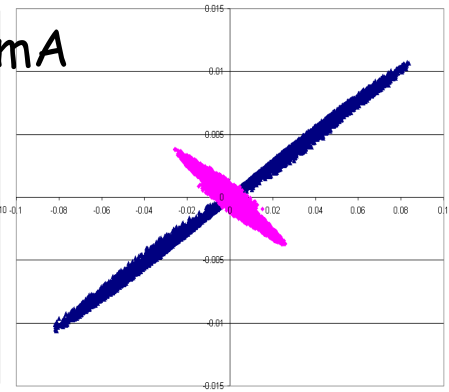
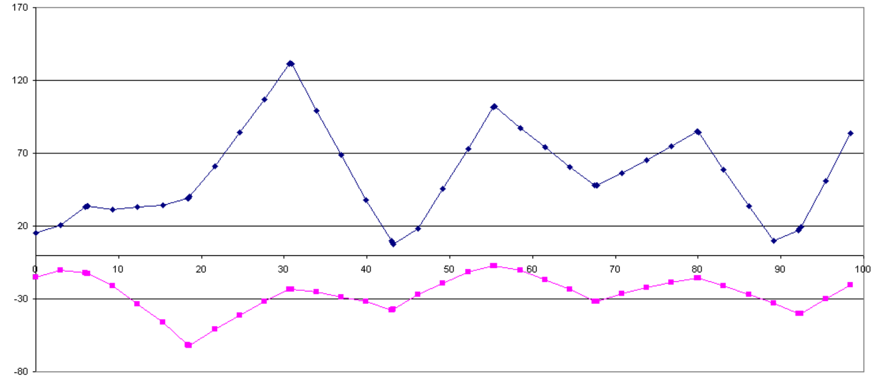
0 mA

OUT

50 MeV

100 mA

Courtesy of Alessandra Lombardi/ CERN, 8/04

Summary

Coherent and incoherent tune shift in a synchrotron:

A high-intensity, un-bunched beam experiences a small deflection by a kicker magnet in one plane and performs betatron oscillations. The machine tune for vanishing intensity is known to be Q_0. A position detector measures the oscillations from which an effective tune Q is derived. Is it:

1. equal to Q_0?
2. equal to $Q_0 + Q_{coherent}$?
3. equal to $Q_0 + Q_{incoherent}$?
4. equal to $Q_0 + Q_{coherent} + Q_{incoherent}$?

Questions?