Electron Cooling Simulation and Benchmark

Andrea Latina
(BE-ABP-LAT)

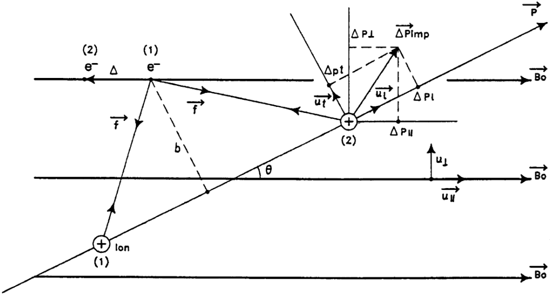
E-coolers and e-lenses simulations - joint BE-ABP / BE-BI meeting

January 10, 2019

Table of Contents

Part 1.

Physics of Electron Cooling



[CERN 92-01 - CAS 1991 - J. Bosser, “Electron Cooling”]

Recap: Electron Cooling as Energy exchange

Collision model: momentum is transferred in the ion-electron collision

- To compute the transferred momentum $\Delta\vec{P}$ it is convenient to make the following two approximations:
  1. The ion does not appreciably change direction, since its mass is much larger than the electron's. This is reasonable if one assumes a uniform density of electrons
  2. The electron remains stationary as the ion passes it, so that the electron receives the entire impulse $\Delta\vec{P}$
- In its reference frame the kinetic energy acquired by the electron, originally at rest, is ΔE_e:

$$\Delta E_e = E'_e - E_e =$$
$$= \sqrt{m_e^2 + \left(\Delta\vec{P}\right)^2} - m_e = K_e \approx \frac{\left(\Delta\vec{P}\right)^2}{2m_e}$$

- This corresponds to the energy lost by one ion to one electron:

$$\Delta E_i = -\Delta E_e = -K_e = -\frac{\left(\Delta\vec{P}\right)^2}{2m_e}$$

Friction force

ΔE_i is the energy that one ion loses to *one electron*. The total energy loss due to *all electrons* is:

$$W = \Delta E_{i,\,total} = \iiint \Delta E_i \cdot n_e \cdot dV$$
$$= \iint \Delta E_i \cdot n_e \cdot 2\pi r \, dr \, dz.$$

where n_e is the volume density of the electrons.
The friction force is the total energy loss per unit of length:

$$F_{\text{friction}} = \frac{dW}{dz} = \int n_e \cdot \Delta E_i \cdot 2\pi r \, dr = -2\pi n_e \int \frac{\left(\Delta \vec{P}\right)^2}{2 m_e} \cdot r \, dr$$

This can be derived in several ways: e.g. from the integration of the Coulomb interaction, or using the Rutherford cross section, ..

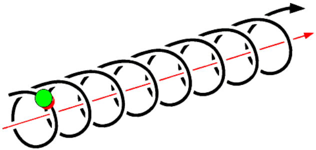
$$F_{friction} = \frac{dW}{dz} = -n_e \frac{2\pi Z^2 e^4}{(4\pi\epsilon_0)^2 \, m_e u^2} \left[\underbrace{\log\left(\frac{r_{\text{max}}}{r_{\text{min}}}\right)}_{\text{Coulomb Logarithm}} - u^2 \left(1 - \frac{r_{\text{min}}}{r_{\text{max}}}\right) \right]$$

- u, the relative velocity.

[See J. D. Jackson, "Classical Electrodynamics", for a derivation starting from the cross section]

Magnetised case: "Fast" and "Adiabatic" collisions

In presence of a solenoidal field $\vec{B}$ the electrons are spiralled by the magnetic field.



If the Larmor radius is much smaller than ρ_{max}, the region of interaction must then be divided in two regions:

$$\underbrace{\rho}_{\text{"fast" collisions}} < r_{\text{Larmor}} < \underbrace{\rho}_{\text{"adiabatic" collisions}} < \rho_{max}$$

The total frictional force is the sum of the "fast" and the "adiabatic" frictional forces:

$$\vec{F} = \vec{F}_0 + \vec{F}_A$$

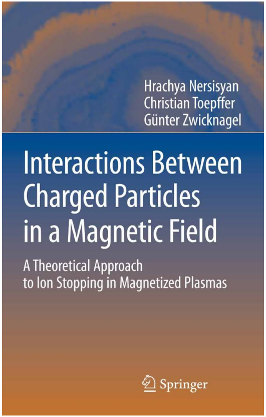
The total cooling force

In this book, a theoretical study of this stopping force is proposed, using two different and independent formalisms:

1. the "dielectric theory" (a continuum theory in which the response of charge and current densities to external perturbations is calculated)

2. the "binary collision approximation" (where the motion of the ion is described as the aggregate of subsequent pairwise interactions with the target electrons).

After detailed calculations, the book demonstrates that the two approaches yield the same mathematical expressions of the cooling force.



The new formulation of the cooling force

[Nersisyan] summarises the force in Eq. (6.1), reducing the intervals to two:

$$\underbrace{r_{\min} \to r_F}_{\text{small impact parameters}} \quad \text{and} \quad \underbrace{r_F \to r_{\max}}_{\text{large impact parameters}}$$

such that

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} \left\{ \underbrace{\iiint \left[L_F \frac{\vec{U}}{U^3} \right] f(\vec{v}_e)\, d\vec{v}_e}_{\mathcal{F}_{\text{Unmagnetized}}} + \underbrace{\int \left[L_M \frac{U_{B\perp}^2}{U_B^5} \left(\vec{U}_{B\parallel} + \frac{\vec{U}_{B\perp}}{2} \left(1 - \frac{U_{B\parallel}^2}{U_{B\perp}^2} \right) \right) \right] f(v_{e\parallel})\, dv_{e\parallel}}_{\mathcal{F}_{\text{Magnetized}}} \right\}$$

with

$$L_F = \frac{1}{2} \log\left(1 + \frac{r_F^2}{r_{\min}^2} \right) \qquad \text{small impact parameters}$$

$$L_M = \frac{1}{2} \log\left(1 + \frac{r_{\max}^2}{r_F^2} \right) \qquad \text{large impact parameters}$$

$$r_{\max} = \min\left(r_{\text{aperture}},\ \lambda_D \sqrt{1 + \frac{U^2}{\Delta_e^2/3}},\ U\Delta t \right) \qquad \text{maximum impact parameter}$$

$$r_F = \frac{\sqrt{U_{B\parallel}^2 + \Delta_\parallel^2}}{\omega_e}, \qquad \text{pitch of the electron helix}$$

$$r_{\min} = \frac{K}{\mu U^2} \qquad \text{minimum impact distance}$$

Effect of the electrons temperature

The electrons are not mono-energetic, and have transverse and longitudinal temperatures. This must be considered when computing the friction force:

$$\vec{F}_{friction} = \iiint \left[\vec{F}_{\text{ion-electron}} \left(v_i , v_e \right) \right] \; f \left(\vec{v}_e \right) d\vec{v}_e$$

where $\vec{u}$ is the ion-electron relative velocity, and $f\left(\vec{v}_e\right)$ is the electron velocity distribution function, which is typically a Maxwellian distribution:

$$f\left(\vec{v}_e\right) = \left[\left(2\pi\right)^{3/2} \Delta_{e\|} \Delta_{e\perp}^2 \, \exp\left(\frac{v_{e_\perp^2}}{2\Delta_{e\perp}^2} + \frac{v_{e_\|^2}}{2\Delta_{e\|}^2} \right) \right]^{-1}$$

This leads to several complex expressions for $\vec{F}_0$ and $\vec{F}_A$.

Due to the acceleration in the electron gun, the thermal velocity distribution of the electrons in their rest frame makes the electron plasma highly anisotropic:

$$\Delta_{e\|} \ll \Delta_{e\perp}$$

with typically $\Delta_{e_\perp} \approx 100 \Delta_{e\|}$.

Considering the thermal effects...

The expected behaviour of the friction force is shown by [Parkhomchuk]

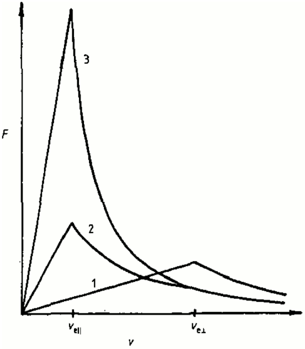


Figure 2. Behaviour plotted against friction force at different values of magnetic field. Curve 1: in the absence of magnetic field, curve 2: under the condition of partial magnetization, curve 3: complete magnetization of transverse electron motion.

Where $v_\perp$ and $v_\parallel$ are the transverse and longitudinal electron velocities.

Fast collisions [Ya.S. Derbenev, A.N. Skrinsky]

A case of practical interest is when $\Delta_{e\|} \ll \Delta_{e\perp}$

a) $v > \Delta_{e_\perp}$; in this case

$$\mathbf{F}^0 \approx -\frac{4\pi n Z^2 e^4}{m} L^0(v) \frac{\mathbf{v}}{v^3} \tag{4.1}$$

b) $v < \Delta_{e_\perp}$; in this region

$$\mathbf{F}^0_\perp \approx -\frac{\pi\sqrt{2\pi} n Z^2 e^4}{m} L^0(\Delta_{e_\perp}) \frac{\mathbf{v}_\perp}{\Delta^3_{e_\perp}} \tag{4.2}$$

$$F^0_\| \approx -\frac{4\pi n Z^2 e^4}{m\Delta^2_{e_\perp}} \times \begin{cases} \dfrac{v_\|}{|v_\||} L^0(v_\|) - \dfrac{v_\|}{\Delta_{e_\perp}} \sqrt{\dfrac{\pi}{2}} L^0(\Delta_{e_\perp}), & v_\| > \Delta_{e_\|} \\ \dfrac{v_\|}{\Delta_{e\|}} \sqrt{\dfrac{2}{\pi}} L^0(\Delta_{e_\|}), & v_\| < \Delta_{e_\|} \end{cases} \tag{4.3}$$

From Ya. S. Derbenev and A. N. Skrinsky, “The Effect of an Accompanying Magnetic Field on Electron Cooling”, Particle Accelerators, 1978, Vol. 8, pp. 235-243

Adiabatic collisions [Ya.S. Derbenev, A.N. Skrinsky]

- When $v_{ion} \gg \Delta_{e\|}$, the distribution $f\left(\vec{v}_e\right)$ can be replaced by $\delta\left(v_{e\|}\right)$ and one obtains:

$$\mathbf{F}_\perp^A = -\frac{2\pi n Z^2 e^4}{m} L^A(v) \frac{v_\perp^2 - 2v_\|^2}{v^2} \cdot \frac{\mathbf{v}_\perp}{v^3}; \quad (3.2)$$

$$F_\|^A = -\frac{6\pi n Z^2 e^4}{m} L^A(v) \frac{v_\perp^2}{v^2} \cdot \frac{v_\|}{v^3}; \quad (3.3)$$

- When $v_{ion} < \Delta_{e\|}$, the distribution $f\left(\vec{v}_e\right)$ is the Maxwellian, and one obtains:

$$\mathbf{F}_\perp^A = -2\sqrt{2\pi} \frac{nZ^2e^4}{m\Delta_{e_\|}^3} \mathbf{v}_\perp \ln\left(\frac{\Delta_{e_\|}}{v_\perp}\right) L^A(\Delta_{e_\|}); \quad (3.7)$$

$$F_\|^A = -2\sqrt{2\pi} \frac{nZ^2e^4}{m\Delta_{e_\|}^3} v_\| L^A(v_\perp); \quad (3.8)$$

From Ya. S. Derbenev and A. N. Skrinsky, "The Effect of an Accompanying Magnetic Field on Electron Cooling", Particle Accelerators, 1978, Vol. 8, pp. 235-243

Electrons thermal effects: numerical integration

One needs to solve the following integrals

$$\mathcal{F}_{\text{Unmagnetized}} = \iiint \left[\frac{\vec{U}}{U^3} \right] f\left(\vec{v}_e\right) d\vec{v}_e \qquad [1/c^2],$$

$$\mathcal{F}_{\text{Magnetized}} = \int \left[\frac{U_{B\perp}^2}{U_B^5} \left(\vec{U}_{B\parallel} + \frac{\vec{U}_{B\perp}}{2} \left(1 - \frac{U_{B\parallel}^2}{U_{B\perp}^2} \right) \right) \right] f\left(v_{e\parallel}\right) dv_{e\parallel} \qquad [1/c^2].$$

I chose to solve them numerically, using a Monte Carlo technique.

Computationally it is relatively fast (about 10 seconds), and it's performed just once. The advantage is that the solution is physically exact.

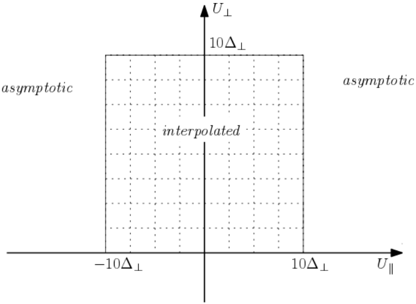
For large relative velocities, $\vec{U}$, the thermal effects of the electrons are negligible and the forces are directly computed using the force components described in the previous section, which can be considered as asymptotic expressions of the force.

Electrons thermal effects: interpolation

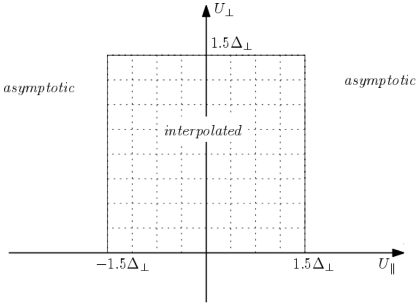
The force is integrated numerically, and stored in a bi-dimensional mesh 160×80. The force is interpolated at run time using a bi-cubic splines.

$$-10\Delta_\perp \le U_\parallel \le 10\Delta_\perp$$
$$0 \le U_\perp \le 10\Delta_\perp$$

$$-1.5\Delta_\perp \le U_\parallel \le 1.5\Delta_\perp$$
$$0 \le U_\perp \le 1.5\Delta_\perp$$



(unmagnetised force)

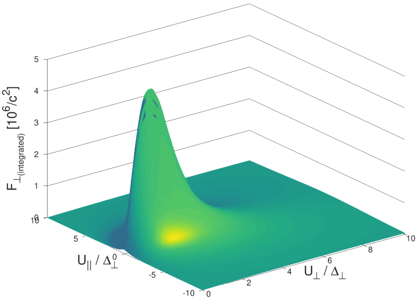


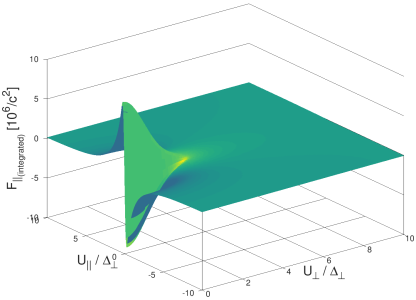
(magnetised force)

Unmagnetised force

The plots show the result of the integration, over the range:

$$-10\mathbf{\Delta}_\perp \leq U_{||} \leq 10\mathbf{\Delta}_\perp$$
$$0 \leq U_\perp \leq 10\mathbf{\Delta}_\perp$$

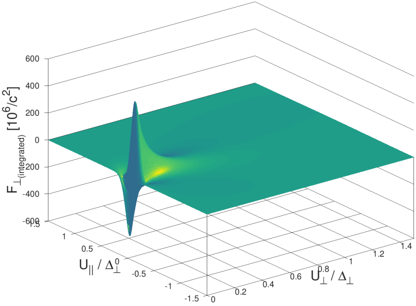


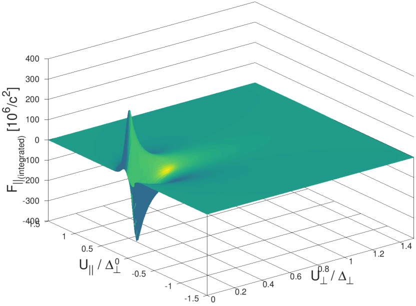


Magnetised force

The plots show the result of the integration, over the range:

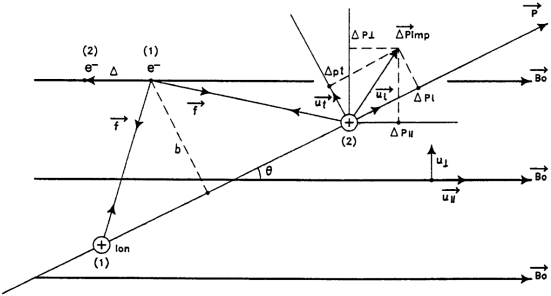
$$-1.5\Delta_\perp \le U_{||} \le 1.5\Delta_\perp$$
$$0 \le U_\perp \le 1.5\Delta_\perp$$





Part 2.

Implementation and Benchmark

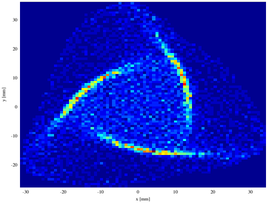


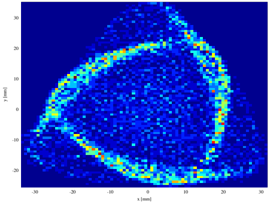
[CERN 92-01 - CAS 1991 - J. Bosser, “Electron Cooling”]

Implementation in RF-Track

- The EC simulation code has been included in RF-Track [1]: a fully relativistic, multi-species tracking code featuring space-charge and beam-beam

- RF-Track includes all tools, computational infrastructure, and physical models needed at the purpose:
  - Flexible beam definition
  - Equations of motion and their integration with a multitude of algorithms
  - Advanced field map interpolation
  - Modular, C++, fast, parallel, with friendly user interfaces
  - A dedicated element "Electron Cooler" has been implemented: it simulates the Coulomb interaction between a bunch of heavy particles and an arbitrary plasma.



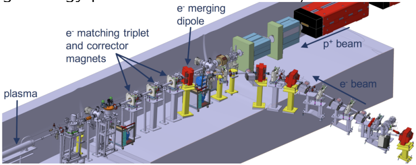


[1] A. Latina, "RF-Track: beam tracking in field maps with space-charge effects. Features and benchmarks", Proceedings of LINAC2016, East Lansing, MI, USA, MOPRC016, 2016.

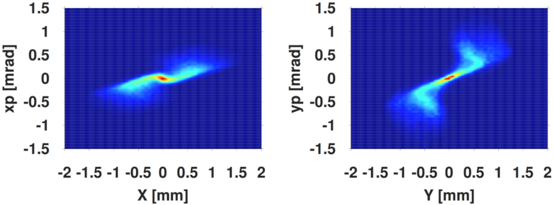
RF-Track example (I): AWAKE primary beam lines

[J. S. Schmidt et al., "Simulations of Beam-beam interactions with RF-Track for the AWAKE Primary Beam lines", Proceedings of IPAC2017, Copenhagen, Denmark, THPAB050]

The AWAKE project uses a high-energy proton beam at 400 GeV/c to drive wakefields in a plasma.



The amplitude of these wakefields will be probed by injecting into the plasma a low-energy electron beam (10-20 MeV/c), which will be accelerated to several GeV.

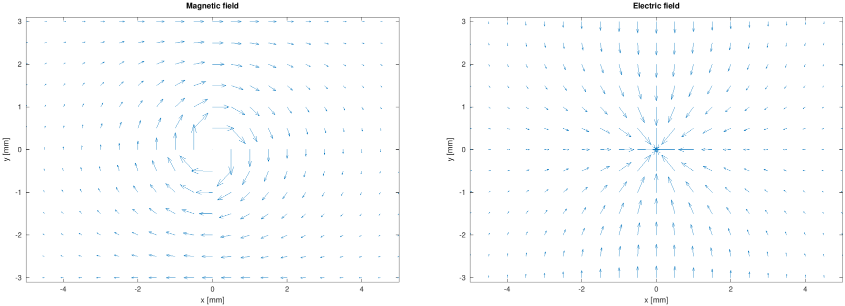


Upstream of the plasma cell the two beams will either be transported coaxially or with an offset of few millimetres for about 6 m. Figure shows the electron phase space at the focal point after propagation on axis with the $3 \cdot 10^{11}$ proton bunch.

RF-Track example (II): Indirect Space-Charge effects

Now RF-Track implements indirect space-charge effects from mirror charges, using the method of the Green's functions w/mirror charges:

- From horizontal parallel plates
- From a long longitudinal cylinder



Example of space-charge fields from horizontal parallel plates located at $y = \pm 3$ mm, due to a negatively charged beam, as calculated by RF-Track.

Electron Cooling "Hybrid" model: fluid electrons, kinetic ions

- The **ion beam** is represented as an ensemble of macro particles
  - full 6d phase space, e.g.
    $$\left(x,\ x',\ y,\ y',\ t,\ P\right)^T$$
    for accurate tracking and for capturing non linearities
  - integrate the effect of cooling force + solenoidal magnetic field, in Δz

- The **electron beam** is represented as a fluid (plasma) on a 3D cartesian mesh
  - it enables to consider arbitrary electron density / velocity distributions
  - each cell $(i,\ j,\ k)$ of the 3D is characterised by

$$\begin{array}{rl} n_{e,\ ijk} & \text{electron density } [\#/\mathrm{m}^3] \\ \vec{v_{ijk}} & \text{average electron velocity } [\mathrm{c}] \\ \Delta_{e\perp,\ ijk} & \text{electron transverse temperature} \\ \Delta_{e\|,\ ijk} & \text{electron longitudinal temperature} \end{array}$$

  - automatic tri-cubic interpolation of each quantities, to work at any arbitrary location (e.g. ion positions)
  - electron evolution follows the Euler equation of fluid dynamics, in Δt (to be implemented)

- Embedded in a solenoidal magnetic field (next step: use a measured / numerical field map)

Example of simulation script (Octave)

```
%% Load RF_Track
RF_Track;

%% Electron beam parameters
Electrons.Ne = 8.27e13; % electron number density #/m^3
Electrons.beta = 0.094; % c
Electrons.Q = -1; % e

%% Cooler parameters
Cooler.L = 3; % m
Cooler.B = 0.07; % T
Cooler.r0 = 0.025; % m, electron beam radius

%% Create an element of type ElectronCooler
% electron mesh size
Nx = Ny = Nz = 16;

% eletron velocity
Vx = Vy = 0;
Vz = Electrons.beta;

EC = ElectronCooler(Cooler.L, Cooler.r0, Cooler.r0);
EC.set_Q(Electrons.Q);
EC.set_electron_mesh(Nx, Ny, Nz, Electrons.Ne, Vx, Vy, Vz);
EC.set_static_Bfield(0.0, 0.0, Cooler.B);

%% Track bunch B0 through the cooler
B1 = EC.track(B0);
```

Benchmark of the cooling force [literature/BETACOOL]

The RF-Track implementation of the cooling has been benchmarked against the experimental results proposed in [Nersisyan]. Their model was checked against measurements and against predictions obtained with the model presented in [Parkhomchuk], which appears in BETACOOL.

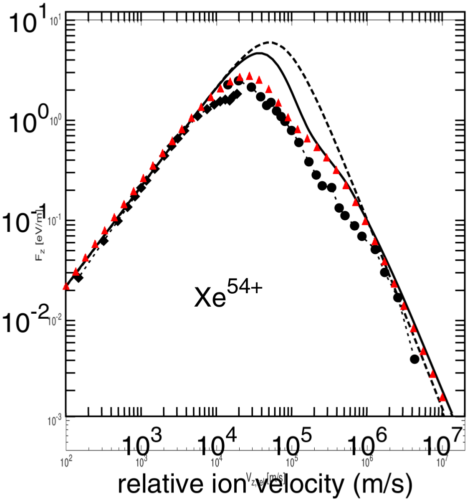
The longitudinal cooling force was evaluated for various fully stripped Xe ions as function of the relative ion velocity with respect to the rest frame of the electron beam, obtained from measurements at the electron cooler of the ESR storage ring.

$$
\begin{aligned}
n_e &= 10^{12}\ \mathrm{m}^{-3} \\
k_B T_\perp &= 0.11\ \mathrm{eV} \\
k_B T_\perp &= 0.1\ \mathrm{meV} \\
B &= 0.1\ \mathrm{T}
\end{aligned}
$$

The transverse ion velocity $v_{i\perp}$ is treated as a free parameter.

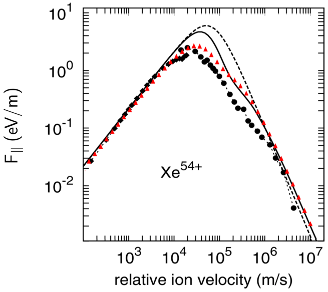
[Parkhomchuk] V. V. Parkhomchuk, *Nucl. Instrum. Meth. Phys. Res. A*, **441**, 9 (2000).

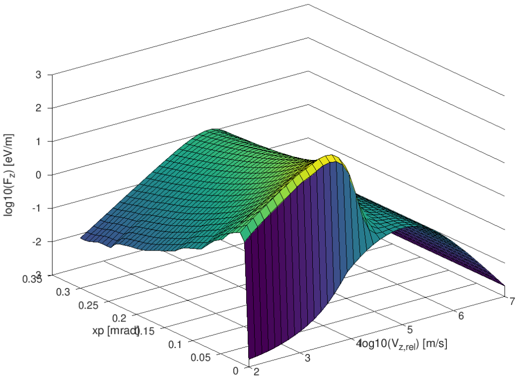
Benchmark of the cooling force



Longitudinal cooling force for various fully stripped Xe^{54+} ions as function of the relative ion velocity with respect to the rest frame of the electron beam. **Black marks**: experimental data. **Solid curve**: binary collision approximation. **Dashed curve**: Parkhomchuk's empiric formula (BETACOOL). **Red triangles**: RF-Track simulation

Dependence on the transverse velocity





The number which best fits the data is $\langle x' \rangle \equiv \left\langle v_{i\perp}/v_{i||} \right\rangle = 0.3$ mrad, compatible with the estimated beam divergence [Nersisyan].

Measurement of the cooling force [LEIR]

We measured the cooling force at LEIR, with Pb ions, $Q = 54^{+}$.

- The cooler was set with electron beam current 210 mA, with temperatures

  $$k_B T_{\perp} = 0.1 \text{ eV and}$$

  $$k_B T_{\parallel} = 1 \text{ meV},$$

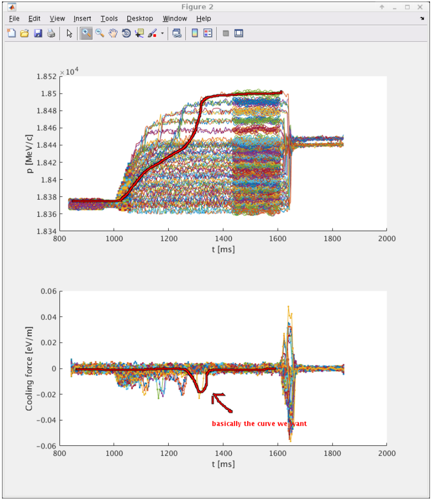
  in a solenoid magnetic field of $B = 0.07$ T

- We performed a velocity scan of the electrons, in order to evaluate the force at different

  $$\Delta V = v_i - v_e$$

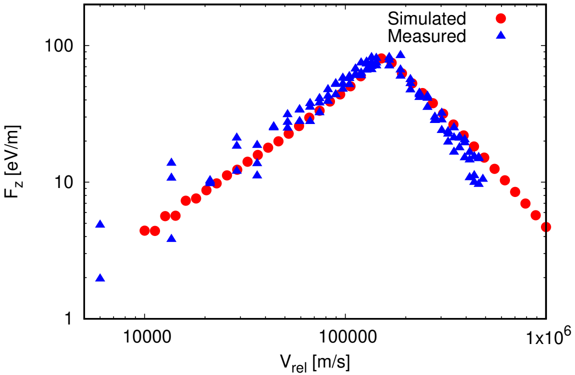
- The cooling force is computed as

  $$F = \frac{\Delta P}{\Delta t}$$



[Measurements taken with the help of N. Biancacci, D. Gamba, and A. Saa Hernandez]

Benchmark of the cooling force at LEIR



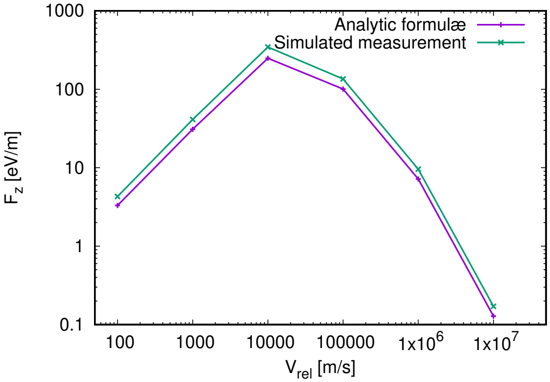
The Cooler's parameters have need to be adjusted to match the experimental curve, anyway to reasonable values:

$N_e = 4 \times 10^{13}\ e^-/\mathrm{m}^3$; $T_\perp = 0.01$ eV ; $T_\parallel = 0.001$ eV (note that the e^- current depends on electron beam size, but the force depends on the electron density)

[Acknowledgments: N. Biancacci, D. Gamba, and A.Saa Hernandez]

Simulated measurement

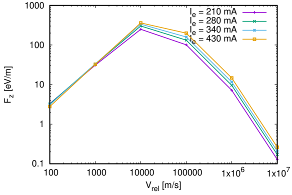
A simulation of the measurement itself has been put in place (self-consistency test)



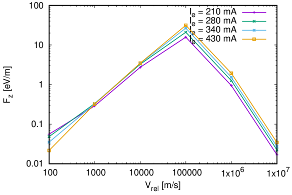
1. 50 ms of cooling before the electron velocity difference is excited
2. The estimate of the cooling force is computed considering the first 1 ms

Simulated measurement: dependence from electron current

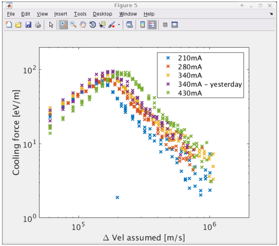
A simulation of the measurement was performed for different electron currents:



uniform distribution (simulation)



hollow distribution (simulation)

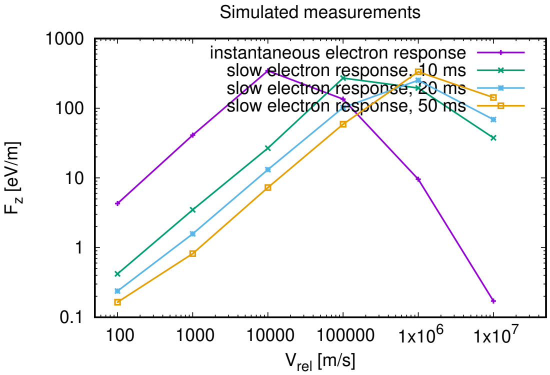


measurement

[Acknowledgments: N. Biancacci, D. Gamba, and A.Saa Hernandez]

Simulated measurement: delayed electron response

A simulation of the measurement was performed including delays in the electron gun's response



Simulation: from E-cooling to E-lens

RF-Track implements E-cooling.

- The simulation considers
  - Scattering ion-electrons (friction force)
  - Ions self-fields in free space
- It ignores:
  - Electrons self-fields
  - Electrons dynamics, electrons are considered as rigid (i.e., their state parameters are constant)

In order to simulate E-lens should also consider

- The self-fields created by the electrons (mostly magnetic)
  - This is already possible, using multi-species beams, but it is impractical for realistic simulations
    $\Rightarrow$ Extended the plasma simulation to feature such effects (some work needed)
  - The effect of the ion bunch on the electron plasma
    $\Rightarrow$ Implement the Euler's equations of fluid dynamics (some work needed)

To complete the picture: the shieling effect of the electrons on the ions self-fields (some work needed)

Conclusions and Outlook

- Electron-cooling has been simulated in RF-Track
  - New model of cooling force
  - New method for including thermal effects

- Benchmark of the cooling force
  - against results in literature/BETACOOL has shown very good matching
  - against measurements at LEIR shows very reasonable results (many unknowns on the experimental conditions)

- Electron-lens simulation:
  - It's within reach, some code development is needed

Extras

New formulation of the cooling force

The book performs critique of the mentioned papers, and even goes in the detail of the equations implemented in BETACOOL:

>
>
> As was already noted above this energy transfer vanishes for symmetry reasons unless the ion has a velocity component transverse to the magnetic field. For the special case $v_{e\|} = 0$ this was previously derived in [120]. Similar expressions with the unexplicable replacements $2\bar{v}_{r\|} \rightarrow 3\bar{v}_{r\|}$ and $1 - \bar{v}_{r\|}^2/\bar{v}_{r\perp}^2 \rightarrow 1 - 2\bar{v}_{r\|}^2/\bar{v}_{r\perp}^2$ are used in [35, 85]. For the energy loss (2.9) an integration with respect to the impact parameter has to be done. The integrand can be approximated piecewise by (2.8) for $\bar{s} < a$ and by (2.32) ($\bar{s} < \delta$) or (2.36) ($\bar{s} > \delta$) for $\bar{s} > a$. This leads to Coulomb log-

[35] Ya.S. Derbenev, A.N. Skrinsky: Part. Accel. **8**, 235 (1978)
[85] I.N. Meshkov: Phys. Part. Nuclei **25**, 631(1994)

The new cooling force: non-magnetised electrons

In the reference frame of the electrons

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} \iiint \left[L_C \frac{\vec{U}}{U^3} \right] f\left(\vec{v}_e\right) d\vec{v}_e$$

where $K = \dfrac{Q_{ion} e^2}{4\pi\epsilon_0}$, $\mu \approx m_e$ is the reduced mass, $\vec{U}$ is the relative ion-electron velocity,

$$\vec{U} = \vec{V}_i - \vec{V}_e$$

and $f\left(\vec{v}_e\right)$ is the electron velocity distribution function, typically a Maxwellian:

$$f\left(\vec{v}_e\right) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{1}{\Delta_\perp^2 \Delta_\parallel} \exp - \left(\frac{v_{e_\perp}^2}{2\Delta_\perp^2} + \frac{v_{e_\parallel}^2}{2\Delta_\parallel^2} \right).$$

If $\vec{V}_i \gg \Delta_e$ (asymptotic behaviour), $\vec{U} \approx \vec{V}_i$ and the integral is no longer needed:

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} L_C \frac{\vec{U}}{U^3}.$$

L_c is the *modified* Coulomb logarithm (see next slide).

The new cooling force: non-magnetised electrons

The *modified* Coulomb logarithm, L_C, is:

$$L_C = \frac{1}{2} \log \left(1 + \frac{r_{\mathrm{max}}^2}{r_{\mathrm{min}}^2} \right)$$

If compared with the standard Coulomb logarithm, $L_C = \log \left(\frac{r_{\mathrm{max}}}{r_{\mathrm{min}}} \right)$, the modified Coulomb logarithm avoids unphysical results for $r_{\mathrm{max}} < r_{\mathrm{min}}$ and accounts for head-on ion-electron collisions. [Nersisyan]

Here:

$$r_{\mathrm{max}} = \min \left(r_{\mathrm{aperture}},\ \lambda_D \sqrt{1 + \frac{U^2}{\Delta_e^2/3}},\ U \Delta t \right) \qquad \text{maximum impact parameter}$$

$$r_{\mathrm{min}} = \frac{K}{\mu U^2} \qquad \text{minimum impact distance}$$

The new cooling force: magnetised electrons

There are three contributions to the cooling force:

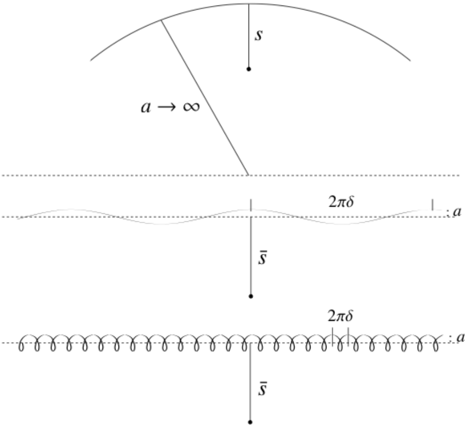


Fig. 2.5. Schematic trajectories of relative motion. Top: Weak field, $s \ll a$. Center: Stretched helices in case of a strong field, $\bar{s} \gg a$, $\bar{s} \ll \delta$. Bottom: Tight helices in case of a strong field, $\bar{s} \gg a$, $\bar{s} \gg \delta$. After [124].

The new cooling force: small impact distance

1. *(small impact parameters)* $r_{\min} \to r_F$

   The effect of the magnetic field is negligible:

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} \iiint \left[L_F \frac{\vec{U}}{U^3} \right] f\left(\vec{v}_e\right) d\vec{v}_e$$

   with

$$\vec{U} = \vec{V}_i - \vec{V}_e,$$

   - $L_F = \frac{1}{2} \log \left(1 + \frac{r_F^2}{r_{\min}^2} \right),$
   - $f\left(\vec{V}_e\right)$ is again a Maxwellian distribution, and
   - $r_F = \frac{\sqrt{U_{B\parallel}^2 + \Delta_\parallel^2}}{\omega_e}$ is the pitch of the electron helix
   - $r_{\min} = \frac{K}{\mu U^2}$ is the minimum impact parameter

The new cooling force: intermediate impact distance

2 *(intermediate impact parameters)* $r_F \to r_L$.

Being $\vec{V}_{e\|}$ the component of the electrons' velocity parallel to the magnetic field, we define:

$$\vec{V}_{e\perp} = \vec{V}_e - \vec{V}_{e\|},$$

the force is

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} \int \left[L_A \frac{\vec{U}_B}{U_B^3} \right] f\left(v_{e\|}\right) dv_{e\|}$$

with

$$\vec{U}_B = \vec{V}_i - \vec{V}_{e\|},$$
$$U_{B\perp} = V_{i\perp}$$
$$U_{B\|} = V_{i\|} - V_{e\|}$$

- The Coulomb logarithm is, $L_A = \frac{1}{2} \log\left(1 + \frac{r_L^2}{r_{\mathrm{F}}^2}\right)$,
- $f\left(v_{e\|}\right) = \frac{1}{\sqrt{2\pi}\Delta_\|} \exp - \frac{v_{e\|}^2}{2\Delta_\|^2}$,
- $r_L = \frac{\sqrt{V_{e\perp}^2 + \Delta_\perp^2}}{\omega_e}$ is the Larmor (or cyclotron) radius of the electron.

The new cooling force: large impact distance

3 *(large impact parameters)* $r_L \to r_{\max}$

Eq. (2.36) in Nersisyan,

$$\vec{F} = -\frac{4\pi n_e K^2}{\mu} \int \left[L_M \frac{U_{B\perp}^2}{U_B^5} \left(\vec{U}_{B\parallel} + \frac{\vec{U}_{B\perp}}{2} \left(1 - \frac{U_{B\parallel}^2}{U_{B\perp}^2} \right) \right) \right] f\left(v_{e\parallel}\right) dv_{e\parallel},$$

- with $\vec{U}_B$ and $f\left(v_{e\parallel}\right)$ like in the previous case,
- $L_M = \log\left(\frac{r_{\max}}{r_L}\right)$,
- $r_{\max}$ is the maximum impact parameter (~Debye length)

The asymptotic behaviour for $U_{B\parallel} \gg \Delta_\parallel$, i.e. $\vec{U}_B \approx \vec{V}_i$, is:

$$F_\parallel = -\frac{4\pi n_e K^2}{\mu} L_M \frac{U_{B\perp}^2}{U_B^5} U_{B\parallel},$$

$$F_\perp = -\frac{4\pi n_e K^2}{\mu} L_M \frac{U_{B\perp}^2}{U_B^5} \frac{U_{B\perp}}{2} \left(1 - \frac{U_{B\parallel}^2}{U_{B\perp}^2} \right).$$