New BKM Lie Algebras from Counting Twisted Dyons

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Work done with Sutapa Samanta (to appear)

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Plan

Introduction

$su(3)$: a toy example

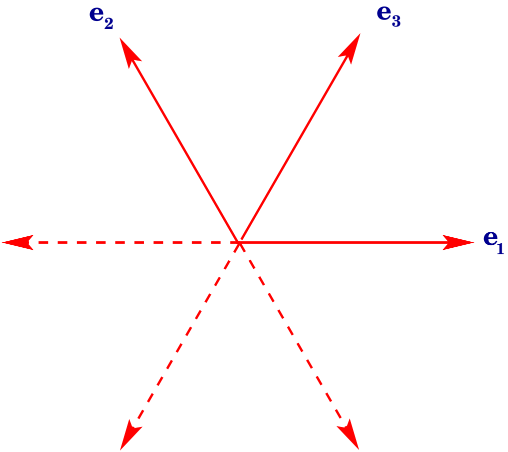
- The Lie algebra of $su(3)$ can be decomposed as

$$su(3) = \mathcal{H} \oplus \mathcal{R}_+ \oplus \mathcal{R}_- \ ,$$

  where $\mathcal{H}$ is the Cartan sub-algebra; $\mathcal{R}_+ = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ are the positive roots; $\mathcal{R}_- = (-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3) = -\mathcal{R}_+$ are the negative roots.$(\mathbf{e}_1, \mathbf{e}_2)$ are simple roots.
- The Cartan matrix which is the matrix of inner proucts of the simple roots is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.
- The Weyl group, $\mathcal{W}$, is generated by elementary reflections of the simple roots. It is isomorphic to S_3.
- The Weyl vector, ρ, is defined as the half the sum of all positive roots. One has $\rho = e_3$ for $su(3)$.

Weyl group and chamber for $su(3)$

The roots can be shown as as vectors in a two-dimensional Euclidean space as $su(3)$ is a rank-two Lie algebra.



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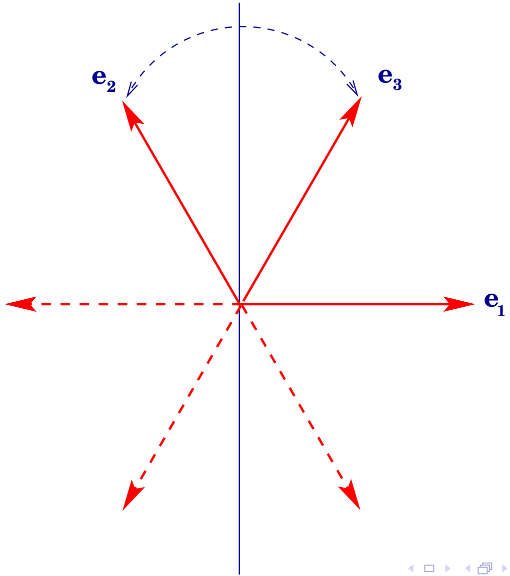

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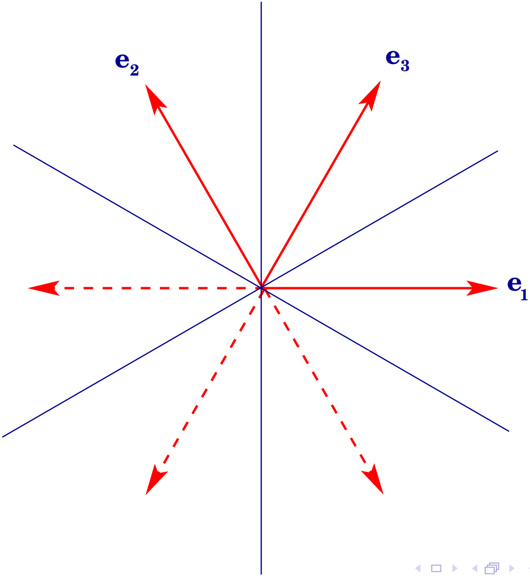

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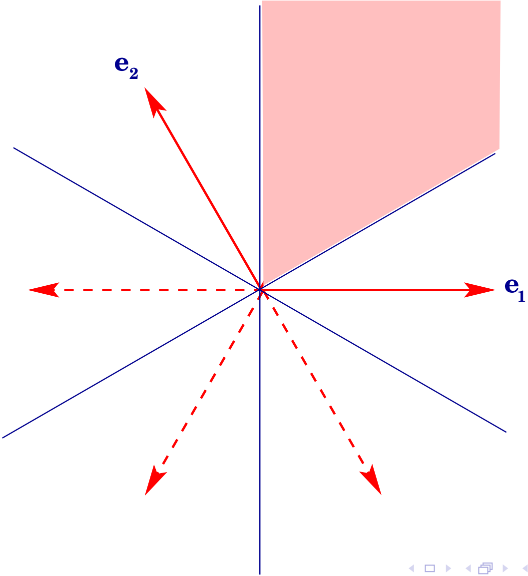


The Weyl denominator formula

- Recall the denominator of the character formula

$$\Sigma \equiv \sum_{w \in \mathcal{W}} (-1)^w \exp[w(\rho)]$$

- For $su(3)$, this reads

$$\Sigma = e^{\mathbf{e}_3} - e^{\mathbf{e}_2} + e^{-\mathbf{e}_1} - e^{-\mathbf{e}_3} + e^{-\mathbf{e}_2} - e^{\mathbf{e}_1}$$

Writing $x = e^{-\mathbf{e}_1}$ and $y = e^{-\mathbf{e}_2}$ and $xy = e^{-\mathbf{e}_3}$, we get

$$\Sigma = (xy)^{-1} - x^{-1} + x - xy + y - y^{-1} = \frac{(1-x)(1-y)(1-xy)}{xy}$$

Note that the appearance of a product running over positive/negative roots.

The Weyl denominator formula

- The appearance of the product is a generic feature of Lie algebras.
- Let

$$\Pi = \prod_{\alpha \in \mathcal{R}_+} \Big(1 - \exp\left[-\alpha\right] \Big)^{\mathrm{mult}(\alpha)} \quad ,$$

where we allow for roots with multiplicity unlike $su(3)$ where all roots appear with multiplicity one.

- Re-defining Σ by multiplying by $e^{-\rho}$, we write

$$\Sigma = \sum_{w \in \mathcal{W}} (-1)^w \exp[w(\rho) - \rho]$$

- One then has the identity: $\Sigma = \Pi$

Lie Algebras – a historical perspective

- The classification of semi-simple Lie algebras was completed by Cartan and the fall into four infinite series (A_n, B_n, C_n, D_n) and the exceptional ones: E_6, E_7, E_8, F_4 and G_2.
- The first generalization due to Kac and Moody (KM) can be understood from the Cartan matrix (the matrix of inner products between the simple roots). The Cartan matrix of the above series is positive definite. If we permit zero eigenvalues, we are lead to the affine Kac-Moody Lie algebras. The Dynkin diagrams are obtained by adding an extra node the one's constructed by Cartan.
- If we complete relax the condition, i.e., permit negative eigenvalues, we get Kac-Moody Lie algebras. These are not well understood. Given such a Cartan matrix, we do no a priori know the multiplicity of roots.
- Borcherds carried out a generalization that enables him to obtain the multiplicities for a new class of Lie algebras – the exceptional BKM Lie algebras.

The models of interest – CHL orbifolds

- This talk will show the appearance of BKM Lie Algebras, some new, from counting BPS states in string theory.
- We will focus on a family of $\mathcal{N}=4$ supersymmetric string theories in $d=4$, the CHL models, that arise as asymmetric $\mathbb{Z}_N$ orbifolds of the heterotic string on T^6
- The vector multiplet moduli space is given by

$$(\lambda, M) \in \left(\Gamma_1(N) \Big\backslash \frac{SL(2)}{U(1)}\right) \times \left(SO(6,p;\mathbb{Z}) \Big\backslash \frac{SO(6,p)}{SO(6)\times SO(p)}\right) ,$$

where p is determined from the orbifold action.

- $SO(6,p;\mathbb{Z})$ is the T-duality symmetry group and the S-duality group $\Gamma_1(N) \subset PSL(2,\mathbb{Z})$ given by

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, c = 0 \bmod N, \quad a = d = 1 \bmod N \right\}$$

The models of interest – CHL orbifolds

- The electric and magnetic charges, $(\mathbf{q}_e, \mathbf{q}_m)$ transform as $(p+6)$-dimensional vectors under the T-duality group.
- The quantization of the charges in terms of T-duality invariants is such that

$$N\frac{\mathbf{q}_e^2}{2} \in \mathbb{Z} \quad , \quad \mathbf{q}_e \cdot \mathbf{q}_m \in \mathbb{Z} \quad , \quad \frac{\mathbf{q}_m^2}{2} \in \mathbb{Z} \ .$$

  We will indicate these integers, respectively, by (n, ℓ, m).
- The three invariants transform as a triplet under $PSL(2, \mathbb{Z})$.
- For these models, one has $\frac{1}{2}\mathbf{q}_e^2 \geq -\frac{1}{N}$ and $\frac{1}{2}\mathbf{q}_m^2 \geq -1$.
- For $\frac{1}{2}$-BPS states one has $\mathbf{q}_m \propto \mathbf{q}_e$ and hence $\frac{1}{4}$-BPS states are necessarily dyonic while states carrying only electric charge are necessarily $\frac{1}{2}$-BPS states.

Generating functions from counting

- Let $d(n)$ denote the microscopic degeneracy of electrically charged $\frac{1}{2}$-BPS states with charge $\mathbf{q}_e^2/2 = n/N$. Let

$$\frac{16}{g(\tau/N)} = \sum_{n=-1}^{\infty} d(n)\ q^{n/N}\ ,$$

  where $q = \exp(2\pi i\tau)$. $g(\tau)$ is a modular form of a level N sub-group of $PSL(2,\mathbb{Z})$

- Similarly, let $D(n,\ell,m)$ denote the microscopic degeneracy of dyonic $\frac{1}{4}$-BPS states with charges (n,ℓ,m). Let

$$\frac{64}{\Phi(\mathbf{Z})} = \sum_{(n,\ell,m)} D(n,\ell,m)\ q^{n/N} r^{\ell} s^{m}\ ,$$

  where $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \sigma \end{smallmatrix}\right)$, $r = \exp(2\pi iz)$ and $s = \exp(2\pi i\sigma)$.

$\Phi^{(N)}(\mathbf{Z})$ turn out to be a genus-two Siegel modular forms.

Walls of Marginal Stability $\mathcal{N}=4$ $d=4$ string theory

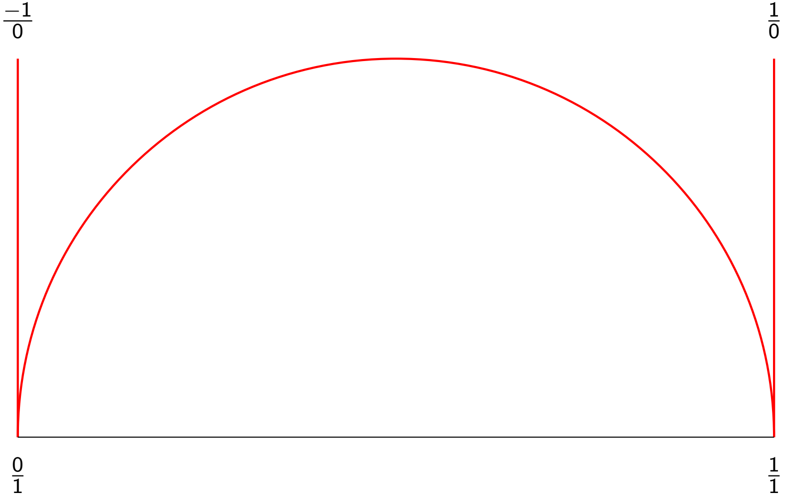
- In $\mathcal{N}=4$ $d=4$ string theory, $\frac{1}{4}$-BPS states can decay into two $\frac{1}{2}$-BPS states as one moves across a wall.
- Let λ denote the complex modulus for the heterotic dilaton-axion field.
- These walls are circular arcs in the upper half-plane given by

$$\left[\mathrm{Re}(\lambda)-\frac{ad+bc}{2ac}\right]^2+\left[\mathrm{Im}(\lambda)+\frac{\mathcal{E}}{2ac}\right]^2=\frac{1+\mathcal{E}^2}{4a^2c^2}\,,\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in PSL(2,\mathbb{Z})$$

  where $\mathcal{E}$ is a real function of all other moduli M.
- The arcs intersect the real λ axis at $\frac{b}{a}$ and $\frac{d}{c}$ for any $\mathcal{E}$.
- When $\mathcal{E}=0$, the arcs are semi-circles centred on the real λ-axis with radius $\frac{1}{2ac}$.
- When either $a=0$ or $c=0$, the circles become straight lines perpendicular to the real axis for $\mathcal{E}=0$.

An example: Heterotic string compactified on T^6

$$\mathcal{F}_1 = \tfrac{-1}{0}, \left(\tfrac{0}{1}, \tfrac{1}{1}\right), \tfrac{1}{0}$$



Counting BPS states: Heterotic string on T^6

- For the heterotic string on T^6, the generating function of (electrically charged) $\frac{1}{2}$-BPS states is

$$\frac{1}{\eta(\tau)^{24}} = \sum_n d(n)\ q^n$$

- Dijkgraaf-Verlinde[2] proposed that the degeneracy of $\frac{1}{4}$-BPS states is generated by a weight ten genus-two Siegel modular form, $\Phi_{10}(\mathbf{Z})$. One has

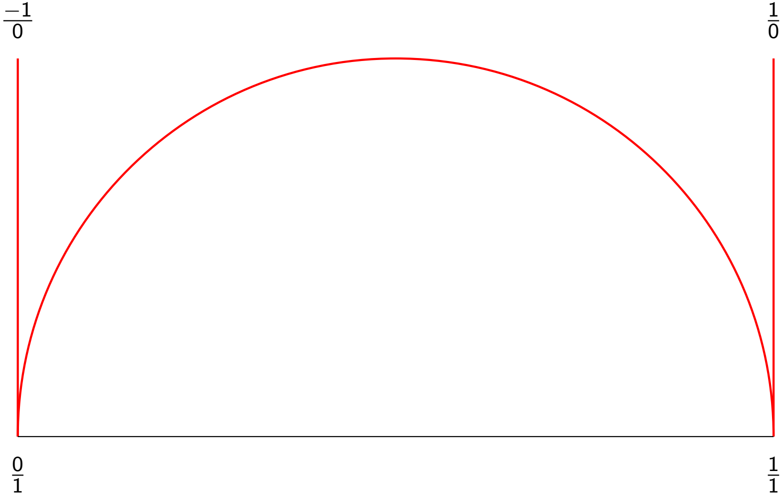
$$\frac{1}{\Phi_{10}(\mathbf{Z})} = \sum_{(n,\ell,m)>0} d(n,\ell,m)\ q^n r^\ell s^m$$

where $\mathbf{Z} = \left(\begin{smallmatrix} \tau & z \\ z & \tau' \end{smallmatrix}\right) \in \mathbb{H}_2$ and $(n,\ell,m) = (\frac{1}{2}\mathbf{q}_e^2, \mathbf{q}_e \cdot \mathbf{q}_m, \frac{1}{2}\mathbf{q}_m^2)$.

- The square root of $\Phi_{10}(\mathbf{Z})$ appears as the denominator formula for a Borcherds-Kac-Moody (BKM) Lie superalgebra.

Walls of the Weyl chamber

Cheng and Verlinde showed that the walls of the Weyl chamber of this BKM Lie superalgebra are mapped to the walls of marginal stablity of $\frac{1}{4}$-BPS dyons! An algebra of BPS states?



Kac-Moody Lie algebras

- The generators of a Lie algebra, $\mathcal{L}$, can be decomposed as:

$$\mathcal{L} = \mathcal{H} \oplus \mathcal{R}_+ \oplus \mathcal{R}_- \ ,$$

  $\mathcal{H}$: the Cartan sub-algebra and $\mathcal{R}_\pm$: the positive/negative roots.

- The simple roots e_{α_i} $(i = 1, 2, \ldots, r)$ provide a basis for $\mathcal{R}_+$.
- The Lie algebra in the Chevalley basis is

$$[h_{\alpha_i}, h_{\alpha_j}] = 0 \ , \ [h_{\alpha_i}, e_{\alpha_j}] = a_{ji} \ e_{\alpha_i} \ , \ [e_{\alpha_i}, e_{-\alpha_i}] = h_{\alpha_i} \ ,$$

  where $A = (a_{ij})$ is the Cartan matrix of the Lie algebra.

- The elements of $\mathcal{R}_+$ are generated by multiple commutators of the simple roots subject to the Serre relations:

$$\overbrace{[e_{\alpha_i}, [e_{\alpha_i}, \cdots, [e_{\alpha_i}}^{(1-a_{ij}) \text{ times}}, e_{\alpha_j}] \cdots]] = 0 \ \text{ for all } i \neq j \ .$$

Borcherds-Kac-Moody Lie algebras

Cartan Matrices and the Weyl group

- Cartan matrices are (i) typically symmetric, (ii) with 2 on the diagonals and (iii) negative/zero entries for the off-diagonal terms such that $a_{ij} = 0$ implies $a_{ji} = 0$.
- The Cartan matrix can be obtained as an inner product on the simple roots α_i via $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$.
- Given a simple root α_i (e_{α_i}) and β (e_β) any root, the Weyl reflection w_i is defined as

$$w_i(\beta) := \beta - 2\,\frac{(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\,\alpha_i \in \mathcal{R}_+ \cup \mathcal{R}_- \ .$$

$w_i(\alpha_i) = -\alpha_i \in \mathcal{R}_-$.

- The Weyl group, W, of a Lie algebra is defined to be the group generated by all Weyl reflections.

$$W = \langle w_1, w_2, \ldots, w_r \rangle \ .$$

The Weyl-Kac Denominator Formula

- The Weyl vector ρ has the property that $(\rho, \alpha_i) = \frac{1}{2}(\alpha_i, \alpha_i)$ for all simple roots. Check for $su(3)$
- It is easy to see that $(w_i(\rho) - \rho) = -\alpha_i \in \mathcal{R}_-$. More generally, for any $w \in W$, one has $w_i(\rho) - \rho) \in \mathcal{R}_-$.
- The Weyl-Kac Denominator formula is given by ($\mathcal{R}_-$ might include imaginary roots)

$$\sum_{w \in W} \det(w)\ e^{w(\rho)} = e^{\rho} \prod_{\alpha \in \mathcal{R}_-} (1 - e^{\alpha})^{\text{mult}(\alpha)}$$

- The LHS knows about the simple roots as they generate the Weyl group.
- The RHS provides details of the space of all roots. However, in general, it is hard to determine the multiplicities of roots.
- For affine KM algebras, the answer is known by connecting the denominator formula to Jacobi forms. [MacDonald]

Borcherds-Kac-Moody Lie algebras

- Borcherds addressed this multiplicity problem by adding imaginary simple roots to KM algebras. Imaginary roots have norm ≤ 0.
- The diagonal elements in the (extended) Cartan matrix now have non-positive entries.
- The denominator formula gets modified leading to the Weyl-Kac-Borcherds denominator formula

$$\Delta = \sum_{w \in W} \det(w) w \Big(\sum_{\alpha \in R_-^{\text{im}} \cup 0} \epsilon(\alpha) e^{\rho+\alpha} \Big) = e^{\rho} \prod_{\alpha \in \mathcal{R}_-} (1 - e^{\alpha})^{\text{mult}(\alpha)}$$

- Borcherds adds imaginary simple roots such that the above sum/product becomes a suitable modular form, Δ, (the automorphic correction).
- Such modular forms admit product formulae ("Borcherd products") leading to an explicit determination of root multiplicities.

A family of examples:

- Consider the rank-three KM Lie algebra with Cartan matrix

$$A^{(1)} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix} .$$

- The Siegel modular form $\Delta_5(\mathbf{Z})$ provides an automorphic correction of the above KM Lie algebra. [Gritsenko-Nikulin]
- The multiplicities of imaginary simple roots are easily determined by the zeroth Fourier-Jacobi coefficient of $\Delta_5(\mathbf{Z})$.
- $\Delta_5(\mathbf{Z})$ is the square-root of the generating function of $\frac{1}{4}$ BPS states in the heterotic string compactified on T^6.

Walls of Marginal Stability in CHL models

CHL orbifolds of heterotic string on T^6

- There exist a family of $\mathbb{Z}_N$ $(N = 1, 2, \ldots, 8)$ asymmetric (CHL) orbifolds of the heterotic string compactified on the six-torus that preserve $\mathcal{N} = 4$ supersymmetry.
- The generating function of $\frac{1}{2}$-BPS states is given by a multiplicative eta product. [SG-Gopalakrishna]
- The electric charges are quantised such that $\frac{N}{2}\mathbf{q}_e^2 \in \mathbb{Z}$.
- The generating function of $\frac{1}{4}$-BPS states is given by a Siegel modular form of the level N sub-group of $Sp(2, \mathbb{Z})$. One has

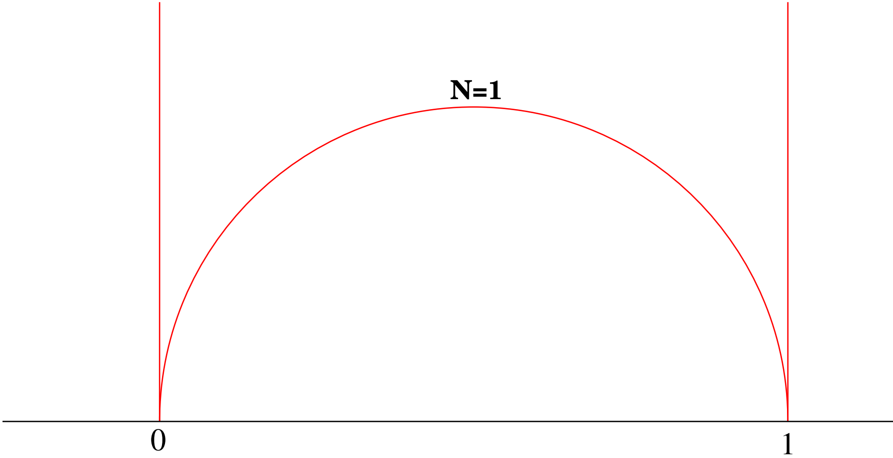
$$\frac{1}{\Phi_k^{(N)}(\mathbf{Z})} = \sum_{(n,\ell,m)>0} d(n,\ell,m)\; q^{n/N} r^{\ell} s^m$$

$(n, \ell, m) = (\frac{N}{2}\mathbf{q}_e^2, \mathbf{q}_e \cdot \mathbf{q}_m, \frac{1}{2}\mathbf{q}_m^2)$. [Jatkar-Sen,SG-Gopalakrishna]

- The square-root of the Siegel modular form makes sense for $N = 1, 2, \ldots, 6$. [Cheng-Dabholkar,SG-Gopalakrishna,SG-Samanta]

Walls of marginal stability for $N = 1, 2, 3$

Sen obtained the following walls of marginal stability.



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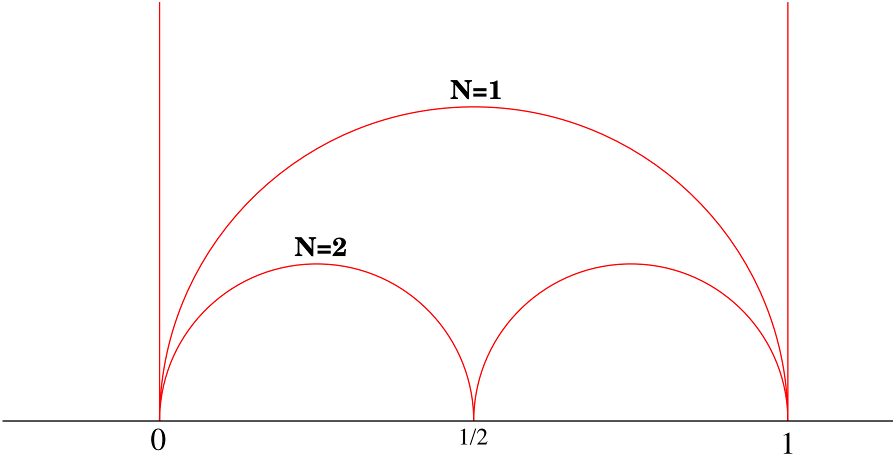

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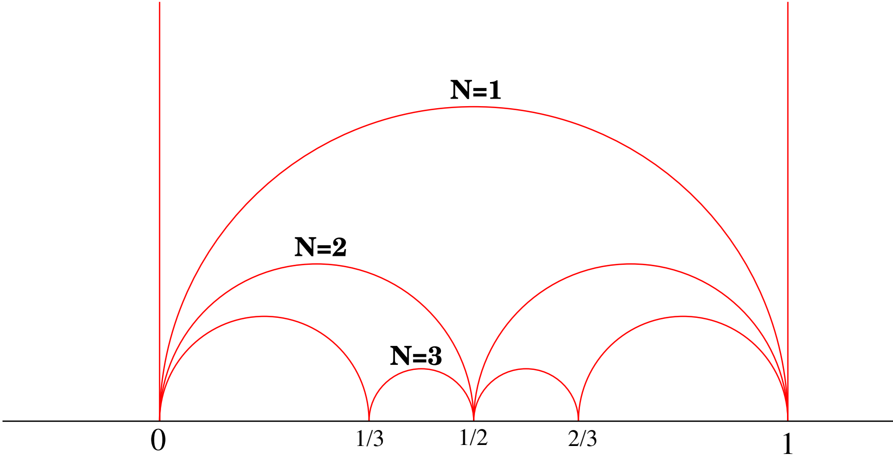
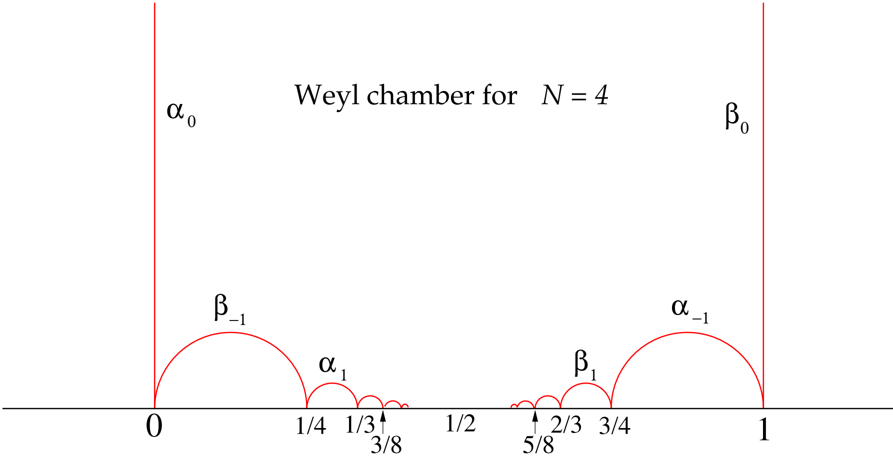


The polygons have finite number of edges: 3,4,6.

Walls of marginal stability for $N = 4$

Sen obtained the following walls of marginal stability.



The polygon has infinite edges. $\frac{1}{2}$ is reached as a limit point.

Walls of marginal stability for $N = 5$

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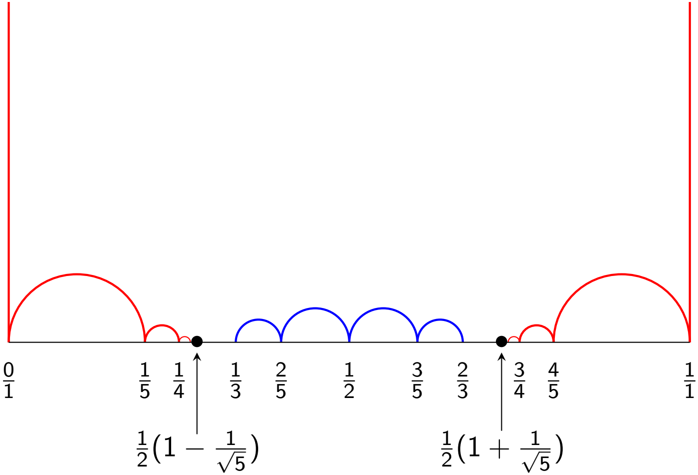
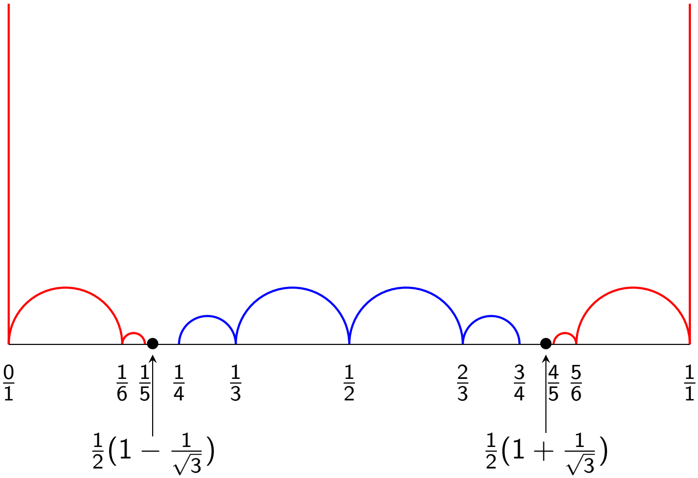


The polygon has infinite edges. $\frac{1}{2}$ is not reached as a limit point.

Walls of marginal stability for $N = 6$

Sen obtained the following walls of marginal stability.



The polygon has infinite edges. $\frac{1}{2}$ is not reached as a limit point.

Walls of Marginal Stability to Lie algebras

Towards KM Lie algebras

- Let us assume that there exists a Lie algebra whose Weyl chamber is identical the one obtained by the interior of the polygon defined by the walls of marginal stability.
- To each edge with vertices $(\frac{b}{a}, \frac{d}{c})$, following Cheng-Verlinde, we can associate a (real) simple root as follows:

$$\left(\frac{b}{a}, \frac{d}{c}\right) \longleftrightarrow \alpha = \begin{pmatrix} 2bd & ad+bc \\ ad+bc & 2ac \end{pmatrix} \in PGL(2,\mathbb{Z}) \ .$$

- The norm of the root is given by $-2\det(\alpha)$.

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- The norm of the root is given by $-2\det(\alpha)$.
- The Cartan matrix of the roots can be determined by the inner product induced by the norm. One obtains:

$$A^{(2)} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix} \ ,$$

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- The Cartan matrix of the roots can be determined by the inner product induced by the norm. One obtains:

$$A^{(3)} = \begin{pmatrix} 2 & -2 & -10 & -14 & -10 & -2 \\ -2 & 2 & -2 & -10 & -14 & -10 \\ -10 & -2 & 2 & -2 & -10 & -14 \\ -14 & -10 & -2 & 2 & -2 & -10 \\ -10 & -14 & -10 & -2 & 2 & -2 \\ -2 & -10 & -14 & -10 & -2 & 2 \end{pmatrix} ,$$

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- The norm of the root is given by $-2\det(\alpha)$.
- The Cartan matrix of the roots can be determined by the inner product induced by the norm. One obtains:

$$A^{(4)} = 2 - 4(n-m)^2 \ , \ \text{with } m,n \in \mathbb{Z} \ .$$

All Cartan matrices, $A^{(N)}$, for $N = 1,2,3,4$ are rank 3.

The BKM Lie algebras for $N \leq 4$

- The Cartan matrices, $A^{(N)}$ can be used to construct Kac-Moody Lie algebras.
- The associated root lattices, $\oplus_i \mathbb{Z}\alpha_i$, fit Nikulin's classification of rank-three hyperbolic lattices which admit a Lattice Weyl vector.
- Further, the square-root of the $\frac{1}{4}$-BPS generating function, $\Delta^{(N)}(\mathbf{Z})$, provides an automorphic correction to these KM Lie algebras leading to BKM algebras.
- The cases of $N = 2, 3$ appear in the work of Gritsenko-Nikulin on rank-three hyperbolic Lie algebras of elliptic type. [Cheng-Dabholkar].
- For $N = 4$, it fits the general classification of Gritensko and Nikulin and corresponds to rank-three hyperbolic Lie algebras of parabolic type. [SG-Gopalakrishna]
- The modularity of the $\Delta^{(N)}(\mathbf{Z})$ is crucial in proving that denominator identity.

The BKM Lie algebras for $N = 5, 6$

- For $N = 5, 6$, we need to include walls from the central part to get a closed polygon.
- The corresponding root lattices are rank-three hyperbolic lattices with a Lattice vector. The norm of the Weyl vector determines the type of lattice and it is of hyperbolic type.
- There is a no-go theorem of the Gritsenko-Nikulin which says that there exist no rank-three BKM Lie algebras associated to such lattices. However, it assumes that the denominator identity is the one due to Borcherds which incorporates the imaginary simple roots on the sum side.
- However, we have been able to show that he square-root of the $\frac{1}{4}$-BPS generating function, $\Delta^{(N)}(\mathbf{Z})$ appears to be the denominator identity of a new kind of Lie algebra.

The BKM Lie algebras for $N = 5, 6$

- The product side obtained as a Borcherds product sees **all** the simple roots that form the closed polygon.
- The sum side obtained as an additive lift is **not** covariant under the Weyl group generated subset of roots shown in red. Again, modularity is useful in proving this fact.
- The equality of the sum side and the product side is highly non-trivial.
- The sum side of the Borcherds denominator formula has to be further modified to interpret the sum side of the answer.
- We have some understanding based on experimentally looking at how the other set of roots must be treated.
- We propose that there are new kinds of (B)KM Lie algebras associated with rank-three hyperbolic lattices with Weyl vector of hyperbolic type.

Concluding Remarks

Periodic Table of BKM Lie superalgebras

N \ M	1	2	3	4	5	6	Cartan matrix
1	1^{24} Δ_5	$1^8 2^8$ ∇_3	$1^6 3^6$ ∇_2	$1^4 2^2 4^4$ $\nabla_{3/2}$	$1^4 5^4$ Δ_1	$1^2 2^2 3^3 6^2$ P_1	$A^{(1)}$
2	$1^8 2^8$ $\widetilde{\nabla}_3$	2^{12} Δ_2	×	$2^4 4^4$ Q_1	×	$2^3 6^3$ $S_{1/2}$	$A^{(2)}$
3	$1^6 3^6$ $\widetilde{\nabla}_2$	×	3^8 Δ_1	×			$A^{(3)}$
4	$1^4 2^2 4^4$ $\widetilde{\nabla}_{3/2}$	$2^4 4^4$ $\widetilde{Q}_1$	×	4^6 $\Delta_{1/2}$	×	×	$A^{(4)}$
5	$1^4 5^4$ $\widetilde{\nabla}_1$	×	×	×	×	×	$A^{(5)}$
6	$1^2 2^2 3^2 6^2$ $\widetilde{P}_1$	$2^3 6^3$ $\widetilde{S}_{1/2}$	×	×	×	6^4 Δ_0	$A^{(6)}$

Conclusion

- We didn't focus on two aspects of the problem: (i) Modular aspects and (ii) Connections to various moonshines.
- Modularity is very important to proving Weyl covariance of the WKB denominator formula. Characters of irreps of these Lie algebras will also have nice modular properties. These are being currently investigated with Viswanath and Shabbir.
- We have already seen glimpses of connections with umbral moonshine that generalises Mathieu moonshine. Are there more new examples that arise here?
- Can we write analogous formulae for $\frac{1}{2}$ BPS states in $\mathcal{N} = 2$ compactifications? The case of type II compactifications on Borcea-Voison threefolds might be a good starting point.
- Can we construct new BKM Lie superalgebras for all rank-three hyperbolic lattices of hyperbolic type in Nikulin's list? In our examples, the physics of wall-crossing gave us the needed modular forms. Is there a nice way to get modular form for these cases.

Thank You