Cosmological Time Crystal: Cyclic Universe with a small Λ in a toy model approach

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Plan of the talk...

- ✓ Brief Introduction
- ✓ Action formulation from Noncommutative FRW metric
- ✓ Cosmological Time Cystal
- ✓ Cosmological Time Cystal and effective small Cosmological Constant
- ✓ Modified Friedmann equation
- ✓ Summary and future prospects

Introduction

- **Classical Time Crystal**: Within a few years of the theoretical conjecture, TC has created an enormous amount of interest, both in theoretical and experimental contexts (see *K. Sacha and J. Zakrzewski, Rep.Prog.Phys. 81, 016401 (2018)* for a recent review).
- Symmetry of a system is spontaneously broken when the ground state (or classically, the lowest energy state) of a system is less symmetrical than the equations of motion that control the system. The name Time Crystal is borrowed from the familiar (space) crystal that has a spatially ordered structure in its ground state. It is a manifestation of breaking of continuous translation symmetry, leaving behind a ground state with discrete translation symmetry.
- A mathematical model for a classical TC was provided in (see *A. Shapere and F. Wilczek, Phys.Rev.Lett. 109,160402 (2012)*) and later in (see *A. Shapere and F. Wilczek, arxiv: 1708.03348*) a more "physical" model was constructed in which the relevant degree of freedom undergoes periodic Sisyphus dynamics in its lowest energy state.
- Classical Time Crystal (CTC) has a non-canonical form of kinetic energy that minimizes (not for zero velocity as in conventional systems but) at a non-zero velocity, much in analogy to Spontaneous Symmetry Breaking (SSB) where a potential function minimizes at a non-zero (but constant) value of the field variable that constitutes the ground state condensate.

- **Noncommutative Geometry**: NC geometry or equivalently a generalized form of canonical (Heisenberg) phase space algebra was introduced by Snyder (see *H. S. Snyder, Phys. Rev. 71, 38 (1947)*) but the idea was not successful. The work of Seiberg and Witten (see *N. Seiberg and E. Witten, J. HEP 09(1999) 032*) showed that string theory in certain low energy limits can be identified with quantum field theory extended to NC spacetime of the form

$$[x_\mu, x_\nu] = i\Theta_{\mu\nu} \tag{1}$$

with constant antisymmetric NC parameter $\Theta_{\mu\nu} = -\Theta_{\nu\mu}$.

- The systematic procedure of extending a field theoretic model living in conventional spacetime to NC spacetime is to replace local products of fields by the Groenewold-Moyal star product given by

$$\star = \exp\left\{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}\right\}, \tag{2}$$

where $\overleftarrow{\partial_\mu}$ and $\overrightarrow{\partial_\mu}$ are left and right derivatives, respectively, with respect to some generic coordinate x^μ. To be explicit, products of local fields like $A(x)B(x)$ are replaced by

$$A(x) \star B(x) = A(x) \exp\left\{\frac{i}{2}\Theta^{\mu\nu}\overleftarrow{\partial_\mu}\,\overrightarrow{\partial_\nu}\right\} B(x)$$

in the action and construct NC extended equations of motion.

- In our model we applied the fascinating concept of Classical Time Crystal (CTC), proposed by Shapere and Wilczek in an extended model of Friedmann-Robertson-Walker (FRW) cosmology. Specifically, the extension is induced by Non-Commutative (NC) gravity contribution with an underlying Quantum Gravity perspective which was derived by Fabi, Harmes and Stern (see *S. Fabi, B. Harms, and A. Stern, Physical Review D 78, 065037 (2008)*).
- Two of our principal results are the following:
  (i) The scale factor borrows the Sisyphus like periodic behavior that characterizes the CTC but, more importantly for our present interest, it can naturally serve as a physically motivated toy model for a cyclic universe.
  (ii) Once again borrowing a CTC feature, the minimum energy state (or ground state) consists of a condensate leading to an arbitrarily small positive cosmological constant Λ.

- The conventional Friedmann-Robertson-Walker metric reads,

$$ds^2 = -dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right) , \qquad (3)$$

where $a(t)$ is the scale factor.
Now we reproduce the NC corrected FRW metric (see *S. Fabi, B. Harms, and A. Stern, Physical Review D 78, 065037 (2008)*),

$$\begin{aligned}
g_{tt} &= -1 + \frac{\theta^2}{16c^4}\left(6\ddot{a}^2 + 5\dot{a}\dddot{a}\right) + O(\theta^4) \\
g_{rr} &= \frac{a^2}{1-kr^2} - \frac{\theta^2}{16c^4}\left(\dot{a}^4 + 13a\ddot{a}\dot{a}^2 + 12a^2\dot{a}\dddot{a} + 16a^2\ddot{a}^2\right) + O(\theta^4) \\
g_{\theta\theta} &= r^2a^2 + \frac{\theta^2}{16c^2}\left[5\dot{a}^2 + 4a\ddot{a} - \frac{a}{c^2}\left(8a\ddot{a}^2 + 9\dot{a}^2\ddot{a} + 4a\dot{a}\dddot{a}\right)r^2\right] + O(\theta^4) \\
g_{\phi\phi} &= \sin^2\theta g_{\theta\theta} ,
\end{aligned} \qquad (4)$$

For simplicity we set all components of $\Theta^{\mu\nu}$ equal to zero except for

$$\Theta^{tr} = -\Theta^{rt} \equiv \theta . \qquad (5)$$

.

- The generic form of the Einstein-Hilbert action $\mathcal{A} = \frac{c^4}{16\pi G}\int (R - 2\Lambda)\sqrt{-g}\; d^4x$ in the minisuperspace reduction reads

$$\mathcal{A} = \sigma \int \, dt \Bigg[\left(-a\dot{a}^2 + \kappa a - \frac{\Lambda}{3} a^3 \right) - \beta_1 \frac{\dot{a}^2}{a} + \beta_2 \frac{\dot{a}^4}{a} - \beta_3 \frac{\dot{a}^6}{a} + \alpha_1 a\dot{a}^2 + \alpha_2 a\dot{a}^4 \Bigg] \quad (6)$$

where NC corrections are introduced via θ in the numerical parameters,

$$\sigma = \frac{c^2 L^3}{2G}, \quad \beta_1 = \left(\frac{5\theta^2}{16L^4\rho}\right), \quad \beta_2 = \left(\frac{\theta^2}{4L^2c^2}\right), \quad (7)$$

$$\beta_3 = \frac{\theta^2}{96c^4}, \quad \alpha_1 = \left(\frac{3\Lambda\theta^2}{16L^2c^2}\right), \quad \alpha_2 = \frac{317}{96c^4}\Lambda\theta^2 \quad . \quad (8)$$

- In our model the dimensions of the physical quantities and parameters in mass (m), length (l) and time (t) units are as follows:

$$[\mathcal{A}] = \frac{ml^2}{t}, \; [G] = \frac{l^3}{mt^2}, \; [a] = 0, \; [\theta] = l^2, \; [\Lambda] = [\kappa] = \frac{1}{t^2}.$$

- The parameter $[L] = l$ denotes the size of the comoving spatial coordinates and L^3 arises from their integration when the field theory is reduced to a quantum mechanical system.
- Some comments about the action in (6) are in order::
  We have considered a closed ($\kappa = 1$) universe in the canonical sector of the action but have not considered κ-dependent terms in the NC $O(\theta^2)$ contributions, primarily for convenience.
- Secondly, we have followed an approximation scheme suggested in (*S. Fabi, B. Harms, and A. Stern, Physical Review D 78, 065037 (2008)*) that drops $\ddot{a}(t)$ and higher time derivatives, again from NC correction terms only.

- Let us identify $L = c/(\Lambda)^{\frac{1}{2}}$ (to be justified presently) and rewrite the action in a compact form,

$$\mathcal{A} = \sigma \int dt \left(-A\dot{a}^2 + B\dot{a}^4 - C\dot{a}^6 + \kappa a - \frac{\Lambda}{3} a^3 \right), \tag{9}$$

where $A = a + \frac{\nu}{16}\left(\frac{5}{\rho a} - 3a\right)$; $B = \frac{\nu}{4\Lambda}\left(\frac{1}{a} + \frac{317a}{24}\right)$; $C = \frac{\nu}{96\Lambda^2 a}$. We have introduced a dimensionless universal parameter $\nu = \frac{\theta^2\Lambda^2}{c^4}$. Defining the canonical momentum

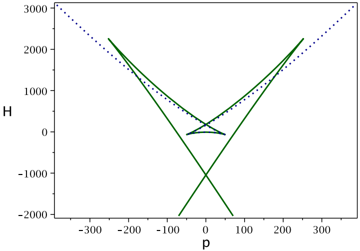
$$p = (\partial\mathcal{L})/(\partial\dot{a}) = \sigma(-2A\dot{a} + 4B\dot{a}^3 - 6C\dot{a}^5), \tag{10}$$

our Hamiltonian yields,

$$H = p\dot{a} - \mathcal{L} = \sigma\left(-A\dot{a}^2 + 3B\dot{a}^4 - 5C\dot{a}^6 - \kappa a + \frac{\Lambda}{3} a^3 \right). \tag{11}$$

This Hamiltonian constitutes our primary result which we now study from the CTC perspective. First of all note that a being the scale factor is positive. This makes B, C positive and A will be positive provided the small parameter ρ is less that $5\Lambda/3$.

- *Cusp structure*: A richer cusp structure is revealed from the p vs. H diagram. The plot of p vs. H shows that because of the $\dot{a}^6$-term, $\partial p/\partial \dot{a} = 0$, is a fourth order equation in $\dot{a}$ having four solutions as seen from the figure: the dotted blue line resembles the profile of (see *A. Shapere and F. Wilczek, Phys.Rev.Lett. 109,160402 (2012)*) where $\dot{a}^6$ contribution is negligible whereas the solid deep-green line shows the new structure with significant $\dot{a}^6$ contribution.
- We name this new structure as *batwing* catastrophe that reduces to the swallowtail catastrophe if the contribution of the $\dot{a}^6$-term becomes negligible.



- *Effective planar model without $\dot{a}^6$-term*: A naive plot of $a(t)$ and $\dot{a}(t)$ from the Lagrangian equation of motion, with $A' = \partial A/\partial a, ..,$

$$2\ddot{a}(6B\dot{a}^2 - A) + \dot{a}^2(3B'\dot{a}^2 - A') - 1 + \Lambda a^2 = 0, \qquad (12)$$

is given in the figure where $a(t)$, $\dot{a}(t)$, H are plotted against the cosmic time t. From this profile we find a single hump before it hits a singularity. However, quite interestingly, we have discovered a particular set of parameter values and initial conditions that give rise to a stable oscillating a that hits a singularity after a very long time.

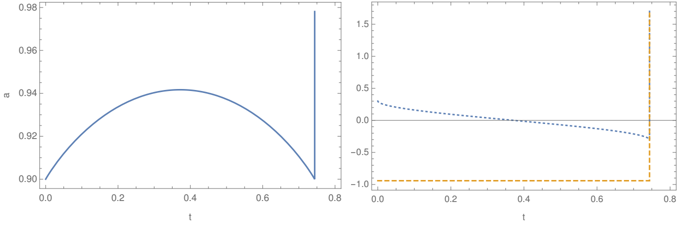


Figure: The dynamics of the eqn. 12 for some particular choices of the model parameters has been presented. Left panel shows the evolution of the scale factor while the right panel shows the expansion rate $\dot{a}$ (blue dotted curve) and the total energy H (brown dashdot curve).

- Thus, one may infer that the cosmological model framed by TC may allow a cyclic nature of the universe. While we believe that further investigations toward this direction is necessary, possibly to understand the nature of entropy transfer from one cycle to another since during the evolution of the universe.

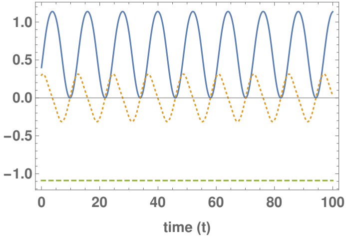


Figure: The profile of the scale factor a (solid curve), its time derivative $\dot{a}$ (dotted curve), and the total energy H (dashdot curve) for the dynamical system in presence of the $\dot{a}^4$-term only (12) have been displayed for some specific choices of the parameters.

- *Regularized CTC*: We follow the elegant route of (see *A. Shapere and F. Wilczek, arxiv: 1708.03348*) that shows a way of removing the singularity by introducing a regulator in the form of μ in an alternative Lagrangian,

$$\mathcal{L} = \frac{\mu}{2}\dot{x}^2 + f(x)J(a)\dot{a} - g(x)(1 + K(a)) - V(a), \tag{13}$$

with

$$f(x) = \frac{x^3}{3} - x, \;\; g(x) = \frac{x^4}{4} - \frac{x^2}{2} \tag{14}$$

and the quantities K, J in (13) are,

$$1 + K(a) = \frac{A^2}{3B}, \;\; J(a) = \sqrt{\frac{2A^3}{3B}}. \tag{15}$$

The Lagrangian (13) mimics a planar charged particle in an external potential. The equations of motion from (13) are

$$\mu\ddot{x} - f'(x)J(a)\dot{a} + (K(a) + 1)g'(x) = 0, \tag{16}$$
$$J(a)f'(x)\dot{x} + g(x)K'(a) + V'(a) = 0. \tag{17}$$

Now to check, In the present case for $\mu = 0$, the coupled set above reduces to (12).

- This figure shows the behavior of the scale factor a (blue solid line), x (brown dotted line) and the total energy of the system H (green dashed line) with the evolution of the cosmic time, t. Although the present model is similar to that of (*A. Shapere and F. Wilczek, arxiv: 1708.03348*) from the TC point of view (both having only $\dot{a}^2$ and $\dot{a}^4$ terms), however, the a-dependence is much more complicated in our case.
- This is reflected in our x profile that contains both positive and negative values (whereas it has either positive or negative values) with the sharp edges separated by smooth curves. Thus our model describes a *doubled* Sisyphus dynamics.

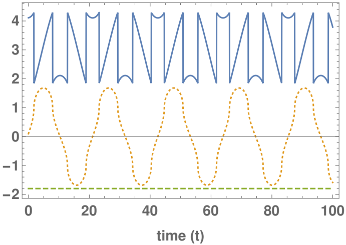


Figure: The profile of the scale factor a (solid curve), x (dotted curve) and H (dashdot curve) have been shown for the system of equations (16) and (17) for some specific choices of the parameters.

- *Effective planar model with $\dot{a}^6$-term*: Generic solutions of the equation of motion,

$$2\ddot{a}(A - 6B\dot{a}^2 + 15C\dot{a}^4) + \dot{a}^2(A' - 3B'\dot{a}^2 + 5C'\dot{a}^4) + 1 - \Lambda a^2 = 0, \tag{18}$$

with $A' = \frac{dA}{da}$, $B' = \frac{dB}{da}$, and $C' = \frac{dC}{da}$, have singularities as expected.
- A very surprising and interesting result is that we have discovered a small parameter window in which the scale factor $a(t)$ oscillates smoothly without any singularity and with constant energy indicating a cyclic universe.

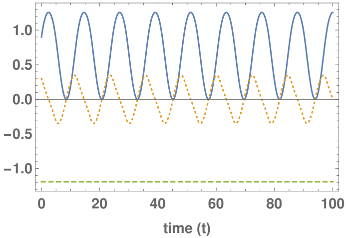


Figure: The profile of the scale factor a (solid), $\dot{a}$ (dotted) and H (dashdot) for the equation (18) have been presented for some specific choices of the parameters.

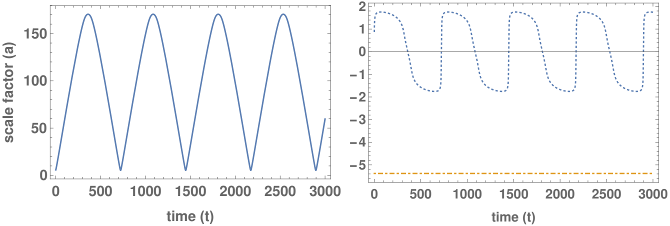
- In order to construct the μ-regularized model we try with a simple generalization. Keeping the same form of $\mathcal{L}$ as in (13) but with new forms of $f(x)$, $g(x)$ as,

$$g(x) = \frac{x^6}{6} - \frac{x^4}{4} + \frac{x^2}{2}, \quad f(x) = \frac{x^5}{5} - \frac{x^3}{3} + x \tag{19}$$

we find the same set of equations of motion with the identifications

$$1 + K(a) = \frac{A^2}{3B}, \quad J(a) = \sqrt{\frac{2A^3}{3B}}, \quad 30C = \frac{J^6}{(1+K)^5}. \tag{20}$$

The last identification shows that the regularized form for $\dot{a}^6$-model is not entirely satisfactory as it fails to generate the independent form of C. This suggests that a more elaborate version of the μ-regularized model is needed to faithfully represent the parent $\dot{a}^6$-model.



- Let us derive the condensate energy where we need the condensate values for a_0 and $\dot{a}_0$ that minimize the Hamiltonian. The Hamiltonian H for the action (9) without the $\dot{a}^6$ term, can be written in the form

$$H = \sigma \left[3B \left(\dot{a}^2 - \frac{A}{6B} \right)^2 + V_{\rm eff} \right], \tag{21}$$

$$V_{\rm eff} = \left(a - \frac{\Lambda}{3} a^3 - \frac{A^2}{12B} \right). \tag{22}$$

Since B is always positive, H will be minimized for

$$\dot{a}_0 = \pm \sqrt{\frac{A(a_0)}{6B(a_0)}} = \pm \sqrt{\frac{A_0}{6B_0}} \tag{23}$$

with a_0 obtained from $\frac{\partial V_{\rm eff}}{\partial a} \, |_{a_0} = 0$. The ground state energy will be

$$H_{condensate} = -\frac{A_0^2}{12B_0} + a_0 - \frac{\Lambda}{3} a_0^3,$$

$$A_0 = A(a_0), B_0 = B(a_0). \tag{24}$$

- In this figure the solid line ($\partial V_{\texttt{eff}}/\partial a$) cuts the a-axis at a_0, which incidentally is greater than $a = 1$ (i.e., in future). We define

$$\Lambda_{\rm eff} = \Lambda - \frac{3}{a_0^2} + \frac{A_0^2}{4B_0 a_0^3} \tag{25}$$

and the value of $\Lambda(a_0) = \Lambda_{\rm eff}$ can be arbitrarily small (as seen from the figure) even with a large value of Λ: the value of Cosmological Constant gets renormalized in a sense.

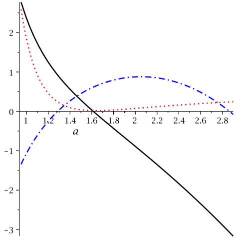


Figure: Qulitative evolutions for the variation of the effective potential,i.e., $\partial V_{\texttt{eff}}/\partial a$ (solid line); the effective cosmological constant $\Lambda_{\texttt{eff}}$ (dotted line) and the Hamiltonian having its ground state energy $H_{\rm condensate}$ (dashed line) have been shown with the evolution of the universe in terms of the scale factor a.

- Friedmann equations are derived from a generic minisuperspace model in a straightforward way: one equation is the Lagrangian equation of motion for $a(t)$ and the other equation comes from simply putting the Hamiltonian to zero since the Hamiltonian being one of the generators of diffeomorphism symmetry, comes as a factor of the lapse function.
- However, in the present case, due to two forms of SSB, (in velocity $\dot{a}(t)$ and coordinate $a(t)$ sectors), one has to shift both $a(t)$, $\dot{a}(t)$ by their respective condensate values to get the action in terms of new variables where the condensate does not appear any more with the degrees of freedom vanishing in the ground state. This amounts to using the variable $b(t)$ defined by the transformations

$$b(t) = a(t) - a_0, \quad \dot{b}(t) = \dot{a}(t) - \dot{a}_0 \tag{26}$$

in the action.

- The new action yields the following set of equations,

$$\frac{\dot{b}^2}{b^2} + \frac{\bar{k}}{b^2} = \frac{\Lambda(b+a_0)^3}{3b^2 f} + \frac{1}{b^2}\frac{g}{f}, \tag{27}$$

$$\begin{aligned} \ddot{b} &= \frac{1}{h}\Bigg[\frac{b^2 f}{b+a_0}\left(\frac{g}{b^2 f} + \frac{\Lambda(b+a_0)^3}{3b^2 f} - \frac{\dot{b}^2}{b^2}\right) - \Lambda(b+a_0)^2 \\ &+ \left(\dot{b}^2 - \dot{a}_0^2\right)\left(1 - \frac{3\nu}{16} - \frac{5\nu}{16\rho(b+a_0)^2}\right) \\ &+ \frac{\nu}{\Lambda}\left(\dot{b}+\dot{a}_0\right)^3\left(\frac{\dot{b}+\dot{a}_0}{4} - \dot{b}\right)\left(\frac{317}{24} - \frac{1}{(b+a_0)^2}\right)\Bigg], \end{aligned} \tag{28}$$

where $\bar{k} = k(b+a_0)/f$ is the effective curvature scalar and the functions f, g , h have some forms and are functions of $b, a_0, \nu, \dot{a}, \Lambda, \rho, \dot{b}, \dot{a}_0$.

- The first one is the Hubble equation and second one is the acceleration equation.

Summary ::

- We have considered an existing model of generalized FRW metric endowed with noncommutativity corrections. This extended FRW model gives rise to a new form of time crystal behavior (*batwing-catastrophe*) that can be compared with the simpler form of singularity – swallowtail catastrophe.
- In addition to that, our model has a doubly Sisyphus dynamics due to the presence of exotic terms involving quartic and sextet order velocity contributions and field dependent factors multiplying the velocity terms.
- Since our time crystal model is a form of generalized FRW model, it is natural to term it as a Cosmological Time Crystal and as the scale factor $a(t)$ is associated with motion periodic in time - Sisyphus dynamics - the model can be interpreted as a form of Cyclic Cosmology in a natural way. This cyclic behavior substantiate our view that noncommutative gravity can represent cyclic cosmologies. However, it should be stressed that our model is purely geometrical in the sense that no (matter) field is introduced from outside.
- The other major success of our toy model approach is that it allows us to generate an arbitrarily small positive cosmological constant in a natural way. We have also derived a modified form of Friedman equation that has noncommutative contributions.

- Detailed analysis of the noncommutativity modified Friedman equation needs to be studied.
- Inclusion of matter degrees of freedom is another area to be looked at since one can conjure up more than one Time Crystal structures of with Time Crystal behavior coming from the matter sector as well as the FRW (scale factor) sector. Also presence of matter will bring these systems closer to realistic cosmological models.
- Another very interesting avenue to explore is to analyze other forms of Time Crystal that can appear naturally in cosmology, in particular in $f(R)$ gravity (that will contain quartic and still powers of velocity).

References

- A. Shapere and F. Wilczek, Phys. Rev. Lett. **109**, 160402 (2012).
- A. Shapere and F. Wilczek, *"Realization of "Time Crystal" Lagrangians and Emergent Sisyphus Dynamics"* [arXiv:1708.03348 [cond-mat.stat-mech]].
- For a recent review and references, see K. Sacha and J. Zakrzewski, Rep. Prog. Phys. **81**, 016401 (2018).
- S. Fabi, B. Harms, A. Stern, Phys. Rev. D **78**, 065037 (2008).
- P. J. Steinhardt and N. Turok, Phys. Rev. D **65**, 126003 (2002); New Astron. Rev. **49**, 43 (2005).
- J. Zakrzewski, Viewpoint: Crystals of Time, Physics **5**, 116, (October 15) 2012.
- H. S. Snyder, Phys. Rev. 71 38 (1947); ibid 72 68 (1947); See also C. N. Yang, Phys. Rev. **72**, 874 (1947).
- N. Seiberg and E. Witten, JHEP **9909**, 032 (1999).
- M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001); R. J. Szabo, Phys. Rep. **378**, 207 (2003); R. Banerjee, B. Chakraborty, S. Ghosh, P. Mukherjee, S. Samanta, Found. Phys. **39**, 1297 (2009).