Three loop QCD corrections to heavy quark form factors

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Success of a theory lies in absolute predictions of the experiments.

The Standard Model is the most successful theory in describing the elementary particles and their fundamental interactions, due to combined effort from both the magnificent experiments like Tevatron, HERA, LHC etc. and precise theory predictions, namely perturbative calculations.

Tevatron	LHC	FCC/ILC
the top quark	the Higgs boson	BSM physics?!
fundamental laws $\sim 10\%$	fundamental laws $\sim 5\%$	more precision!
agreement with NLO theory predictions	agreement with NNLO theory predictions	More precise theory predictions needed!

Top-quark phenomenology

✓ **Top quark is heavy**

- The heaviest SM particle - probes the Higgs sector most - plays unique role in understanding the EW symmetry breaking
- New physics potential : perfect place to manifest it

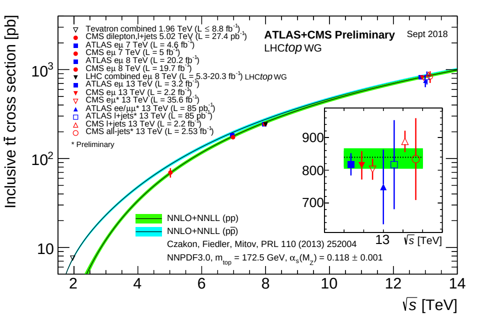
✓ **Top quark is short-lived**

- (Mostly) decays before hadronization -the only 'free' quark - Spin properties, Interaction vertices, Precise description of mass

✓ **Sudakov behavior of the massive form factors**

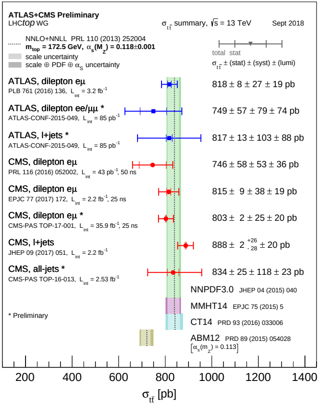
- The form factors are basic building blocks for many physical quantities. Also, they exhibit Sudakov behavior in the asymptotic limit.

Top @ LHC

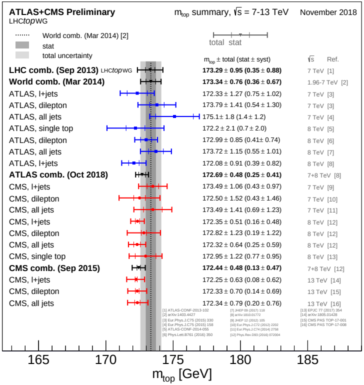


ATLAS+CMS Sep'18

Top @ LHC



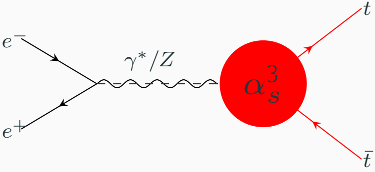
ATLAS+CMS Preliminary
LHCtop WG
σ_tt̄ summary, √s = 13 TeV
Sept 2018
NNLO+NNLL PRL 110 (2013) 252004
m_top = 172.5 GeV, α_s(M_Z) = 0.118±0.001
scale uncertainty
scale ⊕ PDF ⊕ α_S uncertainty
total stat
σ_tt̄ ± (stat) ± (syst) ± (lumi)
ATLAS, dilepton eμ
PLB 761 (2016) 136, L_int = 3.2 fb^-1
818 ± 8 ± 27 ± 19 pb
ATLAS, dilepton ee/μμ *
ATLAS-CONF-2015-049, L_int = 85 pb^-1
749 ± 57 ± 79 ± 74 pb
ATLAS, l+jets *
ATLAS-CONF-2015-049, L_int = 85 pb^-1
817 ± 13 ± 103 ± 88 pb
CMS, dilepton eμ
PRL 116 (2016) 052002, L_int = 43 pb^-1, 50 ns
746 ± 58 ± 53 ± 36 pb
CMS, dilepton eμ
EPJC 77 (2017) 172, L_int = 2.2 fb^-1, 25 ns
815 ± 9 ± 38 ± 19 pb
CMS, dilepton eμ *
CMS-PAS TOP-17-001, L_int = 35.9 fb^-1, 25 ns
803 ± 2 ± 25 ± 20 pb
CMS, l+jets
JHEP 09 (2017) 051, L_int = 2.2 fb^-1
888 ± 2 +26 -28 ± 20 pb
CMS, all-jets *
CMS-PAS TOP-16-013, L_int = 2.53 fb^-1
834 ± 25 ± 118 ± 23 pb
NNPDF3.0 JHEP 04 (2015) 040
MMHT14 EPJC 75 (2015) 5
CT14 PRD 93 (2016) 033006
ABM12 PRD 89 (2015) 054028
[α_s(m_Z) = 0.113]
* Preliminary
200 400 600 800 1000 1200 1400
σ_tt̄ [pb]

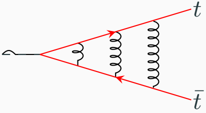


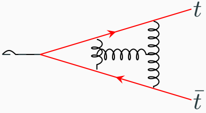
ATLAS+CMS Preliminary
LHCtopWG
m_top summary, √s = 7-13 TeV
November 2018
World comb. (Mar 2014) [2]
stat
total uncertainty
total stat
m_top ± total (stat ± syst)
√s Ref.
LHC comb. (Sep 2013) LHCtopWG 173.29 ± 0.95 (0.35 ± 0.88) 7 TeV [1]
World comb. (Mar 2014) 173.34 ± 0.76 (0.36 ± 0.67) 1.96-7 TeV [2]
ATLAS, l+jets 172.33 ± 1.27 (0.75 ± 1.02) 7 TeV [3]
ATLAS, dilepton 173.79 ± 1.41 (0.54 ± 1.30) 7 TeV [3]
ATLAS, all jets 175.1± 1.8 (1.4 ± 1.2) 7 TeV [4]
ATLAS, single top 172.2 ± 2.1 (0.7 ± 2.0) 8 TeV [5]
ATLAS, dilepton 172.99 ± 0.85 (0.41± 0.74) 8 TeV [6]
ATLAS, all jets 173.72 ± 1.15 (0.55 ± 1.01) 8 TeV [7]
ATLAS, l+jets 172.08 ± 0.91 (0.39 ± 0.82) 8 TeV [8]
ATLAS comb. (Oct 2018) 172.69 ± 0.48 (0.25 ± 0.41) 7+8 TeV [8]
CMS, l+jets 173.49 ± 1.06 (0.43 ± 0.97) 7 TeV [9]
CMS, dilepton 172.50 ± 1.52 (0.43 ± 1.46) 7 TeV [10]
CMS, all jets 173.49 ± 1.41 (0.69 ± 1.23) 7 TeV [11]
CMS, l+jets 172.35 ± 0.51 (0.16 ± 0.48) 8 TeV [12]
CMS, dilepton 172.82 ± 1.23 (0.19 ± 1.22) 8 TeV [12]
CMS, all jets 172.32 ± 0.64 (0.25 ± 0.59) 8 TeV [12]
CMS, single top 172.95 ± 1.22 (0.77 ± 0.95) 8 TeV [13]
CMS comb. (Sep 2015) 172.44 ± 0.48 (0.13 ± 0.47) 7+8 TeV [12]
CMS, l+jets 172.25 ± 0.63 (0.08 ± 0.62) 13 TeV [14]
CMS, dilepton 172.33 ± 0.70 (0.14 ± 0.69) 13 TeV [15]
CMS, all jets 172.34 ± 0.79 (0.20 ± 0.76) 13 TeV [16]
165 170 175 180 185
m_top [GeV]

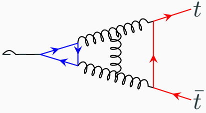
In future colliders, specially in electron-positron colliders, top pair production will play an important role. In this talk, we present the three loop QCD corrections to the massive form factor which is one of the most important element to obtain the full N^3LO corrections.

Specifically, what we compute









+ many more

One-loop and beyond

$F^{(1)}_{V,i}, F^{(1)}_{A,i}$ [Arbuzov, Bardin, Leike '92; Djouadi, Lampe, Zerwas '95]

$F^{(1)}_{S}, F^{(1)}_{P}$ [Braaten, Leveille '80; Sakai '80; Drees, Hikasa '90]

$F^{(2)}_{V,i}, F^{(2)}_{A,i}$ [Altarelli, Lampe '93; Ravindran, van Neerven '98; Catani, Seymour '99]

$F^{(2)}_{S}, F^{(2)}_{P}$ [Gorishnii *et. al.* '91; Chetyrkin, Kwiatkowski '95; Harlander, Steinhauser '97]

Two-loop

$F^{(2)}_{I}$ [Bernreuther, Bonciani, Gehrmann, Heinesch, Leineweber, Mastrolia, Remiddi '04,'05]

$F^{(2)}_{V,i}(\mathcal{O}(\epsilon))$ [Gluza, Mitov, Moch, Riemann '09]

$F^{(2)}_{I}(\mathcal{O}(\epsilon^2))$ [Ablinger, Behring, Blümlein, Falcioni, Freitas, Marquard, Rana, Schneider '17]

Three-loop

$F^{(3)}_{V,i}|_{\mathrm{large\,N}}$ [Henn, Smirnov, Smirnov, Steinhauser '16]

$F^{(3)}_{V,i}, F^{(3)}_{A,i}, F^{(3)}_{S}, F^{(3)}_{P}|_{\mathrm{large\,N+full}\,n_l}$ [Lee, Smirnov, Smirnov, Steinhauser '18]

$F^{(3)}_{V,i}, F^{(3)}_{A,i}, F^{(3)}_{S}, F^{(3)}_{P}|_{\mathrm{large\,N+full}\,n_l}$ [Ablinger, Blümlein, Marquard, Rana, Schneider '18]

J. Ablinger, A. Behring, J. Blümlein, G. Falcioni, A. De Freitas, P. Marquard, NR and C. Schneider,
`Heavy quark form factors at two loops`,
Phys.Rev. D97 (2018) 094022 (arXiv:1712.09889 [hep-ph]).

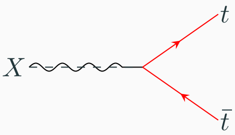
J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
`Heavy Quark Form Factors at Three Loops in the Planar Limit`,
Phys.Lett. B782 (2018) 528-532 (arXiv:1804.07313 [hep-ph]).

J. Ablinger, J. Blümlein, P. Marquard, NR and C. Schneider,
`Automated Solution of First Order Factorizable Systems of Differential Equations in One Variable`,
(to appear in Nucl.Phys. B) (arXiv:1810.12261 [hep-ph]).

Notation

The process

We consider the decay of a color neutral massive particle to a pair of heavy quark of mass m.



Notation

$$X(q) \to t(q_1) + \bar{t}(q_2)$$

$$X = V, A, S, P$$

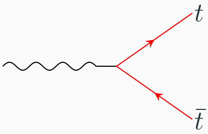
$$s = \frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$$

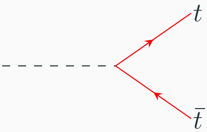
The general structure

Vector and Axial Vector

V: $-i\delta_{ij}v_Q\left(\gamma^\mu\, F_{V,1} + \frac{i}{2m}\sigma^{\mu\nu}q_\nu\, F_{V,2}\right)$

A: $-i\delta_{ij}a_Q\left(\gamma^\mu\gamma_5\, F_{A,1} + \frac{1}{2m}q^\mu\gamma_5\, F_{A,2}\right)$





Scalar and Pseudo Scalar

$-\frac{m}{v}\delta_{ij}\left[s_Q F_S + i p_Q \gamma_5 F_P\right]$

Computational details

The generic procedure

$d = 4 - 2\epsilon$

- Draw the Feynman diagrams -> QGRAF [Nogueira]

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- Now the general form

$$\int \frac{d^d l_1}{(2\pi)^{d/2}} \frac{d^d l_2}{(2\pi)^{d/2}} \frac{d^d l_3}{(2\pi)^{d/2}} \frac{\mathrm{const} \times \mathrm{Poly}(s) \times \mathrm{DotProduct}(l_1, l_2, l_3, q_1, q_2)}{l_1^2 l_2^2 l_3^2 \cdots (l_2 - p_1)^2 \cdots}$$

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- Decomposition of the dot products to obtain scalar integrals

$$\frac{2l.p}{l^2(l-p)^2} = \frac{l^2 - (l-p)^2 + p^2}{l^2(l-p)^2} = \frac{1}{(l-p)^2} - \frac{1}{l^2} + \frac{p^2}{l^2(l-p)^2}$$

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- Numerous identity relations (IBPs) among scalar integrals : reduce number of integrals drastically $\Rightarrow$ compute the remaining integrals

Integration by parts identities (IBPs)

- *IBPs* : reduce the number of integrals to compute [Tkachov, Chetyrkin]

Generalization of Gauss's theorem in d dimension

Within dimensional regularization, all integrals in d dimension are well-defined and convergent $\Rightarrow$ integrand must be zero at boundary

$$\int \prod_{i=1}^{l} \mathcal{D}^d l_i \frac{\partial}{\partial l_j^\mu} \left(\frac{v^\mu}{D_1^{n_1} \dots D_m^{n_m}} \right) = 0 \quad \Big|_{v \equiv l,p}$$

Example : Consider

$$\mathcal{I}(n) = \int \frac{d^d l}{(2\pi)^{d/2}} \frac{1}{l^2 (l-p_1)^2 ((l-p_1-p_2)^2)^n}$$

The identity for $v \equiv l$ gives a recursion relation

$$\mathcal{I}(n+1) = (-1)^n \frac{(d-(n+3)) \cdots (d-4)(d-3)}{n! s^n} \mathcal{I}(1)$$

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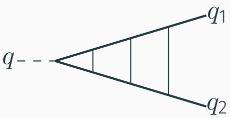
- $\sim 10^6$ identities for our process
- So the problem boils down to solve an algebraic linear system of equations relating the integrals - several programs (AIR, FIRE, Kira, Reduze, LiteRed . . .) available - we use CRUSHER [Marquard, Seidel]

Computing the master integrals

A scalar integral can be expressed as

$$J(\nu_1,\ldots,\nu_n)=\left((4\pi)^{2-\epsilon}e^{\epsilon\gamma_E}\right)^3\int\frac{d^dl_1}{(2\pi)^d}\,\frac{d^dl_2}{(2\pi)^d}\,\frac{d^dl_3}{(2\pi)^d}\,\frac{1}{D_1^{\nu_1}\ldots D_n^{\nu_n}}$$

For example



$$\begin{aligned}&l_1^2-m^2,(l_1-q)^2-m^2,(l_1-l_2)^2,\\&l_2^2-m^2,(l_2-q)^2-m^2,(l_2-l_3)^2,\\&l_3^2-m^2,(l_3-q)^2-m^2,(l_1-q_1)^2\end{aligned}$$

To evaluate the integral $\rightarrow$ Feynman parametrization, Mellin-Barnes …

We use the method of differential equations!

Using differential equations

$\frac{q^2}{m^2} = -\frac{(1-x)^2}{x}$

The integral is a function of d, q^2 and m^2.

$$J(1,1,1,1,1,1,1,1,1) \equiv f(d,\, q^2,\, m^2) \equiv f(d,\, x)$$

The idea is to obtain a differential eqn. for the integral $w.r.t.$ x and solve it.

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$$\frac{d}{dx} J_i = \text{some combinations of integrals}$$

$$\Downarrow \text{ IBP identities}$$

$$= \sum_j c_{ij} J_j$$

c_{ij}'s are rational function of d and x.

Using differential equations

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$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

$$d_x \mathbb{J} = \mathbb{A}(d, x)\mathbb{J}$$

Canonical Basis

The choice of MIs is not unique!

The idea is to find a basis such that the deqs decouple as $\epsilon \to 0$

[Kotikov; Henn]

$$d_x \tilde{\mathbb{J}} = \epsilon \tilde{\mathbb{A}}(x) \tilde{\mathbb{J}}$$

Now one can perform Laurent series expansion in ϵ and trivially solve the deqs as at each order the homogeneous solutions are constants.

Algorithm to find such basis by [Lee]

and implementation in `Epsilon, Canonica` ...

Drawback : Finding such a basis is not always possible.

Generic Basis

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

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To solve such a system, it would be best to organize it in such a way that it diagonalizes, or at least it takes a block-triangular form.

$$d_x \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix} = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & \bullet & \bullet & \bullet & \cdots & \bullet \\ 0 & 0 & 0 & \bullet & \cdots & \bullet \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \bullet \end{bmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ J_4 \\ \vdots \\ J_n \end{pmatrix}$$

Let's consider the 12^{th} blob from below

$$\frac{d}{dx}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} + \begin{pmatrix} R_1(\epsilon, x) \\ R_2(\epsilon, x) \\ R_3(\epsilon, x) \end{pmatrix},$$

$$c_{11} = \frac{(7 + 6x + 7x^2 - 2d(1 + x + x^2))}{x(1 + x)^2},$$

$$c_{12} = \frac{(-4 + d)(-10 + 3d)}{2(-3 + d)^2(1 + x)^2},$$

$$c_{13} = \frac{(d^2(15 + 8x + 15x^2) + 8(20 + 9x + 20x^2) - 2d(49 + 24x + 49x^2))}{4(-3 + d)^2 x(1 + x)^2}, \ldots$$

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c_{13} &= \frac{(d^2(15 + 8x + 15x^2) + 8(20 + 9x + 20x^2) - 2d(49 + 24x + 49x^2))}{4(-3 + d)^2 x(1 + x)^2}, \dots
\end{aligned}$$

Each order in ϵ-expansion gives a much simpler form

$$\frac{d}{dx}\begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} = \begin{bmatrix} \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \\ -\frac{1}{x} + \frac{2}{1+x} & \frac{1}{1+x} - \frac{1}{x} - \frac{1}{1-x} & \frac{1}{x} - \frac{2}{1+x} \\ \frac{1}{x} + \frac{2}{1-x} & 0 & \frac{1}{1+x} - \frac{2}{x} - \frac{3}{1-x} \end{bmatrix}\begin{pmatrix} J_1^{-3} \\ J_2^{-3} \\ J_3^{-3} \end{pmatrix} + \begin{pmatrix} R_1^{-3}(x) \\ R_2^{-3}(x) \\ R_3^{-3}(x) \end{pmatrix},$$

Algorithm

- A natural first step is to reduce the system to a higher order equation in a single unknown.
  `Note that, the inverse operation is trivial!`
- The classical/naive method to achieve this uncoupling is the cyclic vector algorithm. But, it gives a complicated decoupled equation.
- Use smarter uncoupling algorithms, e.g. `Zürcher` algorithm.
- The homogeneous solutions and uncoupling procedure are similar for each order.
- Now, what remains is to integrate the nonhomogeneous parts.

`Sigma` [Schneider], `OreSys` [Gerhold, Schneider]
and `HarmonicSums` [Ablinger, Blümlein, Schneider]

Iterated integrals and Harmonic polylogarithms (HPLs)

The nonhomogeneous parts have the following structure

$$\int dx \; K_{m_n}(x) \; f(x, \ln(x), \mathrm{Li}_n(x), \ldots)$$

$$K_m(x) = \left\{\frac{1}{x}, \frac{1}{1-x}, \frac{1}{1+x}, \frac{1}{1-x+x^2}, \frac{x}{1-x+x^2}\right\} \equiv \left\{0, 1, -1, \{6,0\}, \{6,1\}\right\}$$

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On the other hand, given a set of integration kernels $K_i(t)$, one can define

$$\mathcal{I}(m_n, \ldots m_1, x) = \int_{x_0}^{x} K_{m_n}(t) \mathcal{I}(m_{n-1}, \ldots m_1, t) dt$$

For example,

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t}\, \mathrm{Li}_{n-1}(t)$$

Iterated integrals and Harmonic polylogarithms (HPLs)

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Hence, one defines a new set of functions, called the HPLs

[Remiddi, Vermaseren, Gehrmann, Goncharov]

$$H(m_n, \ldots, m_1, x) = \int_0^x K_{m_n}(t) H(m_{n-1}, \ldots, m_1, t) dt$$

Some important properties :

Shuffle algebra, Scaling invariance and integration-by-parts identities

There exists a basis $\Rightarrow$ Fast numerical evaluation

Results

Results & Checks

- Computational : Automation to solve a single scale and first order factorizable system of differential equations.
- Phenomenological : We have performed UV renormalization and obtained all the form factors which are necessary to obtain N^3LO QCD corrections to top pair productions at electron-positron colliders.

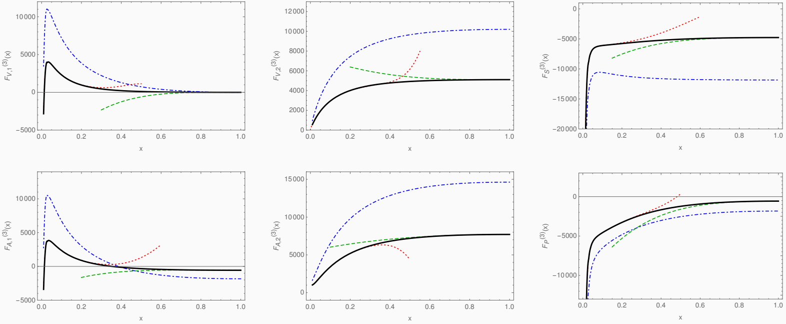
$$F_{V,1}^{(3)}, F_{V,2}^{(3)}, F_{A,1}^{(3)}, F_{A,2}^{(3)}, F_{S}^{(3)}, F_{P}^{(3)}$$

✓ We agree with the results from Lee $et\,al.$ obtained using different method

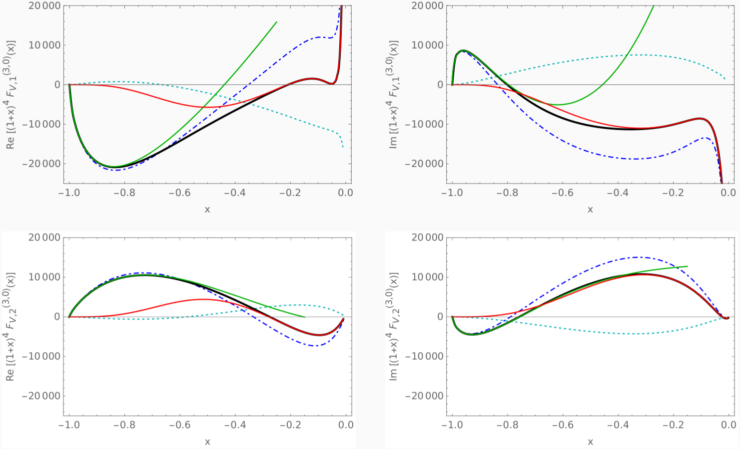
✓ The results reproduce the universal infrared structure

✓ Chiral Ward identity is satisfied between $F_{A,i}^{(3)}$ and $F_{P}^{(3)}$

THE FINITE COMPONENT OF THE FORM FACTORS and THEIR EXPANSIONS IN DIFFERENT REGION



The finite component of the VECTOR FORM FACTORS
threshold Expansions



Remarks

- For non-planar contributions, large differential equations $\sim$ 50 MBs!
- Also first-order non-factorizable differential equations appear!
  $\Rightarrow$ Solutions are Elliptic polylogarithms
- A proper generalization for iterative integrals over elliptic polylogarithms is needed!

Remarks

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- Also first-order non-factorizable differential equations appear!
  $\Rightarrow$ Solutions are Elliptic polylogarithms
- A proper generalization for iterative integrals over elliptic polylogarithms is needed!

We are at a crossroad!

New mathematical structures are emerging in loop calculations! An entirely new branch of research is opening, putting together computer scientists, mathematicians, particle physicists and string theorists. Maybe what we need now, is a fresh look at the problem with fresh ideas!

Thank you for your attention!