The QCD Equation of state at finite density from lattice

Sayantan Sharma

December 11, 2018
1. The QCD phase diagram: outstanding issues from lattice QCD

2. Equation of state at finite $\mu_B$

3. Critical-end point search from lattice
Outline

1. The QCD phase diagram: outstanding issues from lattice QCD
2. Equation of state at finite $\mu_B$
3. Critical-end point search from lattice
Perspectives from Lattice QCD

In view of the RHIC Beam Energy Scan-II in 2019-20, it is important to have control over the Equation of State for $\mu_B/T \leq 3$. 

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Measure the curvature of chiral and freezeout curves expected from QCD thermodynamics.
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Measure the curvature of chiral and freezeout curves expected from QCD thermodynamics.

Look for possible existence and bracket the position of critical end-point in the phase diagram.
Lattice techniques at finite $\mu_B$-l

- Conventional Monte-Carlo methods suffer from **sign problem** at finite $\mu_q$.
- Two methods presently allow to go to thermodynamic and continuum limits.

[From arXiv:1811.02494]
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$Z_{\text{QCD}}(\mu_q/T) = Z_{\text{QCD}}(\mu_q/T + 2ni\pi/3)$ implies Roberge-Weiss end-points at $\mu_q/T = (2n + 1)i\pi/3$.

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- Fitting it to a polynomial in $\mu_q$ analytically continue in the real-$\mu$ plane.

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[Diagram showing intermediate quark mass vs. temperature with transition points at RW trans. 1st order and Z(2) 2nd order (3d Ising) with crossover and TRW point.

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- Fitting it to a polynomial in $\mu_q$ analytically continue in the real-$\mu$ plane.
- Limited due to discontinuities at Roberge-Weiss end-points!.

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Lattice techniques at finite $\mu_B$-II

- **Taylor expansion** of physical observables around $\mu = 0$ in powers of $\mu/T$  
  [Bi-Swansea collaboration, 02]

\[
\frac{P(\mu_B, T)}{T^4} = \frac{P(0, T)}{T^4} + \left(\frac{\mu_B}{T}\right)^2 \frac{\chi^B_2(0, T)}{2T^2} \chi^B_2(0, T) + \left(\frac{\mu_B}{T}\right)^4 \frac{\chi^B_4(0)}{4!} + \ldots
\]

- The series for $\chi^B_2(\mu_B)$ should diverge at the critical point. On finite lattice $\chi^B_2$ peaks, ratios of Taylor coefficients equal, indep. of volume  
  [Gavai & Gupta, 03]
Challenges for Taylor expansion

- The fluctuations of conserved charges can be expressed in terms of Quark number susceptibilities (QNS).
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- QNS $\chi_{ij}$'s can be written as derivatives of the Dirac operator.

Example:

$$\chi^u_{2} = \frac{T}{V} \langle Tr(D_u^{-1}D''_u - (D_u^{-1}D'_u)^2) + (Tr(D_u^{-1}D'_u))^2\rangle.$$ $$\chi^{us}_{11} = \frac{T}{V} \langle Tr(D_u^{-1}D'_u D_s^{-1}D'_s)\rangle.$$
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- Higher derivatives $\rightarrow$ more inversions
  - Inversion is the most expensive step on the lattice!
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- Higher derivatives $\rightarrow$ more inversions
  Inversion is the most expensive step on the lattice!
- Why extending to higher orders so difficult?
  - Matrix inversions increasing with the order
  - Delicate cancellation between a large number of terms for higher order QNS.
Recent developments: A new method to introduce $\mu$

- The staggered fermion matrix used at finite $\mu$ [Hasenfratz, Karsch, 83]

$$D(\mu)_{xy} = \sum_{i=1}^{3} \eta_i(x) \left[ U_i^\dagger(y) \delta_{x,y+i} - U_i(x) \delta_{x,y-i} \right]$$

$$+ \eta_4(x) \left[ e^{\mu a} U_4^\dagger(y) \delta_{x,y+\hat{4}} - e^{-\mu a} U_4(x) \delta_{x,y-\hat{4}} \right]$$
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$$+ \eta_4(x) \left[ e^{\mu a} U_4^\dagger(y) \delta_{x,y+4} - e^{-\mu a} U_4(x) \delta_{x,y-4} \right]$$

- One can also add $\mu$ coupled to the conserved number density as in the continuum.

$$D(0)_{xy} - \frac{\mu a}{2} \eta_4(x) \left[ U_4^\dagger(y) \delta_{x,y+4} + U_4(x) \delta_{x,y-4} \right]$$
Pros and Cons

- Linear method: \( D' = \sum_{x,y} N(x, y) \), and 
  \( D'' = D''' = D'''' \ldots = 0 \)

  in contrast to the Exp-prescription, all derivatives are non-zero.
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  For 8th order QNS the no. of matrix inversions reduced from 20 to 8
  for staggered fermions. [Gavai & Sharma, 12]
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  [Gavai & Sharma, 15]
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- **In Exp method:** counter terms already at the Lagrangian level. We use this method for $\chi_n^B$, $n = 2, 4$.  
  
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Our Set-up

- $V = N^3 a^3$, Box size: $m_\pi V^{1/3} > 4$.
- $T = \frac{1}{N_\tau a}$
  
  We use $N_\tau = 6, 8, 12, 16$ lattices for $\chi_{2,4}$ and $N_\tau = 6, 8$ for higher order fluctuations.

- Input $m_s$ physical and $m_\pi^G = 160$ MeV for $T > 175$ MeV and $m_\pi^G = 140$ MeV for $T \leq 175$ MeV.
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EoS in the constrained case

- In most central heavy-ion experiments typically:
  \[ n_S = 0, \text{ Strangeness neutrality}, \]
  \[ \frac{n_Q}{n_B} = \frac{n_P}{n_B+n_N} = 0.4. \]
  [Bi-BNL collaboration, 1208.1220]

- For lower \( \sqrt{s} \) collisions: Need to understand baryon stopping!
- Imposes non-trivial constraints on the variation of \( \mu_S \) and \( \mu_Q \).
- Possible to vary them by only varying \( \mu_B \) through

\[
\begin{align*}
\mu_S &= s_1 \mu_B + s_3 \mu_B^3 + s_5 \mu_B^5 + \ldots \\
\mu_Q &= q_1 \mu_B + q_3 \mu_B^3 + q_5 \mu_B^5 + \ldots
\end{align*}
\]
Central values of $P_4, P_6$ already deviate from Hadron Resonance gas model at $T > 145$ MeV $\rightarrow$ need to analyze the errors on $P_6$ better.

$P_6$ has characteristic structure at $T > T_c$ $\rightarrow$ remnant of the chiral symmetry due to the light quarks. Effects of $U_A(1)$ anomaly?

Essentially non-perturbative $\rightarrow$ cannot be predicted within Hard Thermal Loop perturbation theory.
EoS in the constrained case

- The EoS is well under control for $\mu_B/T \sim 2.5$ with $\chi_6$.
- Full parametric dependence for $N_B$ on $T$ available in arxiv: 1701.04325.
- Expanding to $\mu_B/T = 3$, need to calculate $\chi_8$!
Summary for the EoS

- Continuum estimates from two different fermion discretization agree for \( \mu_B/T \leq 2 \).
  [Bielefeld-BNL-CCNU collaboration, 1701.04325, Borsanyi et. al, 1606.07494].

- Steeper EoS for RHIC energies compared to LHC energy.
\( \chi_6 \) contribution is 30-times larger than in pressure.

\[
\frac{N(\mu_B)}{T^3} = \frac{\mu_B}{T} \chi_2^B(0) + \frac{1}{2} \left( \frac{\mu_B}{T} \right)^4 \chi_4^B(0) + \frac{1}{4!} \left( \frac{\mu_B}{T} \right)^6 \chi_6^B(0) + ... 
\]

Strongly sensitive to the singular part of \( \chi_6^B \).

For strangeness neutral system, effect is milder.
Can $T < T_c$ be described by Hadron Resonance Gas?

- Effects of interactions between hadrons can be mimicked by free gas of resonances + hadrons?
- Some baryon channels do not have resonances. \textit{In-medium modification of baryons}?
- Lattice data for higher order baryon no. fluc. are precise enough to distinguish between diff. scenarios → support additional resonances from quark-models + interactions

$$\chi^B_4$$

![Graph and Table Image]

[P. Huovinen, P. Petreczky, 1811.09330]
Can $T < T_c$ be described by Hadron Resonance Gas?

Including Van der Waal’s interaction for baryons + non-interacting mesons + resonances, new versions of HRG has been studied → significant deviation from non-interacting HRG.

Lattice data can constrain such models strongly! Currently none of these models are perfect to describe QCD at freezeout.

[F. Karsch, QM17 proceedings, 1706.01620]

[V. Vovchenko, M. I. Gorenstein and H. Stoecker 1609.03975]
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Curvature of the chiral crossover line

\[ \frac{T_c(\mu X)}{T_c(0)} = 1 - \kappa_2^X \frac{\mu_X^2}{T_c(0)^2} - \kappa_4^X \frac{\mu_X^4}{T_c(0)^4} \]

For strangeness neutral system, continuum results available!
\( \kappa_2^B = 0.012(4) , \kappa_4^B \sim 0 \) with Taylor expansions and HISQ fermions.

[HotQCD collaboration, 1807.05607, and in prep.]
Curvature of the chiral crossover line

\[
\frac{T_c(\mu x)}{T_c(0)} = 1 - \kappa_2^X \frac{\mu^2}{T_c(0)^2} - \kappa_4^X \frac{\mu^4}{T_c(0)^4}
\]

Consistent with imaginary chemical potential method and stout fermions
\(\kappa_2^B = 0.0135(20)\) [C. Bonati et. al., 1805.02960]
removes earlier tension between two methods! [courtesy M. D’Elia Quark Matter 18]

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Curvature of freeze-out line vs chiral crossover line

- Different LCP’s agree within 2 MeV for $\mu_B/T \leq 2$ for 3 initial choices of $T_0$.
- For lines $P = \text{const}$, the entropy density changes by 15% $\rightarrow$ better description of LCP for viscous medium formed in heavy-ion collisions.

[HotQCD collaboration, 1701.04325].

- STAR results give a steeper curvature.
  [arXiv:1412.0499].
- Agreement with the recent ALICE results. [arXiv:1408.6403].
- Consistent with phenomenological models. [Becattini et. al., 1605.09694].
The Taylor series for $\chi^B_2(\mu_B)$ should diverge at the critical point. On finite lattice $\chi^B_2$ peaks, ratios of Taylor coefficients equal, indep. of volume.

The radius of convergence determines location of the critical point.

Definition: $r_{2n} \equiv \sqrt{2n(2n - 1) \left| \frac{\chi^B_{2n}}{\chi^B_{2n+2}} \right|}$.

- Strictly defined for $n \to \infty$. How large $n$ could be on a finite lattice?
- Signal to noise ratio deteriorates for higher order $\chi^B_n$. 
Critical-end point search from Lattice

- Current bound for CEP: $\frac{\mu_B}{T} > 3$ for $142 \leq T \leq 150$ MeV
  [HotQCD coll., 1701.04325, update 2018].
- The $r_n$ extracted by analytic continuation of imaginary $\mu_B$ data
  [D’Elia et. al., 1611.08285] consistent with this bound.
- Results with a lower bound? [Datta et. al., 1612.06673, Fodor and Katz, 04] → need to understand the systematics in these studies. Ultimately all estimates will agree in the continuum limit!

![Graph showing the critical-end point search from Lattice with various data points and estimates.](image-url)
Critical end-point and Chiral Crossover line: current status

Steeper curvature would imply slow convergence of $r_n$ with order $n$
Critical end-point and Chiral Crossover line: current status

For current trends
\( \kappa_4 = \kappa_6 = \kappa_8 \ldots \sim 0 \),
radius of curvature estimates tell us
\( T_{CEP} \sim 0.92 T_c(0) \) and
\( \mu_B / T_{CEP} > 3 \).

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For current trends $\kappa_4 = \kappa_6 = \kappa_8 \ldots \sim 0$, radius of curvature estimates tell us $T_{CEP} \sim 0.92 T_c(0)$ and $\mu_B / T_{CEP} > 3$.

If $\kappa_4 \sim 0.1 \kappa_2$, only significantly contributes when $\mu_B / T_{CEP} > 3$ so its precise determination is imp.

Steeper curvature would imply slow convergence of $r_n$ with order $n$
Preparing for BES-II runs: LQCD EoS important for hydrodynamic modeling of QGP. For $\frac{\mu_B}{T} \leq 2 \rightarrow \sqrt{s_{NN}} \geq 11$ GeV already under control with $\chi_6^B$. 
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Lines of constant $\epsilon, p$ consistent with LQCD estimates of curvature of chiral crossover line.
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Higher order cumulants of baryon no. will also help in bracketing the possible CEP. Most LQCD calculations suggest $\mu_B(\text{CEP}) / T \geq 3$. 