Using A4 to ameliorate popular lepton mixings: A model for realistic neutrino masses and mixing based on see-saw

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The A4 Group

Group of of even permutations of four objects comprising of $4!/2 = 12$ elements.

Generated by S and T satisfying $S^2 = T^3 = (ST)^3 = \mathbb{I}$.

Has four inequivalent irreducible representations.

$$1 \quad S = 1 \quad T = 1.$$
$$1' \quad S = 1 \quad T = \omega.$$
$$1'' \quad S = 1 \quad T = \omega^2.$$
$$3: \quad S = diag(1, -1, -1) \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Multiplication Rules

$$1' \times 1'' = 1, \quad 1' \times 1' = 1'', \quad 1'' \times 1'' = 1',$$
$$\boxed{3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}.}$$
$$\Downarrow$$
$$1 = a_1b_1 + a_2b_2 + a_3b_3 \equiv \rho_{1ij}a_ib_j, \quad 1' = a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3 \equiv \rho_{3ij}a_ib_j,$$
$$1'' = a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3 \equiv \rho_{2ij}a_ib_j.$$
$$3 \sim (a_2b_3, a_3b_1, a_1b_2), \quad \bar{3} \sim (a_3b_2, a_1b_3, a_2b_1).$$
$$3_{sym} \equiv \frac{1}{2}(3 + \bar{3}) = \alpha_{ijk}a_jb_k$$
$$\text{and} \quad 3_{antisym} \equiv \frac{1}{2}(3 - \bar{3}) = \beta_{ijk}a_jb_k$$

Three flavor mixing

Neutrinos are massive and they mix !!

Three flavors: $\boldsymbol{\nu_e, \nu_\mu, \nu_\tau}$

Oscillation probability

$$P_{\nu_\alpha \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{j>i} U_{\alpha i} U_{\beta i} U^*_{\alpha j} U^*_{\beta j} \sin^2\left(\frac{\pi L}{\lambda_{ij}}\right)$$

2 independent Δm^2, 3 mixing angles, 1 phase

$$U^T M_\nu U = \mathrm{diag}(m_1, m_2, m_3)$$

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

The Pontecorvo, Maki, Nakagawa, Sakata – PMNS – matrix

A measure of CP-violation is given by the basis-independent leptonic Jarlskog(J) parameter:

$$J = \mathrm{Im}[U_{e1}U_{\mu 2}U^*_{e2}U^*_{\mu 1}]$$

Popular lepton mixings

Recall:

$$U_{PMNS} \equiv V_l^\dagger U_\nu = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}.$$

$\downarrow$

θ_{13}=0, θ_{23}=$\pi/4$

$\downarrow$

$$U^0 = \begin{pmatrix} \cos\theta_{12}^0 & \sin\theta_{12}^0 & 0 \\ -\frac{\sin\theta_{12}^0}{\sqrt{2}} & \frac{\cos\theta_{12}^0}{\sqrt{2}} & \sqrt{\frac{1}{2}} \\ \frac{\sin\theta_{12}^0}{\sqrt{2}} & -\frac{\cos\theta_{12}^0}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \to \theta_{12}^0 = 0^\circ(\mathrm{NSM}) \to U^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

$\Downarrow$

$\theta_{12}^0 : 35.26^\circ$ [TriBimaximal(TBM)], 45° [Bimaximal (BM)], 31.7° [Golden Ratio(GR)].

$$\begin{aligned} \Delta m_{21}^2 &= (7.02 - 8.08)\times 10^{-5}\,\mathrm{eV}^2,\ \ \theta_{12} = (31.52 - 36.18)^\circ, \\ |\Delta m_{31}^2| &= (2.351 - 2.618)\times 10^{-3}\,\mathrm{eV}^2,\ \ \theta_{23} = (38.6 - 53.1)^\circ, \\ \theta_{13} &= (7.86 - 9.11)^\circ,\ \ \delta = (0 - 360)^\circ\ . \end{aligned}$$

NuFIT2.1 of 2016

Amendment Required!!

Objectives

$A4$ Model

Type II Seesaw (dominant) | **Type I + II Seesaw**

Case 1: $\Delta m^2_\odot = 0$; NSM

Case 2: $\Delta m^2_\odot = 0$; TBM

Case 3: $\Delta m^2_\odot = 0$; BM

Case 4: $\Delta m^2_\odot = 0$; GR

$\longrightarrow$ $\Delta m^2_\odot \neq 0$; $\theta_{13}, \theta_{12} \neq 0$; $\theta_{23} \neq \pi/4$ allowed by data

Recall: **Popular Lepton Mixings**

Mixings	TBM	BM	GR	NSM
θ^0_{12}	35.3°	45.0°	31.7°	0.0°
θ_{13}	0.0°	0.0°	0.0°	0.0°
θ_{23}	45.0°	45.0°	45.0°	45.0°

Lepton Catalogue

Fields	Notations	$A4$	$SU(2)_L\,(Y)$	L
Left-handed leptons	$(\nu_i, l_i)_L$	3	2 (-1)	1
Right-handed charged leptons	l_{1R}	1	1 (-2)	1
	l_{2R}	$1'$		
	l_{3R}	$1''$		
Right-handed neutrinos	N_{iR}	3	1 (0)	-1

Purpose	Notations	$A4$	$SU(2)_L$ (Y)	L	vev
Charged fermion mass	$\Phi = \begin{pmatrix} \phi_1^+ & \phi_1^0 \\ \phi_2^+ & \phi_2^0 \\ \phi_3^+ & \phi_3^0 \end{pmatrix}$	3	2 (1)	0	$\langle\Phi\rangle = \frac{v}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$
Neutrino Dirac mass	$\eta = (\eta^0, \eta^-)$	1	2 (-1)	2	$\langle\eta\rangle = \begin{pmatrix} u, 0 \end{pmatrix}$
Type-II see-saw mass	$\hat{\Delta}_a^L = \begin{pmatrix} \hat{\Delta}_{1a}^{++} & \hat{\Delta}_{1a}^{+} & \hat{\Delta}_{1a}^{0} \\ \hat{\Delta}_{2a}^{++} & \hat{\Delta}_{2a}^{+} & \hat{\Delta}_{2a}^{0} \\ \hat{\Delta}_{3a}^{++} & \hat{\Delta}_{3a}^{+} & \hat{\Delta}_{3a}^{0} \end{pmatrix}^L$	3	3 (2)	-2	$\langle\hat{\Delta}_a^L\rangle = v_{La} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
Type-II see-saw mass	$\hat{\Delta}_b^L = \begin{pmatrix} \hat{\Delta}_{1b}^{++} & \hat{\Delta}_{1b}^{+} & \hat{\Delta}_{1b}^{0} \\ \hat{\Delta}_{2b}^{++} & \hat{\Delta}_{2b}^{+} & \hat{\Delta}_{2b}^{0} \\ \hat{\Delta}_{3b}^{++} & \hat{\Delta}_{3b}^{+} & \hat{\Delta}_{3b}^{0} \end{pmatrix}^L$	3	$3(2)$	-2	$\langle\hat{\Delta}_b^L\rangle = v_{Lb} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$
Type-II see-saw mass	$\Delta_\zeta^L = (\Delta_\zeta^{++}, \Delta_\zeta^{+}, \Delta_\zeta^{0})^L$	1	$3(2)$	-2	$\langle\Delta_1^L\rangle = \begin{pmatrix} 0, 0, u_L \end{pmatrix}$
		$1'$	$3(2)$	-2	$\langle\Delta_2^L\rangle = \begin{pmatrix} 0, 0, u_L \end{pmatrix}$
		$1''$	$3(2)$	-2	$\langle\Delta_3^L\rangle = \begin{pmatrix} 0, 0, u_L \end{pmatrix}$
Right-handed neutrino mass	$\hat{\Delta}_a^R = \begin{pmatrix} \hat{\Delta}_{1a}^0 \\ \hat{\Delta}_{2a}^0 \\ \hat{\Delta}_{3a}^0 \end{pmatrix}^R$	3	$1(0)$	2	$\langle\hat{\Delta}_a^R\rangle = v_{Ra} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$
Right-handed neutrino mass	$\hat{\Delta}_b^R = \begin{pmatrix} \hat{\Delta}_{1b}^0 \\ \hat{\Delta}_{2b}^0 \\ \hat{\Delta}_{3b}^0 \end{pmatrix}^R$	3	$1(0)$	2	$\langle\hat{\Delta}_b^R\rangle = v_{Rb} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}$
Right-handed neutrino mass	$\hat{\Delta}_c^R = \begin{pmatrix} \hat{\Delta}_{1c}^0 \\ \hat{\Delta}_{2c}^0 \\ \hat{\Delta}_{3c}^0 \end{pmatrix}^R$	3	$1(0)$	2	$\langle\hat{\Delta}_c^R\rangle = v_{Rc} \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$
Right-handed neutrino mass	$\Delta_1^R = (\Delta_1^0)^R$	1	$1(0)$	2	$\langle\Delta_1^R\rangle = u_{1R}$
Right-handed neutrino mass	$\Delta_2^R = (\Delta_2^0)^R$	$1'$	$1(0)$	2	$\langle\Delta_2^R\rangle = u_{2R}$
Right-handed neutrino mass	$\Delta_3^R = (\Delta_3^0)^R$	$1''$	$1(0)$	2	$\langle\Delta_3^R\rangle = u_{3R}$

3HDM talk by SP
Dec 11, 2018
BSM-IIB (16:15 hrs)

Mass Model

The $SU(2)_L$ and $A4$ conserving Lagrangian:

$$
\begin{aligned}
\mathscr{L}_{mass} &= y_j \rho_{jik} \bar{l}_{Li} l_{Rj} \Phi^0_k \ \ \text{(charged lepton mass)} \\
&+ f \rho_{1ik} \bar{\nu}_{Li} N_{Rk} \eta^0 \ \ \text{(neutrino Dirac mass)} \\
&+ \frac{1}{2}\left(\sum_{n=a,b} \hat{Y}^L_n \ \alpha_{ijk} \nu^T_{Li} C^{-1} \nu_{Lj} \hat{\Delta}^{L0}_{nk} + Y^L_\zeta \ \rho_{\zeta ij} \nu^T_{Li} C^{-1} \nu_{Lj} \Delta^{L0}_\zeta \right) (\text{Type} - \text{II seesaw}) \\
&+ \frac{1}{2}\left(\sum_{p=a,b,c} \hat{Y}^R_p \ \alpha_{ijk} N^T_{Ri} C^{-1} N_{Rj} \hat{\Delta}^{R0}_{kp} + Y^R_\gamma \ \rho_{\gamma ij} N^T_{Ri} C^{-1} N_{Rj} \Delta^{R0}_\gamma \right) (\text{rh } \nu \text{ mass}) + h.c.
\end{aligned}
$$

Choose: $Y^L_2 = Y^L_3$

$$
M_{e\mu\tau} = \frac{v}{\sqrt{3}} \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1 & \omega y_2 & \omega^2 y_3 \\ y_1 & \omega^2 y_2 & \omega y_3 \end{pmatrix}, \quad M_{\nu L} = \begin{pmatrix} (Y^L_1 + 2Y^L_2) u_L & \frac{1}{2}\hat{Y}^L_b v_{Lb} & \frac{1}{2}\hat{Y}^L_b v_{Lb} \\ \frac{1}{2}\hat{Y}^L_b v_{Lb} & (Y^L_1 - Y^L_2) u_L & \frac{1}{2}(\hat{Y}^L_a v_{La} + \hat{Y}^L_b v_{Lb}) \\ \frac{1}{2}\hat{Y}^L_b v_{Lb} & \frac{1}{2}(\hat{Y}^L_a v_{La} + \hat{Y}^L_b v_{Lb}) & (Y^L_1 - Y^L_2) u_L \end{pmatrix}
$$

$$
M_D = f u \, \mathbb{I} \ , \quad M_{\nu R} = m_R \begin{pmatrix} \chi_1 & \chi_6 & \chi_5 \\ \chi_6 & \chi_2 & \chi_4 \\ \chi_5 & \chi_4 & \chi_3 \end{pmatrix} \quad \text{where, } \chi_i s \text{ are :}
$$

$$
\begin{aligned}
m_R \chi_1 &\equiv (Y^R_1 u_{1R} + Y^R_2 u_{2R} + Y^R_3 u_{3R}) \\
m_R \chi_2 &\equiv (Y^R_1 u_{1R} + \omega Y^R_2 u_{2R} + \omega^2 Y^R_3 u_{3R}) \\
m_R \chi_3 &\equiv (Y^R_1 u_{1R} + \omega^2 Y^R_2 u_{2R} + \omega Y^R_3 u_{3R}) \\
m_R \chi_4 &\equiv \frac{1}{2}(\hat{Y}^R_a v_{Ra} + \hat{Y}^R_b v_{Rb} + \hat{Y}^R_c v_{Rc}) \\
m_R \chi_5 &\equiv \frac{1}{2}(\hat{Y}^R_a v_{Ra} + \omega \hat{Y}^R_b v_{Rb} + \omega^2 \hat{Y}^R_c v_{Rc}) \\
m_R \chi_6 &\equiv \frac{1}{2}(\hat{Y}^R_a v_{Ra} + \omega^2 \hat{Y}^R_b v_{Rb} + \omega \hat{Y}^R_c v_{Rc}).
\end{aligned}
$$

Mass Matrices

- Apply U_L on the left handed fermion doublets and V_R on right handed neutrinos keeping the right handed charged fermions intact.

$$U_L = V_R = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} .$$

Identify:

$$Y_1^R u_{1R} = m_R(r_{11} + 2r_{23}), \;\; Y_2^R u_{2R} = m_R(r_{22} + 2r_{13}), \;\; Y_3^R u_{3R} = m_R(r_{33} + 2r_{12})$$

$$\hat{Y}_a^R v_{Ra} = 2m_R(r_{11} - r_{23}), \;\; \hat{Y}_b^R v_{Rb} = 2m_R(r_{22} - r_{13}) \;\text{ and }\; \hat{Y}_c^R v_{Rc} = 2m_R(r_{33} - r_{12}).$$

$$3(Y_1^L + 2Y_2^L)u_L = (m_3^{(0)} + m^+), 6(Y_1^L - Y_2^L)u_L = \hat{Y}_a^L v_{La} = m^+, 3\hat{Y}_b^L v_{Lb} = -2m^-.$$

- The resultant mass matrices:

$$M_{e\mu\tau}^{flavour} = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \;,\;\; M_{\nu L}^{flavour} = \frac{1}{2} \begin{pmatrix} 2m_1^{(0)} & 0 & 0 \\ 0 & m^+ & m^- \\ 0 & m^- & m^+ \end{pmatrix} \;,$$

$$M_D = f u \mathbb{I}, \;\; (M_{\nu R}^{flavour})_{ij} = \frac{m_R}{4} r_{ij} \Rightarrow$$

$$\begin{aligned}
r_{11} &\equiv \sqrt{2}b \sin 2\theta_{12}^0 + a \sin^2 \theta_{12}^0, \\
r_{22} &\equiv -\sqrt{2}b \sin \theta_{12}^0 - \frac{b}{2} \sin 2\theta_{12}^0 - a \cos \theta_{12}^0 + \frac{a}{2} \cos^2 \theta_{12}^0 + \frac{a}{2} \\
r_{33} &\equiv -\frac{b}{\sqrt{2}} \sin 2\theta_{12}^0 - \sqrt{2}b \sin \theta_{12}^0 + a \cos \theta_{12}^0 + \frac{a}{2} \cos^2 \theta_{12}^0 + \\
r_{12} &\equiv b \cos 2\theta_{12}^0 + \frac{a}{2\sqrt{2}} \sin^2 \theta_{12}^0 + b \cos \theta_{12}^0 - \frac{a}{\sqrt{2}} \sin \theta_{12}^0, \\
r_{13} &\equiv -b \cos 2\theta_{12}^0 - \frac{a}{2\sqrt{2}} \sin^2 \theta_{12}^0 + b \cos \theta_{12}^0 - \frac{a}{\sqrt{2}} \sin \theta_{12}^0 \\
r_{23} &\equiv \frac{b}{2} \sin 2\theta_{12}^0 - \frac{a}{2} \cos^2 \theta_{12}^0 + \frac{a}{2} \;\; .
\end{aligned}$$

Charged lepton mass matrix is diagonal in the resultant basis $\Rightarrow$ $Flavour\ basis$.

$$\boxed{m^\pm \equiv (m_3^{(0)} \pm m_1^{(0)})} \Rightarrow m^- \text{ is +ve (-ve) for NO (IO)}$$

Mass matrices and Seesaw

- The $M_{\nu L}^{flavour}$ originates from **Type II Seesaw** and is diagonalized by U^0:

$$M^0 = M_{\nu L}^{mass} = U^{0T} M_{\nu L}^{flavour} U^0 = diag(m_1^{(0)}, m_1^{(0)}, m_3^{(0)}) \Rightarrow \Delta m_\odot^2 = 0$$

Here,

$$U^0 = \begin{pmatrix} \cos\theta_{12}^0 & \sin\theta_{12}^0 & 0 \\ -\frac{\sin\theta_{12}^0}{\sqrt{2}} & \frac{\cos\theta_{12}^0}{\sqrt{2}} & \sqrt{\frac{1}{2}} \\ \frac{\sin\theta_{12}^0}{\sqrt{2}} & -\frac{\cos\theta_{12}^0}{\sqrt{2}} & \sqrt{\frac{1}{2}} \end{pmatrix} \rightarrow \text{Columns : unperturbed flavour basis}$$

Demand $M_D \propto \mathbb{I}$ in this basis. $\Rightarrow$ Apply $U^{0\dagger}$ on right-handed neutrinos.

$$M_D = m_D \mathbb{I} \text{ and } M_{\nu R}^{mass} = \frac{m_R}{2\sqrt{2}ab} \begin{pmatrix} 0 & b & b \\ b & \frac{a}{\sqrt{2}} & \frac{-a}{\sqrt{2}} \\ b & \frac{-a}{\sqrt{2}} & \frac{a}{\sqrt{2}} \end{pmatrix} .$$

$m_D \rightarrow$ Dirac neutrino mass scale $m_R \rightarrow$ right-handed neutrino mass scale.

- Put $a = xe^{-i\phi_1}; \;\; b = ye^{-i\phi_2}$

  Type I Seesaw: $M'^{mass} = [M_D^T (M_{\nu R}^{mass})^{-1} M_D]$

$$M'^{mass} = \frac{m_D^2}{m_R} \begin{pmatrix} 0 & y\, e^{i\phi_1} & y\, e^{i\phi_1} \\ y\, e^{i\phi_1} & \frac{x\, e^{i\phi_2}}{\sqrt{2}} & \frac{-x\, e^{i\phi_2}}{\sqrt{2}} \\ y\, e^{i\phi_1} & \frac{-x\, e^{i\phi_2}}{\sqrt{2}} & \frac{x\, e^{i\phi_2}}{\sqrt{2}} \end{pmatrix} .$$

Analysis

$$M'^{mass} = \frac{m_D^2}{m_R}\begin{pmatrix} 0 & ye^{i\phi_1} & ye^{i\phi_1} \\ ye^{i\phi_1} & \frac{xe^{i\phi_2}}{\sqrt{2}} & \frac{-xe^{i\phi_2}}{\sqrt{2}} \\ ye^{i\phi_1} & \frac{-xe^{i\phi_2}}{\sqrt{2}} & \frac{xe^{i\phi_2}}{\sqrt{2}} \end{pmatrix}.$$

Unperturbed term: $M^{0\dagger}M^0$, Perturbation: $(M^{0\dagger}M' + M'^\dagger M^0)$; CP violation allowed.

$$(M^{0\dagger}M' + M'^\dagger M^0)^{mass} = \frac{m_D^2}{m_R}\begin{pmatrix} 0 & 2ym_1^{(0)}\cos\phi_1 & yf(\phi_1) \\ 2ym_1^{(0)}\cos\phi_1 & \sqrt{2}xm_1^{(0)}\cos\phi_2 & -\frac{x}{\sqrt{2}}f(\phi_2) \\ yf^*(\phi_1) & -\frac{x}{\sqrt{2}}f^*(\phi_2) & \sqrt{2}xm_3^{(0)}\cos\phi_2 \end{pmatrix}.$$

where, $f(\varphi) = m^+\cos\varphi - im^-\sin\varphi$.

- **Solar Sector** :

$$\theta_{12} = \theta_{12}^0 + \zeta \ , \ \tan 2\zeta = 2\sqrt{2}\,\frac{y}{x}\,\frac{\cos\phi_1}{\cos\phi_2} \ .$$

$$\sin\epsilon = \frac{y\cos\phi_1}{\sqrt{y^2\cos^2\phi_1 + x^2\cos^2\phi_2/2}} \ , \ \cos\epsilon = \frac{x\cos\phi_2/\sqrt{2}}{\sqrt{y^2\cos^2\phi_1 + x^2\cos^2\phi_2/2}} \ , \ \tan\epsilon = \frac{1}{2}\tan 2\zeta \ .$$

$$\Delta m^2_{solar} = \frac{2m^-m_1^{(0)}\sin\theta_{13}\cos\delta\cos\epsilon}{\sin(\epsilon - \theta_{12}^0)\ cos2\zeta}$$

Model (θ_{12}^0)	TBM (35.3°)	BM (45.0°)	GR (31.7°)	NSM (0.0°)
ζ	-4.0° $\leftrightarrow$ 0.6°	-13.7° $\leftrightarrow$ -9.1°	-0.4° $\leftrightarrow$ 4.2°	31.3° $\leftrightarrow$ 35.9°
ϵ	-4.0° $\leftrightarrow$ 0.6°	-14.5° $\leftrightarrow$ -9.3°	-0.4° $\leftrightarrow$ 4.2°	44.0° $\leftrightarrow$ 56.7°
$\epsilon - \theta_{12}^0$	-39.2° $\leftrightarrow$ -34.6°	-59.5° $\leftrightarrow$ -54.4°	-39.2° $\leftrightarrow$ -30.0°	44.0° $\leftrightarrow$ 56.7°

Analysis Continued .. .

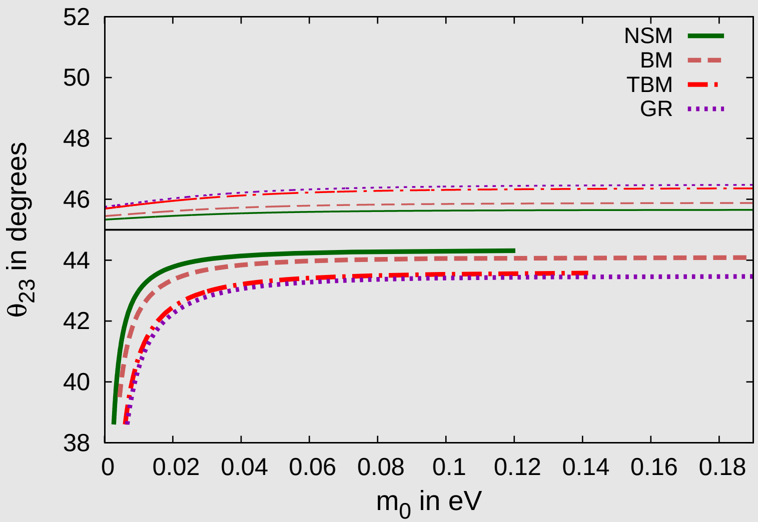
$$\sin\theta_{13}\cos\delta = \kappa_c\ \sin(\epsilon - \theta_{12}^0)\ \text{ where }\ \kappa_c = \frac{m_D^2}{m_R m^-}\sqrt{y^2\cos^2\phi_1 + x^2\cos^2\phi_2/2}\ ,$$

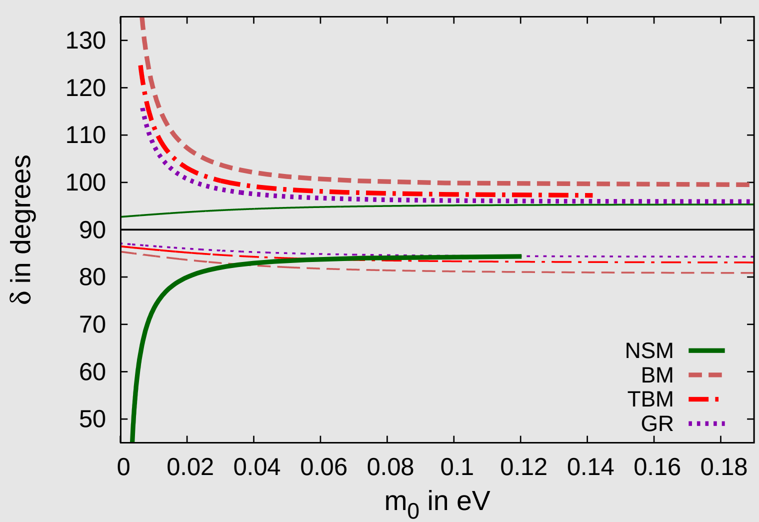
$$\sin\theta_{13}\sin\delta = \kappa_c\ \frac{m^-}{m^+\cos\phi_1\cos\phi_2}\left[\sin\epsilon\sin\phi_1\cos\phi_2\cos\theta_{12}^0 - \cos\epsilon\cos\phi_1\sin\phi_2\sin\theta_{12}^0\right]\ .$$

$$\tan\theta_{23} \equiv \tan(\pi/4-\omega) \Rightarrow \tan\omega = \frac{\sin\theta_{13}\cos\delta}{\tan(\epsilon-\theta_{12}^0)}$$

Mixing Pattern	Normal Ordering		Inverted Ordering	
	δ quadrant	θ_{23} octant	δ quadrant	θ_{23} octant
NSM	First/Fourth	First	Second/Third	Second
BM, TBM, GR	Second/Third	First	First/Fourth	Second

Results





Conclusions

- **General discussion:** a) Discrete flavour symmetry $A4$.
  b) Neutrino masses and mixings.
- $A4$ **conserving neutrino mass model.**
- Model had two components: dominant *Type II Seesaw* part and subdominant *Type I Seesaw* contribution.
- *Type II Seesaw* $\rightarrow \Delta m^2_{solar} = 0$ for TBM, BM, GR and NSM case.
- *Type II +I Seesaw* $\rightarrow \Delta m^2_{solar} \neq 0$; $\theta_{12} \neq 0, \;\; \theta_{13} \neq 0, \;\; \theta_{23} \neq \pi/4$ (allowed by data).
- Model has testable predictions.

THANK YOU!!

Backup Slides

Some more on A4 .. .

$A4$ **invariants come from:** a) $1 \times 1 = \boxed{1}$ (trivial)

b) $1' \times 1'' = \boxed{1}$

c) $3 \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

- **Consider four $A4$ triplets:** $X_1, \; X_2, \; X_3$ and X_4

Combine: $X_i X_j$ and $X_k X_l$

$\Downarrow$ $\qquad$ $\Downarrow$

$3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}$ $\qquad$ $3 \times 3 = 1 + 1' + 1'' + 3 + \bar{3}$

First way: $1 \times 1 = \boxed{1}$

Second way: $1' \times 1'' = \boxed{1}$

Third way: $1'' \times 1' = \boxed{1}$

Fourth way: (a) $3 \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(b) $\bar{3} \times 3 = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(c) $3 \times \bar{3} = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

(d) $\bar{3} \times \bar{3} = \boxed{1} + 1' + 1'' + 3 + \bar{3}$

Scalar Potential

Notations:

$$A = (a_1, a_2, a_3)^T; B = (b_1, b_2, b_3)^T \quad \in \quad 3 \ \text{of} \ \ A4$$

$$3_A \otimes 3_B = 1 \oplus 1' \oplus 1'' \oplus 3 \oplus 3 \quad .$$

$$\rho_{1ij} a_i b_j \equiv O_1(A,B); \qquad \rho_{2ij} a_i b_j \equiv O_3(A,B); \qquad \rho_{3ij} a_i b_j \equiv O_2(A,B);$$

$3_{sym} \equiv T_s(A,B)$ and $3_{antisym} \equiv T_a(A,B)$

Guiding Principles:

a) All couplings are real.

b) No two scalars have all quantum numbers same. So terms in the potential cannot be replicated by replacing one field by another.

c) vev of singlet is much larger than that of other scalars.

Scalar Potential for A4

Total Potential $\Rightarrow \boxed{V = V_{singlet} + V_{\mathscr{D}} + V_{\mathscr{T}}}$

with $V_{\mathscr{T}} = V_{triplet} + V_{ts}$ and $V_{\mathscr{D}} = V_{doublet} + V_{ds}$.

them. These three triplets possess identical quantum numbers, their *vev* being the only discriminating criterion. Also there are three more fields viz. Δ_1^R, Δ_2^R and Δ_3^R transforming as 1, $1'$ and $1''$ under $A4$. From Eq. (B.1) we can see that two same $\hat{\Delta}_p^R$ triplets can combine to produce several $A4$ irreducible representations. For notational simplicity let us define:

$$O_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_p^R);\ O_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_p^R);\ T_{sp}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_p^R)\ ,\ (p = a, b, c). \tag{B.4}$$

Using two different triplets $\hat{\Delta}_p^R$ and $\hat{\Delta}_q^R$ where $p \neq q$ analogous combinations can be defined:

$$\hat{O}_{1pq}^{ss} \equiv O_1(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_q^R);\ \hat{O}_{2pq}^{ss} \equiv O_2(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_q^R);\ \hat{T}_{spq}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_q^R)\ ,\ (p, q = a, b, c \text{ and } p \neq q). \tag{B.5}$$

Generically, it is convenient to use $\widetilde{O}_{ip}$ or $\widetilde{T}_{sp}$ if the second triplet in the argument is replaced by its hermitian conjugate. As an example,

$$\widetilde{O}_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_p^{R\dagger}),\ \widetilde{O}_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_p^{R\dagger}),\ \widetilde{O}_{3p}^{ss} \equiv O_3(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_p^{R\dagger}) \text{ and } \widetilde{T}_{sp}^{ss} \equiv T_s(\hat{\Delta}_p^R, \hat{\Delta}_p^{R\dagger}), \tag{B.6}$$

One can also consider:

$$\widetilde{O}_{1pq}^{ss} \equiv O_1(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_q^{R\dagger}),\ \widetilde{O}_{2pq}^{ss} \equiv O_2(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_q^{R\dagger}),\ \widetilde{O}_{3pq}^{ss} \equiv O_3(\hat{\Delta}_p^{R\dagger}, \hat{\Delta}_q^{R\dagger}). \tag{B.7}$$

Also the following combinations are required:

$$\mathscr{O}_{1p}^{ss} \equiv O_1(\hat{\Delta}_p^R, T_{sp}^{ss\dagger}),\ \mathscr{O}_{2p}^{ss} \equiv O_2(\hat{\Delta}_p^R, T_{sp}^{ss\dagger}),\ \mathscr{O}_{3p}^{ss} \equiv O_3(\hat{\Delta}_p^R, T_{sp}^{ss\dagger})\ ,\ (p = a, b, c). \tag{B.8}$$

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$SU(2)_L$ singlet sector

The $A4$ singlets Δ_i^R $(i = 1,2,3)$ can be combined to yield

$$Q_i^{ss} \equiv \Delta_i^{R\dagger} \Delta_i^R \ , \ \ (i = 1,2,3). \tag{B.9}$$

Needless to mention that such terms are singlets of all the symmetries under consideration.

Having devised the essential notations one can write the most general scalar potential for the $SU(2)_L$ singlet sector of this model as:

$$\begin{aligned}
V_{singlet} \ = \ & \sum_{i=1}^{3} m^2_{\Delta_i^R} Q_i^{ss} + \sum_{p=a}^{c} m^2_{\hat{\Delta}_p^R} O_{1p}^{ss} + \left[\sum_{p\neq q;\, p,q=a}^{c} m^2_{\hat{\Delta}_{pq}^R} \hat{O}_{1pq}^{ss} \ + \ \text{all possible permutations} \right] \\
+ \ & \frac{1}{2} \sum_{i=1}^{3} \lambda_{1i}^s \left[Q_i^{ss}\right]^2 + \frac{1}{2} \sum_{k<j;\, k\neq j;\, k=1}^{2} \sum_{j=2}^{3} \lambda_{2jk}^s \left\{ Q_j^{ss} Q_k^{ss} \right\} \\
+ \ & \frac{1}{2} \sum_{p=a}^{c} \lambda_{3p}^s \left\{ [O_{1p}^{ss}]^2 + (O_{2p}^{ss})^\dagger O_{2p}^{ss} + O_{1p}(T_{sp}^{ss}, T_{sp}^{ss\dagger}) \right\} \\
+ \ & \sum_{p\neq q;\, p,q=a}^{c} \lambda_{3pq}^s \left\{ [\hat{O}_{1pq}^{ss}]^2 + (\hat{O}_{2pq}^{ss})^\dagger \hat{O}_{2pq}^{ss} + h.c. \right\} + \frac{1}{2} \sum_{p\neq q;\, p,q=a}^{c} \widetilde{\lambda}_{3pq}^s \left\{ (\hat{O}_{1pq}^{ss})^\dagger \hat{O}_{1pq}^{ss} + O_1(\hat{T}_{spq}^{ss}, \hat{T}_{spq}^{ss\dagger}) \right\} \\
+ \ & \sum_{p=a}^{c} \sum_{i=1}^{3} \left[\frac{1}{2} \lambda_{4ip}^s \left(Q_i^{ss} O_{1p}^{ss} \right) \right] + \sum_{i=1}^{3} \sum_{p\neq q;\, p,q=a}^{c} \lambda_{4ipq}^s \left[\left(Q_i^{ss} \hat{O}_{1pq}^{ss} \right) + h.c. \right] + \frac{1}{2} \sum_{p=a}^{c} \lambda_{5p}^s \left(\mathscr{O}_{1p}^{ss} \Delta_1^R + \ \text{h.c.} \right) \\
+ \ & \sum_{p\neq q;\, p,q=a}^{c} \lambda_{5pq}^s \left[\left\{ \left(\Delta_1^R O_1(\hat{\Delta}_p^R, \hat{T}_{spq}^{ss\dagger} \right) + \left(\Delta_1^R O_1(\hat{\Delta}_q^R, \hat{T}_{spq}^{ss\dagger} \right) \right\} + h.c. \right] \\
+ \ & \frac{1}{2} \sum_{p=a}^{c} \lambda_{6p}^s \left(\mathscr{O}_{3p}^{ss} \Delta_2^R + \ \text{h.c.} \right) + \sum_{p\neq q;\, p,q=a}^{c} \lambda_{6pq}^s \left[\left\{ \left(\Delta_2^R O_3(\hat{\Delta}_p^R, \hat{T}_{spq}^{ss\dagger} \right) + \left(\Delta_2^R O_3(\hat{\Delta}_q^R, \hat{T}_{spq}^{ss\dagger} \right) \right\} + h.c. \right] \\
+ \ & \frac{1}{2} \sum_{p=a}^{c} \lambda_{7p}^s \left(\mathscr{O}_{2p}^{ss} \Delta_3^R + \ \text{h.c.} \right) + \sum_{p\neq q;\, p,q=a}^{c} \lambda_{7pq}^s \left[\left\{ \left(\Delta_3^R O_2(\hat{\Delta}_p^R, \hat{T}_{spq}^{ss\dagger} \right) + \left(\Delta_3^R O_2(\hat{\Delta}_q^R, \hat{T}_{spq}^{ss\dagger} \right) \right\} + h.c. \right] \\
+ \ & \sum_{p=a}^{c} \sum_{i=1}^{3} \lambda_{8ip}^s \left(\Delta_i^{R^2} \widetilde{O}_{ip} + h.c. \right) + \sum_{p\neq q;\, p,q=a}^{c} \sum_{i=1}^{3} \lambda_{8ipq}^s \left(\Delta_i^{R^2} \widetilde{O}_{ipq} + h.c. \right) \\
+ \ & \sum_{p=a}^{c} \left[\lambda_{91p}^s \Delta_2^R \Delta_3^R \widetilde{O}_{1p} + \lambda_{92p}^s \Delta_1^R \Delta_3^R \widetilde{O}_{2p} + \lambda_{93p}^s \Delta_1^R \Delta_2^R \widetilde{O}_{3p} + h.c. \right] \\
+ \ & \sum_{p\neq q;\, p,q=a}^{c} \left[\lambda_{91pq}^s \Delta_2^R \Delta_3^R \widetilde{O}_{1pq} + \lambda_{92pq}^s \Delta_1^R \Delta_3^R \widetilde{O}_{2pq} + \lambda_{93pq}^s \Delta_1^R \Delta_2^R \widetilde{O}_{3pq} + h.c. \right].
\end{aligned} \tag{B.10}$$

Here λ_{3p}^s, λ_{3pq}^s and $\widetilde{\lambda}_{3pq}^s$ are taken as the common coefficient of the different $A4$ invariants generated by combining two $\hat{\Delta}^R$ and two $(\hat{\Delta}^R)^\dagger$ fields. Similar policy will be adopted for the fields with other $SU(2)_L$ properties.

$SU(2)_L$ doublet sector

B.3 $SU(2)_L$ Doublet Sector:

The $SU(2)_L$ doublet scalar precinct consists of the two fields Φ and η transforming as 3 and 1 of $A4$ respectively. Opposite hypercharges are assigned to Φ and η. The $A4$ triplet Φ combinations are denoted as:

$$O_1^{dd} \equiv O_1(\Phi^\dagger, \Phi);\ O_2^{dd} \equiv O_2(\Phi^\dagger, \Phi);\ T_s^{dd} \equiv T_s(\Phi, \Phi), \tag{B.11}$$

and that of the $A4$ singlet η are:

$$Q_\eta^{dd} \equiv \eta^\dagger \eta\ . \tag{B.12}$$

The potential for the $SU(2)_L$ doublet sector is given by:

$$\begin{aligned} V_{doublet} &= m_\eta^2 Q_\eta^{dd} + m_\Phi^2 O_1^{dd} + \frac{1}{2}\lambda_1^d \left[Q_\eta^{dd}\right]^2 + \frac{1}{2}\lambda_2^d \left\{[O_1^{dd}]^2 + \{O_2^{dd}\}^\dagger O_2^{dd} \right. \\ &+ \left. O_1(T_s^{dd}, T_s^{dd\dagger})\right\} + \frac{1}{2}\lambda_3^d \left[Q_\eta^{dd} O_1^{dd}\right]. \end{aligned} \tag{B.13}$$

B.4 $SU(2)_L$ Triplet Sector:

The $SU(2)_L$ triplet sector comprises of five fields. There are two $A4$ triplets $\hat{\Delta}_a^L$ and $\hat{\Delta}_b^L$ together with the fields the Δ_1^L, Δ_2^L and Δ_3^L transforming as 1, $1'$, $1''$ of $A4$ respectively.

It is useful to define:

$$O_{1n}^{tt} \equiv O_1(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_n^L);\ O_{2n}^{tt} \equiv O_2(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_n^L);\ T_{sn}^{tt} \equiv T_s(\hat{\Delta}_n^L, \hat{\Delta}_n^L)\ ,\ (n = a, b), \tag{B.14}$$

$$\hat{O}_{1nl}^{tt} \equiv O_1(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_l^L);\ \hat{O}_{2nl}^{tt} \equiv O_2(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_l^L);\ \hat{O}_{3nl}^{tt} \equiv O_3(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_l^L);\ \hat{T}_{snl}^{tt} \equiv T_s(\hat{\Delta}_n^L, \hat{\Delta}_l^L),\ (n, l = a, b \text{ and } n \neq l), \tag{B.15}$$

$$Q_i^{tt} \equiv \Delta_i^{L\dagger}\Delta_i^L\ ,\ (i = 1, 2, 3), \tag{B.16}$$

and

$$\mathscr{O}_{\gamma n}^{tt} \equiv O_\gamma(\hat{\Delta}_n^L, T_{sn}^{tt\dagger});\ \mathscr{O}_{\gamma nl}^{tt} \equiv O_\gamma(\hat{\Delta}_n^L, \hat{T}_{sl}^{tt\dagger}),\ (\gamma = 1, 2, 3) \text{ and } (n, l = a, b), \tag{B.17}$$

$$\widetilde{O}_{jn}^{tt} \equiv O_j(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_n^{L\dagger});\ \widetilde{O}_{jnl}^{tt} \equiv O_j(\hat{\Delta}_n^{L\dagger}, \hat{\Delta}_l^{L\dagger}),\ (j = 1, 2, 3) \text{ and } (n, l = a, b \text{ and } n \neq l). \tag{B.18}$$

$SU(2)_L$ triplet sector

$$
\begin{aligned}
V_{triplet} &= \sum_{i=1}^{3} m^2_{\Delta_i^L}\, Q_i^{tt} + \sum_{n=a}^{b} m^2_{\hat{\Delta}_n^L}\, O_{1n}^{tt} + \left(\sum_{n\neq l;\, n,l=a}^{b} m^2_{\hat{\Delta}_{nl}^L}\, \hat{O}_{1nl}^{tt} + \text{ all possible permutations} \right) \\
&+ \frac{1}{2}\sum_{i=1}^{3} \lambda_{1_i}^t \left[Q_i^{tt}\right]^2 + \frac{1}{2} \sum_{k<j,\, k=1}^{2} \sum_{j=2}^{3} \lambda_{2jk}^t Q_j^{tt} Q_k^{tt} + \frac{1}{2}\sum_{n=a}^{b} \lambda_{3n}^t \left\{ [O_{1n}^{tt}]^2 + \{O_{2n}^{tt}\}^\dagger O_{2n}^{tt} + O_1(T_{sn}^{tt}, T_{sn}^{tt\dagger}) \right\} \\
&+ \frac{1}{2} \sum_{n\neq l;\, n,l=a}^{b} \lambda_{3nl}^t \left\{ [\hat{O}_{1nl}^{tt}]^2 + \{\hat{O}_{2nl}^{tt}\}^\dagger \hat{O}_{2nl}^{tt} + h.c. \right\} + \frac{1}{2} \sum_{n\neq l;\, n,l=a}^{b} \widetilde{\lambda}_{3nl}^t \left\{ [\hat{O}_{1nl}^{tt}]^\dagger \hat{O}_{1nl}^{tt}] + O_1(\hat{T}_{sn}^{tt}, \hat{T}_{sn}^{tt\dagger}) \right\} \\
&+ \frac{1}{2}\sum_{j=1}^{3}\sum_{n=a}^{b} \lambda_{4_{jn}}^t \left[\left(\Delta_j^{L\dagger}\Delta_j^L\right) O_{1n}^{tt} \right] + \sum_{j=1}^{3} \sum_{n\neq l;\, n,l=a}^{b} \lambda_{4_{1nl}}^t \left[\left(\Delta_j^{L\dagger}\Delta_j^L\right) \hat{O}_{1n}^{tt} + h.c. \right]
\end{aligned}
$$

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$$
\begin{aligned}
&+ \frac{1}{2}\sum_{n=a}^{b} \lambda_{5n}^t \left[\{\Delta_1^L \mathscr{O}_{1n}^{tt}\} + h.c. \right] + \sum_{n,l=a}^{b} \lambda_{5nl}^t \left[\{\Delta_1^L \mathscr{O}_{1nl}^{tt}\} + h.c. \right] + \frac{1}{2}\sum_{n=a}^{b} \lambda_{6n}^t \left[\{\Delta_2^L \mathscr{O}_{3n}^{tt}\} + h.c. \right] \\
&+ \sum_{n,l=a}^{b} \lambda_{6nl}^t \left[\{\Delta_2^L \mathscr{O}_{3nl}^{tt}\} + h.c. \right] + \frac{1}{2}\sum_{n=a}^{b} \lambda_{7n}^t \left[\{\Delta_3^L \mathscr{O}_{2n}^{tt}\} + h.c. \right] + \sum_{n,l=a}^{b} \lambda_{7nl}^t \left[\{\Delta_3^L \mathscr{O}_{2nl}^{tt}\} + h.c. \right] \\
&+ \sum_{n=a}^{b}\sum_{j=1}^{3} \lambda_{8jn}^t \left[\left(\Delta_j^{L^2} \widetilde{O}_{jn}^{tt} \right) + h.c. \right] + \sum_{n\neq l;\, n,l=a}^{b} \sum_{j=1}^{3} \lambda_{8jnl}^t \left[\left(\Delta_j^{L^2} \widetilde{O}_{jnl}^{tt} \right) + h.c. \right] \\
&+ \sum_{n=a}^{b} \left[\left\{ \lambda_{91n}^t \left(\Delta_2^L \Delta_3^L \widetilde{O}_{1n}^{tt} \right) \right\} + \left\{ \lambda_{92n}^t \left(\Delta_1^L \Delta_3^L \widetilde{O}_{2n}^{tt} \right) \right\} + \left\{ \lambda_{93n}^t \left(\Delta_1^L \Delta_2^L \widetilde{O}_{3n}^{tt} \right) \right\} + h.c. \right] \\
&+ \sum_{n\neq l;\, n,l=a}^{b} \left[\left\{ \lambda_{91nl}^t \left(\Delta_2^L \Delta_3^L \widetilde{O}_{1nl}^{tt} \right) \right\} + \left\{ \lambda_{92nl}^t \left(\Delta_1^L \Delta_3^L \widetilde{O}_{2nl}^{tt} \right) \right\} + \left\{ \lambda_{93nl}^t \left(\Delta_1^L \Delta_2^L \widetilde{O}_{3nl}^{tt} \right) \right\} + h.c. \right]
\end{aligned}
$$

. (B.19)

Doublet singlet inter-sector

B.5.1 Singlet-Doublet inter-sector terms:

Let us consider the combinations:

$$\widetilde{T}^{ss}_{sp} \equiv T_s(\hat{\Delta}^R_p, \hat{\Delta}^{R\dagger}_p); \;\; \widetilde{T}^{ss}_{spq} \equiv T_s(\hat{\Delta}^R_p, \hat{\Delta}^{R\dagger}_q) \text{ and } \widetilde{T}^{dd}_s \equiv T_s(\Phi, \Phi^\dagger), \;\; (p,q = a,b,c \text{ and } p \neq q) \qquad \text{(B.20)}$$

and

$$O^{sd}_{1sp} \equiv O_1(\widetilde{T}^{dd}_s, \widetilde{T}^{ss}_{sp}); \;\; \hat{O}^{sd}_{1spq} \equiv O_1(\widetilde{T}^{dd}_s, \widetilde{T}^{ss}_{spq}); \;\; \mathscr{O}^{sd}_{p\gamma} \equiv O_\gamma(\hat{\Delta}^R_p, \widetilde{T}^{dd}_s) \;, \;\; (\gamma = 1,2,3) \text{ and } (p,q = a,b,c \text{ with } p \neq q). \qquad \text{(B.21)}$$

Using this notations:

$$\begin{aligned} V_{sd} &= \frac{1}{2}\sum_{i=1}^{3}\left[\lambda^{sd}_{1i}\left(Q^{ss}_i Q^{dd}_\eta\right) + \left(\lambda^{sd}_{2i} Q^{ss}_i O^{dd}_1\right)\right] + \frac{1}{2}\sum_{p=a}^{c}\lambda^{sd}_{3p}\left[Q^{dd}_\eta O^{ss}_{1p}\right] + \frac{1}{2}\sum_{p\neq q;\, p,q=a}^{c}\left[Q^{dd}_\eta \hat{O}^{ss}_{1pq}\right] \\ &+ \sum_{p=a}^{c}\left[\lambda^{sd}_{4p}\left(\{\mathscr{O}^{sd}_{1p}\}\Delta^R_1 + h.c.\right) + \lambda^{sd}_{5p}\left(\{\mathscr{O}^{sd}_{2p}\}\Delta^R_2 + h.c.\right) + \lambda^{sd}_{6p}\left(\{\mathscr{O}^{sd}_{3p}\}\Delta^R_3 + h.c.\right)\right] \\ &+ \frac{1}{2}\sum_{p=a}^{c}\lambda^{sd}_{7p}\left[O^{dd}_1 O^{ss}_{1p} + \{O^{ss}_{2p}\}^\dagger O^{dd}_2 + \{O^{dd}_2\}^\dagger O^{ss}_{2p} + O^{sd}_{1sp}\right] \end{aligned}$$

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$$+ \frac{1}{2}\sum_{p\neq q;\, p,q=a}^{c}\lambda^{sd}_{7pq}\left[O^{dd}_1 \hat{O}^{ss}_{1pq} + \{\hat{O}^{ss}_{2pq}\}^\dagger O^{dd}_2 + \{O^{dd}_2\}^\dagger \hat{O}^{ss}_{2pq} + \hat{O}^{sd}_{1spq}\right]. \qquad \text{(B.22)}$$

Triplet singlet inter-sector

B.5.2 Singlet-Triplet inter-sector terms:

In this case the following combinations comes into play:

$$\begin{aligned}
\widetilde{T}^{tt}_{sn} &\equiv T_s(\hat{\Delta}^L_n, \hat{\Delta}^{L\dagger}_n); \;\; \widetilde{T}^{tt}_{snl} \equiv T_s(\hat{\Delta}^L_n, \hat{\Delta}^{L\dagger}_l); \;\; O^{ts}_{1snp} \equiv O_1(\widetilde{T}^{tt}_{sn}, \widetilde{T}^{ss}_{sp}); \;\; \hat{O}^{ts}_{1snpq} \equiv O_1(\widetilde{T}^{tt}_{sn}, \widetilde{T}^{ss}_{spq}); \\
\hat{O}^{ts}_{1snlp} &\equiv O_1(\widetilde{T}^{tt}_{snl}, \widetilde{T}^{ss}_{sp}); \;\; \hat{O}^{ts}_{1snlpq} \equiv O_1(\widetilde{T}^{tt}_{snl}, \widetilde{T}^{ss}_{spq}); \;\; O^{ts}_{\gamma np} \equiv O_\gamma(\hat{\Delta}^{R\dagger}_p, \hat{\Delta}^L_n); \;\; \widetilde{O}^{ts}_{\gamma np} \equiv O_\gamma(\hat{\Delta}^R_p, \hat{\Delta}^L_n); \\
\mathcal{O}^{ts}_{\gamma np} &\equiv O_\gamma(\widetilde{T}^{ss}_{sp}, \hat{\Delta}^L_n); \;\; \widetilde{\mathcal{O}}^{ts}_{\gamma np} \equiv O_\gamma(\widetilde{T}^{tt}_{sn}, \hat{\Delta}^R_p); \;\; \hat{\mathcal{O}}^{ts}_{\gamma npq} \equiv O_\gamma(\widetilde{T}^{ss}_{spq}, \hat{\Delta}^L_n); \;\; \widetilde{\mathcal{O}}^{ts}_{\gamma nlp} \equiv O_\gamma(\widetilde{T}^{tt}_{snl}, \hat{\Delta}^R_p).
\end{aligned} \tag{B.23}$$

where $(\gamma = 1,2,3)$; $(p,q = a,b,c)$ and $(n,l = a,b)$. Needless to mention $p \neq q$ and $n \neq l$.

Following the convention introduced already:

$$O^{ts}_{\gamma np} \equiv O_\gamma(\hat{\Delta}^{R\dagger}_p, \hat{\Delta}^L_n); \;\; \widetilde{O}^{ts}_{\gamma np} \equiv O_\gamma(\hat{\Delta}^R_p, \hat{\Delta}^L_n), \; (\gamma = 1,2,3); \;\; (p = a,b,c) \;\text{ and }\; (n = a,b). \tag{B.24}$$

The inter-sector potential for this case is given by:

$$\begin{aligned}
V_{ts} \;=\;& \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\lambda^{ts}_{1ij}\left[Q^{ss}_i Q^{tt}_j\right] + \frac{1}{2}\sum_{j=1}^{3}\sum_{n=a}^{b}\lambda^{ts}_{2jn}\left[(Q^{ss}_j O^{tt}_{1n}) + h.c.\right] + \frac{1}{2}\sum_{j=1}^{3}\sum_{n\neq l;\, n,l=a}^{b}\lambda^{ts}_{2jnl}\left[\left(Q^{ss}_j \hat{O}^{tt}_{1nl}\right) + h.c.\right] \\
+\;& \frac{1}{2}\sum_{i=1}^{3}\sum_{p=a}^{c}\lambda^{ts}_{3ip}\left[Q^{tt}_i O^{ss}_{1p}\right] + \sum_{i=1}^{3}\sum_{p\neq q;\, p,q=a}^{c}\lambda^{ts}_{3ipq}\left[Q^{tt}_i \hat{O}^{ss}_{1pq}\right] \\
+\;& \frac{1}{2}\sum_{p=a}^{c}\sum_{n=a}^{b}\lambda^{ts}_{41nnpp}\left[O^{tt}_{1n}O^{ss}_{1p} + \{O^{ss}_{2p}\}^\dagger O^{tt}_{2n} + \{O^{tt}_{2n}\}^\dagger O^{ss}_{2p} + O^{ts}_{1snp}\right] \\
+\;& \frac{1}{2}\sum_{p\neq q;\, p,q=a}^{c}\sum_{n=a}^{b}\lambda^{ts}_{42nnpq}\left[O^{tt}_{1n}\hat{O}^{ss}_{1pq} + \{\hat{O}^{ss}_{2pq}\}^\dagger O^{tt}_{2n} + \{O^{tt}_{2n}\}^\dagger \hat{O}^{ss}_{2pq} + \hat{O}^{ts}_{1snpq}\right] \\
+\;& \frac{1}{2}\sum_{p=a}^{c}\sum_{n\neq l;\, n,l=a}^{b}\lambda^{ts}_{43nlpp}\left[\hat{O}^{tt}_{1nl}O^{ss}_{1p} + \{O^{ss}_{2p}\}^\dagger \hat{O}^{tt}_{2nl} + \{\hat{O}^{tt}_{2nl}\}^\dagger O^{ss}_{2p} + \hat{O}^{ts}_{1snlp}\right] \\
+\;& \frac{1}{2}\sum_{p\neq q;\, p,q=a}^{c}\sum_{n\neq l;\, n,l=a}^{b}\lambda^{ts}_{44nlpq}\left[\hat{O}^{tt}_{1nl}\hat{O}^{ss}_{1pq} + \{\hat{O}^{ss}_{2pq}\}^\dagger \hat{O}^{tt}_{2nl} + \{\hat{O}^{tt}_{2nl}\}^\dagger \hat{O}^{ss}_{2pq} + \hat{O}^{ts}_{1snlpq}\right] \\
+\;& \sum_{i=1}^{3}\sum_{p=a}^{c}\sum_{n=a}^{b}\lambda^{ts}_{5ippn}\left(\mathcal{O}^{ts}_{inp}\Delta^{L\dagger}_i + h.c.\right) + \sum_{i=1}^{3}\sum_{p\neq q;\, p,q=a}^{c}\sum_{n=a}^{b}\lambda^{ts}_{5ipqn}\left(\hat{\mathcal{O}}^{ts}_{inpq}\Delta^{L\dagger}_i + h.c.\right) \\
+\;& \sum_{i=1}^{3}\sum_{p=a}^{c}\sum_{n=a}^{b}\lambda^{ts}_{6innp}\left(\widetilde{\mathcal{O}}^{ts}_{inp}\Delta^{R\dagger}_i + h.c.\right) + \sum_{i=1}^{3}\sum_{p=a}^{c}\sum_{n\neq l;\, n,l=a}^{b}\lambda^{ts}_{6inlp}\left(\widetilde{\mathcal{O}}^{ts}_{inlp}\Delta^{R\dagger}_i + h.c.\right)
\end{aligned}$$

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$$\begin{aligned}
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_7 O^{ts}_{1np}\left(\Delta^{L\dagger}_1\Delta^R_1 + \Delta^{L\dagger}_2\Delta^R_2 + \Delta^{L\dagger}_3\Delta^R_3\right) + h.c.\right] \\
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_8 O^{ts}_{2np}\left(\Delta^{L\dagger}_1\Delta^R_3 + \Delta^{L\dagger}_2\Delta^R_1 + \Delta^{L\dagger}_3\Delta^R_2\right) + h.c.\right] \\
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_9 O^{ts}_{3np}\left(\Delta^{L\dagger}_3\Delta^R_1 + \Delta^{L\dagger}_1\Delta^R_2 + \Delta^{L\dagger}_2\Delta^R_3\right) + h.c.\right] \\
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_{10} \widetilde{O}^{ts}_{3np}\left(\Delta^{L\dagger}_3\Delta^{R\dagger}_1 + \Delta^{L\dagger}_1\Delta^{R\dagger}_3 + \Delta^{L\dagger}_2\Delta^{R\dagger}_2\right) + h.c.\right] \\
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_{11} \widetilde{O}^{ts}_{2np}\left(\Delta^{L\dagger}_2\Delta^{R\dagger}_1 + \Delta^{L\dagger}_1\Delta^{R\dagger}_2 + \Delta^{L\dagger}_3\Delta^{R\dagger}_3\right) + h.c.\right] \\
+\;& \sum_{p=a}^{c}\sum_{n=a}^{b}\left[\lambda^{ts}_{12} \widetilde{O}^{ts}_{1np}\left(\Delta^{L\dagger}_1\Delta^{R\dagger}_1 + \Delta^{L\dagger}_3\Delta^{R\dagger}_2 + \Delta^{L\dagger}_2\Delta^{R\dagger}_3\right) + h.c.\right] \;.
\end{aligned} \tag{B.25}$$

Singlet minimiztion

B.6.1 $SU(2)_L$ **singlet sector:**

The $SU(2)_L$ singlet *vevs* are much larger than those of the doublet and triplet scalars. Thus it is safe to neglect the contributions to the minimization equations from the inter-sector terms.

Let us remind ourselves that v_{Rp} $(p = a, b, c)$ are real and define:

$$
\begin{aligned}
&\tilde{v}_{Ra_1} \equiv v_{Ra}, \quad \tilde{v}_{Ra_2} \equiv v_{Ra}, \quad \tilde{v}_{Ra_3} \equiv v_{Ra};\\
&\tilde{v}_{Rb_1} \equiv v_{Rb}, \quad \tilde{v}_{Rb_2} \equiv v_{Rb}\omega, \quad \tilde{v}_{Rb_3} \equiv v_{Ra}\omega^2;\\
&\tilde{v}_{Rc_1} \equiv v_{Rc}, \quad \tilde{v}_{Rc_2} \equiv v_{Rc}\omega^2, \quad \tilde{v}_{Rc_3} \equiv v_{Rc}\omega.
\end{aligned} \tag{B.29}
$$

For ease of presentation, let us set the following masses and couplings equal:

$$
\begin{aligned}
m^2_{\Delta^R_1} &= m^2_{\Delta^R_2} = m^2_{\Delta^R_3} = m^2_{R1}; \quad m^2_{\hat{\Delta}^R_a} = m^2_{\hat{\Delta}^R_b} = m^2_{\hat{\Delta}^R_c} = m^2_{R2}; \quad m^2_{ab} = m^2_{ac} = m^2_{bc} = m^2_{R3};\\
\lambda^s_{1i} &= \lambda^s_1 \;\forall\; (i = 1, 2, 3); \quad \lambda^s_{221} = \lambda^s_{231} = \lambda^s_{223} = \lambda^s_2; \quad \lambda^s_{3a} = \lambda^s_{3b} = \lambda^s_{3c} = \lambda^s_3; \quad \widetilde{\lambda}^s_{3a} = \widetilde{\lambda}^s_{3b} = \widetilde{\lambda}^s_{3c} = \widetilde{\lambda}^s_3;\\
\lambda^s_{3ab} &= \lambda^s_{3ac} = \lambda^s_{3bc} = \hat{\lambda}^s_3; \quad \lambda^s_{4ip} = \lambda^s_4 \;\forall\; (p = a, b, c) \text{ and } (i = 1, 2, 3);\\
\lambda^s_{41ab} &= \lambda^s_{41ac} = \lambda^s_{41bc} = \lambda^s_{42ab} = \lambda^s_{42ac} = \lambda^s_{42bc} = \lambda^s_{43ab} = \lambda^s_{43ac} = \lambda^s_{43bc} = \widetilde{\lambda}^s_4;\\
\lambda^s_{5a} &= \lambda^s_{5b} = \lambda^s_{5c} = \lambda^s_5; \quad \lambda^s_{5ab} = \lambda^s_{5ac} = \lambda^s_{5bc} = \widetilde{\lambda}^s_5; \quad \lambda^s_{6a} = \lambda^s_{6b} = \lambda^s_{6c} = \lambda^s_6; \quad \lambda^s_{6ab} = \lambda^s_{6ac} = \lambda^s_{6bc} = \widetilde{\lambda}^s_6;\\
\lambda^s_{7a} &= \lambda^s_{7b} = \lambda^s_{7c} = \lambda^s_7; \quad \lambda^s_{7ab} = \lambda^s_{7ac} = \lambda^s_{7bc} = \widetilde{\lambda}^s_7; \quad \lambda^s_{8ip} = \lambda^s_8 \;\forall\; (p = a, b, c) \text{ and } (i = 1, 2, 3);\\
\lambda^s_{81ab} &= \lambda^s_{81ac} = \lambda^s_{81bc} = \lambda^s_{82ab} = \lambda^s_{82ac} = \lambda^s_{82bc} = \lambda^s_{83ab} = \lambda^s_{83ac} = \lambda^s_{83bc} = \widetilde{\lambda}^s_8;\\
\lambda^s_{9ip} &= \lambda^s_9 \;\forall\; (p = a, b, c) \text{ and } (i = 1, 2, 3);\\
\lambda^s_{91ab} &= \lambda^s_{91ac} = \lambda^s_{91bc} = \lambda^s_{92ab} = \lambda^s_{92ac} = \lambda^s_{92bc} = \lambda^s_{93ab} = \lambda^s_{93ac} = \lambda^s_{93bc} = \widetilde{\lambda}^s_9.
\end{aligned} \tag{B.30}
$$

With the help of the singlet sector potential in Eq. (B.10), the equalities in Eq. (B.30) and the *vev* in Eqs. (3), (4) and (5) one can obtain:

$$
\begin{aligned}
\frac{\partial V_{singlet}|_{min}}{\partial u^*_{1R}} = 0 \;\Rightarrow\;& m^2_{R1} u_{1R} + \lambda^s_1 (u^*_{1R} u^2_{1R}) + \lambda^s_2 \left[(u^*_{2R} u_{2R}) + (u^*_{3R} u_{3R})\right]\\
+\;& \frac{3\lambda^s_4}{2} u_{1R} \left[v^2_{Ra} + v^2_{Rb} + v^2_{Rc}\right] + 3\lambda^s_5 v^3_{Ra}\\
-\;& 3\widetilde{\lambda}^s_5 v_{Ra} \left(v^2_{Rb} + v^2_{Rc}\right) + 6\lambda^s_8 u^*_{1R} v^2_{Ra} + 6\widetilde{\lambda}^s_8 u^*_{1R} v_{Rb} v_{Rc}\\
+\;& 3\lambda^s_9 \left(u^*_{3R} v^2_{Rb} + u^*_{2R} v^2_{Rc}\right) + 3\widetilde{\lambda}^s_9 \left[v_{Ra} \left(u^*_{2R} v_{Rb} + u^*_{3R} v_{Rc}\right)\right] = 0 \;,
\end{aligned} \tag{B.31}
$$

$$
\frac{\partial V_{singlet}|_{min}}{\partial \tilde{v}^*_{Ra_1}} = \frac{\partial V_{singlet}|_{min}}{\partial v^*_{Ra_1}} = 0
$$

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$$
\begin{aligned}
\Rightarrow\;& m^2_{R2} v_{Ra} + m^2_{R3} (v_{Rb} + v_{Rc}) + \frac{7}{2}\lambda^s_3 v^3_{Ra} + \left(3\hat{\lambda}_3 + \frac{\widetilde{\lambda}^s_3}{8}\right) v_{Ra} \left(v^2_{Rb} + v^2_{Rc}\right)\\
+\;& \left[\frac{\lambda^s_4}{2} v_{Ra} + \widetilde{\lambda}^s_4 (v_{Rb} + v_{Rc})\right] (u^*_{1R} u_{1R} + u^*_{2R} u_{2R} + u^*_{3R} u_{3R}) + \lambda^s_5 v^2_{Ra} \left(u^*_{1R} + 2u_{1R}\right)\\
-\;& \widetilde{\lambda}^s_5 \left[(v_{Ra} v_{Rb} + v_{Ra} v_{Rc})(2u_{1R} + u^*_{1R})\right] + \lambda^s_6 v^2_{Ra} (u^*_{2R} - u_{2R})\\
+\;& \widetilde{\lambda}^s_6 \left[u_{2R} \left(2v^2_{Rb} - v^2_{Rc} - v_{Ra} v_{Rb} + 2v_{Rb} v_{Rc}\right) - u^*_{2R} v_{Ra} (v_{Rb} + v_{Rc})\right] + \lambda^s_7 v^2_{Ra} (u^*_{3R} - u_{3R})\\
+\;& \widetilde{\lambda}^s_7 \left[u_{3R} \left(2v_{Ra} v_{Rb} - v_{Ra} v_{Rc} - v^2_{Rb} + 2v^2_{Rc}\right) - u^*_{3R} v_{Ra} (v_{Rb} + v_{Rc})\right]\\
+\;& \left(u^2_{1R} + u^2_{2R} + u^2_{3R}\right) \left[2\lambda^s_8 v_{Ra} + \widetilde{\lambda}^s_8 (v_{Rb} + v_{Rc})\right]\\
+\;& \left(u^*_{1R} u_{2R} + u^*_{1R} u_{3R} + u^*_{2R} u_{3R}\right) \left[2\lambda^s_9 v_{Ra} + \widetilde{\lambda}^s_9 (v_{Rb} + v_{Rc})\right] = 0 \;,
\end{aligned} \tag{B.32}
$$

Doublet minimiztion

The following couplings are set to be equal:

$$
\begin{aligned}
\lambda^{sd}_{1i} &= \lambda^{sd}_1, \;\; \lambda^{sd}_{2i} = \lambda^{sd}_2 \;\; \forall \;\; (i = 1, 2, 3); \\
\lambda^{sd}_{3p} &= \lambda^{sd}_3, \;\; \lambda^{sd}_{4p} = \lambda^{sd}_4, \;\; \lambda^{sd}_{5p} = \lambda^{sd}_5, \;\; \lambda^{sd}_{6p} = \lambda^{sd}_6, \;\; \lambda^{sd}_{7p} = \lambda^{sd}_7 \;\; \forall \;\; (p = a, b, c); \\
\lambda^{sd}_{3ab} &= \lambda^{sd}_{3ac} = \lambda^{sd}_{3bc} = \widetilde{\lambda}^{sd}_3; \;\; \lambda^{sd}_{7ab} = \lambda^{sd}_{7ac} = \lambda^{sd}_{7bc} = \widetilde{\lambda}^{sd}_7.
\end{aligned} \tag{B.36}
$$

For the *vev*s in Eqs. (B.26), (B.27) and (B.28) correspond to the minimum of the scalar potential it is necessary to satisfy the following conditions:

$$
\frac{\partial V_{\mathscr{D}}|_{min}}{\partial u^*} = 0 \Rightarrow u \left[m^2_\eta + \lambda^d_1 u^* u + \lambda^d_3 v^2 + \lambda^{sd}_1 \sum_{i=1}^{3} (u^*_{iR} u_{iR}) + \frac{3}{2} \lambda^{sd}_3 \sum_{p=a}^{c} v^2_{Rp} \right] = 0. \tag{B.37}
$$

and

$$
\begin{aligned}
\frac{\partial V_{\mathscr{D}}|_{min}}{\partial v^*_1} &= 0 \\
&\Rightarrow \frac{v}{\sqrt{3}} \left[m^2_\Phi + 2\lambda^d_2 \frac{v^2}{3} + \frac{\lambda^d_3}{2} (u^* u) + \frac{\lambda^{sd}_2}{2} \sum_{i=1}^{3} (u^*_{iR} u_{iR}) \right. \\
&+ \frac{\lambda^{sd}_4}{2} (2v_{Ra} - v_{Rb} - v_{Rc}) (u_{1R} + u^*_{1R}) \\
&+ \frac{\lambda^{sd}_7}{2} \left[\left(\sum_{p=a}^{c} 3 v^2_{Rp} \right) + \frac{1}{2} (2v^2_{Ra} - v^2_{Rb} - v^2_{Rc}) \right] \\
&+ \left. \frac{\widetilde{\lambda}^{sd}_7}{2} \left[6 (v_{Ra} v_{Rb} + v_{Ra} v_{Rc} + v_{Rb} v_{Rc}) + \frac{1}{2} (v_{Ra} v_{Rb} + v_{Ra} v_{Rc} - 2 v_{Rb} v_{Rc}) \right] \right] = 0.
\end{aligned} \tag{B.38}
$$

B.6.3 $SU(2)_L$ **triplet sector:**

In analogy to the doublet sector, let us define $V_{\mathscr{T}} = V_{triplet} + V_{ts}$ using Eqs. (B.19) and (B.25). Let us also recall, $v_{La1} = v_{La}$, $v_{La2} = v_{La3} = 0$ and $v_{Lb1} = v_{Lb2} = v_{Lb3} = v_{Lb}$.

This sector has several couplings involved. For simplicity of presentation, let us implement the following choices:

$$
\begin{aligned}
m_{\Delta^L_1} &= m_{\Delta^L_2} = m_{\Delta^L_3} = m_{t1}; \;\; m_{\hat{\Delta}^L_a} = m_{\hat{\Delta}^L_b} = m_{t2}; \;\; m_{ab} = m_{t3}; \;\; \lambda^t_{1i} = \lambda^t_1, \;\; \forall \;\; (i = 1, 2, 3); \\
\lambda^t_{221} &= \lambda^t_{232} = \lambda^t_{231} = \lambda^t_2 \;\; ; \;\; \lambda^t_{3a} = \lambda^t_{3b} = \lambda^t_3; \;\; \lambda^t_{3ab} = \hat{\lambda}^t_3; \;\; \widetilde{\lambda}^t_{3ab} = \widetilde{\lambda}^t_3; \\
\lambda_{4jn} &= \lambda_{4jnl} = \lambda_4; \;\; \lambda_{8jn} = \lambda_{8jnl} = \lambda_8; \;\; \lambda_{9jn} = \lambda_{9jnl} = \lambda_9, \;\; \forall \;\; (j = 1, 2, 3), \;\; (n, l = a, b) \;\; \text{and} \;\; n \neq l; \\
\lambda^t_{5a} &= \lambda^t_{5b} = \lambda^t_{5ab} = \lambda^t_5; \;\; \lambda^t_{6a} = \lambda^t_{6b} = \lambda^t_{6ab} = \lambda^t_6; \;\; \lambda^t_{7a} = \lambda^t_{7b} = \lambda^t_{7ab} = \lambda^t_7; \\
\lambda^{ts}_{1ij} &= \lambda^{ts}_1, \;\; \forall \;\; (i, j = 1, 2, 3) \;\; \text{and} \;\; i \neq j; \;\; \lambda^{ts}_{2jn} = \lambda^{ts}_{2jnl}, \;\; \forall \;\; (j = 1, 2, 3), \;\; (n, l = a, b) \;\; \text{and} \;\; n \neq l;
\end{aligned}
$$

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Triplet minimiztion

$$
\begin{aligned}
\lambda^{ts}_{3pi} &= \lambda^{ts}_{3pqi}, \ \forall \ (i=1,2,3), \ (p,q=a,b,c) \ \text{and} \ p \neq q; \\
\lambda^{ts}_{4jnnpp} &= \lambda^{ts}_{4jnlpp} = \lambda^{ts}_{4jnnpq} = \lambda^{ts}_{4jnlpq} = \lambda^{ts}_{4}, \forall\,(j=1,2,3), \ (p,q=a,b,c), \ (n,l=a,b) \ \text{and} \ p \neq q, \ n \neq l; \\
\lambda^{ts}_{5jppn} &= \lambda^{ts}_{5jpqn} = \lambda^{ts}_{5}, \ \forall \ (j=1,2,3), \ (p=a,b,c), \ (n=a,b); \\
\lambda^{ts}_{6jnnp} &= \lambda^{ts}_{6jnlp} = \lambda^{ts}_{6}, \ \forall \ (j=1,2,3), \ (p=a,b,c), \ (n,l=a,b) \ \text{and} \ n \neq l.
\end{aligned} \tag{B.39}
$$

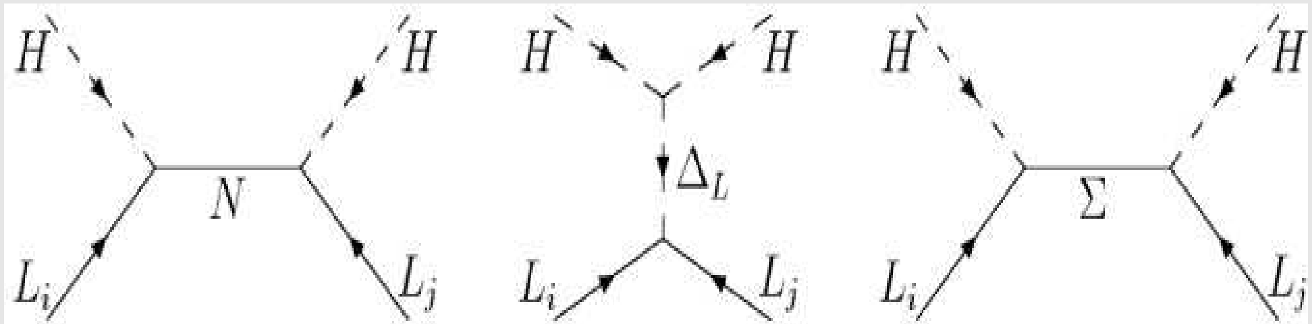
In order to minimize $V_{\mathscr{T}}$ such that one can arrive to *vevs* furnished in Eqs. (B.26), (B.27) and (B.28) the following conditions are to be ensured:

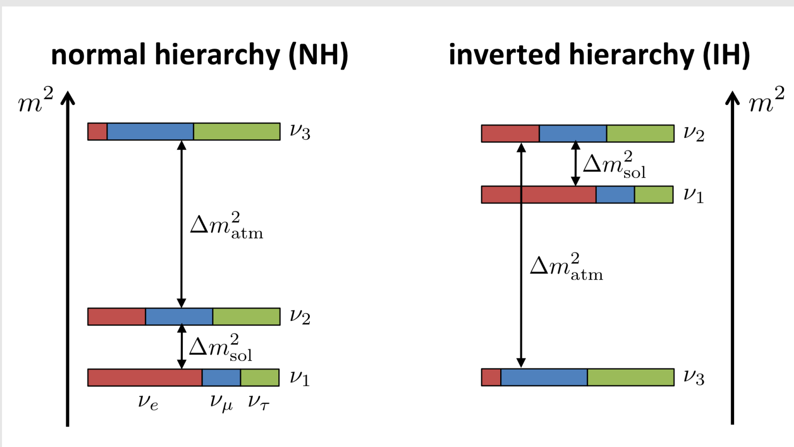
$$
\begin{aligned}
\frac{\partial V_{\mathscr{T}}|_{min}}{\partial u^*_{L1}} &= 0 \\
&\Rightarrow u_L \left[m^2_{t1} + (u^*_L u_L)\left(\lambda^t_1 + \lambda^t_2\right) + \frac{\lambda^t_4}{2}\left(v^2_{La} + 3v^2_{Lb} + 2v_{La}v_{Lb}\right)\right] + \lambda^t_5 v^2_{Lb}\left(3v_{Lb} + v_{La}\right) \\
&+ 2\lambda^t_8 u^*_L \left(v^2_{La} + 3v^2_{Lb} + v_{La}v_{Lb}\right) + 2\lambda^t_9 v_{La}\left(v_{La} + v_{Lb}\right) + \lambda^{ts}_1 u_L \sum_{i=1}^{3}\left(u^*_{iR}u_{iR}\right) \\
&+ \frac{3}{2}\lambda^{ts}_3 u_L \left[v^2_{Ra} + v^2_{Rb} + v^2_{Rc} + v_{Rb}v_{Rc}\right] \\
&+ \lambda^{ts}_5 \left(v_{La} + 3v_{Lb}\right)\left[2v^2_{Ra} - v^2_{Rb} - v^2_{Rc} - 2v^2_{Ra}(v_{Ra} + v^2_{Rb}) + 4v_{Rb}v_{Rc}\right] \\
&+ \left[\left(\lambda^t_7 s u_{1R} + \lambda^t_{12} s u^*_{1R}\right)\left[v_{La}\left(v_{Ra} + v_{Rb} + v_{Rc}\right) + 3v_{Ra}v_{Lb}\right]\right] \\
&+ \left[\left(\lambda^t_8 s u_{3R} + \lambda^t_{10} s u^*_{3R}\right)\left[v_{La}\left(v_{Ra} + v_{Rb} + v_{Rc}\right) + 3v_{Rc}v_{Lb}\right]\right] \\
&+ \left[\left(\lambda^t_9 s u_{2R} + \lambda^t_{11} s u^*_{2R}\right)\left[v_{La}\left(v_{Ra} + v_{Rb} + v_{Rc}\right) + 3v_{Rb}v_{Lb}\right]\right] = 0.
\end{aligned} \tag{B.40}
$$

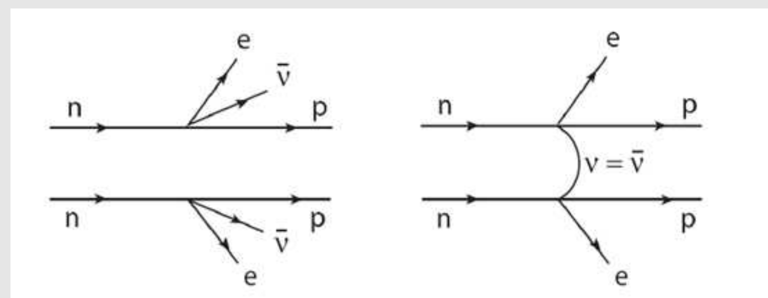
Also one gets;

$$
\begin{aligned}
\frac{\partial V_{\mathscr{T}}|_{min}}{\partial v^*_{La_1}} &= 0 \\
&\Rightarrow v_{La}\left[m^2_{t1} + m^2_{t2} + 2\lambda^t_3 v^2_{La} + 4\hat{\lambda}^t_3 v^2_{Lb} + \frac{3}{2}\widetilde{\lambda}^t_3 v^2_{Lb}\right] + \frac{3}{2}\lambda^t_4\left(u^*_L u_L\right)\left(v_{La} + v_{Lb}\right) \\
&+ \left(2\lambda^t_5 - \lambda^t_6 - \lambda^t_7\right) v^2_{Lb} u_L + \lambda^t_8 u^2_L\left(2v_{La} + 3v_{Lb}\right) + \lambda^t_9 u^2_L\left(2v_{La} + v_{Lb}\right) \\
&+ \frac{\lambda^{ts}_2}{2}\left[\left(v_{La} + v_{Lb}\right)\sum_{i=1}^{3} u^*_{iR}u_{iR}\right] + \frac{\lambda^{ts}_4}{2}\left[3\left(v_{La} + v_{Lb}\right)\sum_{p=a}^{c} v^2_{Rp} + \frac{1}{2}v_{Lb}\left(2v^2_{Ra} - v^2_{Rb} - v^2_{Rc}\right)\right] \\
&+ \widetilde{\lambda}^{ts}_4\left[3\left(v_{La} + v_{Lb}\right)\left(v_{Ra}v_{Rb} + v_{Rb}v_{Rc}\right) + \frac{1}{2}v_{Lb}\left(v_{Ra}v_{Rb} + v_{Ra}v_{Rc} + v_{Rb}v_{Rc}\right)\right] \\
&+ \lambda^{ts}_5\left[3u_L\left(2v^2_{Ra} - v^2_{Rb} - v^2_{Rc} - 2v_{Ra}v_{Rb} - 2v_{Ra}v_{Rc} + 4v_{Rb}v_{Rc}\right)\right] \\
&+ \left[u_L\left(v_{Ra} + v_{Rb} + v_{Rc}\right)\right]\left[\left[\left(u^*_{1R} + u^*_{2R} + u^*_{3R}\right)\left(\lambda^{ts}_7 + \lambda^{ts}_8 + \lambda^{ts}_9\right)\right]\right. \\
&+ \left.\left[\left(u_{1R} + u_{2R} + u_{3R}\right)\left(\lambda^{ts}_{10} + \lambda^{ts}_{11} + \lambda^{ts}_{12}\right)\right]\right] = 0.
\end{aligned} \tag{B.41}
$$

Miscellaneous







Golden Ratio:

$$\frac{a+b}{a} = \frac{a}{b} \stackrel{\text{def}}{=} \varphi, \qquad \varphi = \frac{1+\sqrt{5}}{2} = 1.6180339887$$

$\theta_{12} = 31.7^\circ$ for GR mixing $\Rightarrow \frac{\cos 31.7^\circ}{\sin 31.7^\circ} = 1.618..$

Seesaw in brief

Extend the SM by a singlet RH neutrino N_R per family.

Neutrino Majorana mass term: $m\psi^T_{L(R)}C^{-1}\psi_{L(R)}$

$$\Downarrow$$

$$\mathscr{L}_{mass} = \frac{1}{2}\alpha_L^T C^{-1}\mathscr{M}_{D+M}\alpha_L + h.c.$$

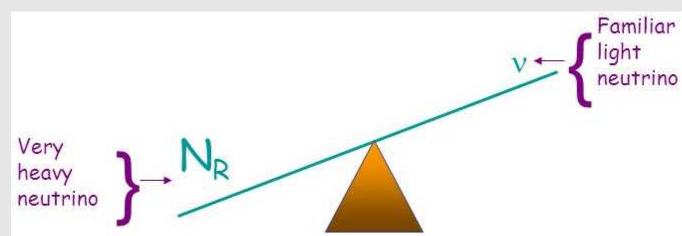
where, $\mathscr{M}_{D+M} = \begin{pmatrix} 0 & M_D^T \\ M_D & M_R \end{pmatrix} \Rightarrow 6 \times 6$ matrix and $\alpha_L = \begin{pmatrix} \nu_L \\ C(\bar{N}_R)^T \end{pmatrix}$

$\searrow$

Diagonalize: $W^T\mathscr{M}_{D+M}W = \begin{pmatrix} M_{light} & 0 \\ 0 & M_{heavy} \end{pmatrix}$

$M_{light} = M_D^T M_R^{-1} M_D$ and $M_{heavy} = M_R$

$\Rightarrow$



Can be done in three ways:

	Type I	Type II	Type III
	Fermion Singlet	Scalar Triplet	Fermion Triplet

Real M'

$$M' = \frac{m_D^2}{m_R}\begin{pmatrix} 0 & y & y \\ y & \frac{x}{\sqrt{2}} & -\frac{x}{\sqrt{2}} \\ y & -\frac{x}{\sqrt{2}} & \frac{x}{\sqrt{2}} \end{pmatrix} \to M'_{2\times 2} = \frac{m_D^2}{m_R}\begin{pmatrix} 0 & y \\ y & x/\sqrt{2} \end{pmatrix}$$

$$\theta_{12} = \theta_{12}^0 + \zeta \quad , \quad \tan 2\zeta = 2\sqrt{2}\left(\frac{y}{x}\right), \ \sin\epsilon = \frac{y}{\sqrt{y^2 + x^2/2}} \ \text{ and } \cos\epsilon = \frac{x/\sqrt{2}}{\sqrt{y^2 + x^2/2}} \quad , \ \text{i.e., } \tan\epsilon = \frac{1}{2}\tan 2\zeta \ .$$

$$\Delta m^2_{solar} = \frac{\sqrt{2} m_D^2}{m_R}\, m_1^{(0)} \sqrt{x^2 + 8y^2} = \frac{\sqrt{2} m_D^2}{m_R}\, m_1^{(0)} \frac{x}{\cos 2\zeta} \quad , \kappa_r \equiv \frac{m_D^2}{m_R m^-}\sqrt{y^2 + x^2/2} = \frac{m_D^2}{m_R m^-}\frac{x}{\sqrt{2}\cos\epsilon}$$

$$|\psi_3\rangle = \begin{pmatrix} \kappa_r \sin(\epsilon - \theta_{12}^0) \\ \frac{1}{\sqrt{2}}[1 - \kappa_r \cos(\epsilon - \theta_{12}^0)] \\ \frac{1}{\sqrt{2}}[1 + \kappa_r \cos(\epsilon - \theta_{12}^0)] \end{pmatrix} \Rightarrow \sin\theta_{13}\cos\delta = \kappa_r \sin(\epsilon - \theta_{12}^0), \tan(\pi/4 - \theta_{23}) \equiv \tan\omega = \kappa_r \cos(\epsilon - \theta_{12}^0) \ .$$

$$\Delta m^2_{solar} = 2\, m^-\, m_1^{(0)} \frac{\sin\theta_{13}\cos\delta\cos\epsilon}{\cos 2\zeta \ \sin(\epsilon - \theta_{12}^0)}$$

$$z \equiv m^- m_1^{(0)}/\Delta m^2_{atmos} \quad \text{and} \quad \tan\xi \equiv m_0/\sqrt{|\Delta m^2_{atmos}|} \quad ,$$

$$z = \left(\frac{\Delta m^2_{solar}}{|\Delta m^2_{atmos}|}\right)\left(\frac{\cos 2\zeta \ \sin(\epsilon - \theta_{12}^0)}{2\sin\theta_{13}|\cos\delta|\cos\epsilon}\right) \quad .$$

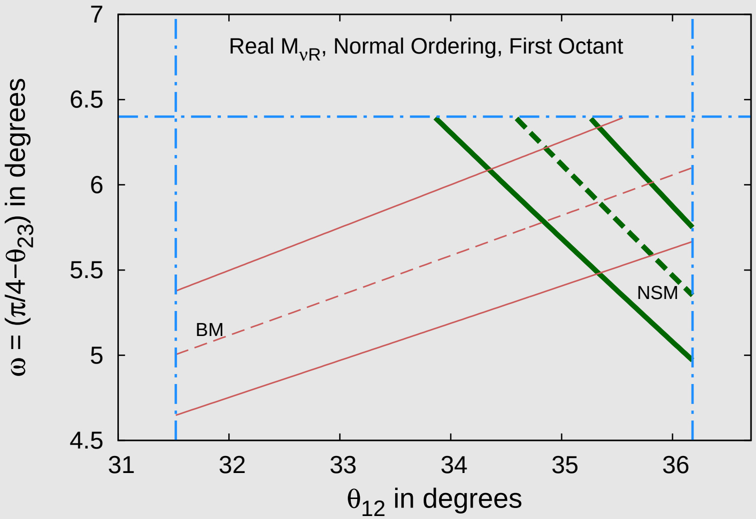
$$z = \sin\xi/(1 + \sin\xi) \quad \text{i.e.,} \quad 0 \le z \le \frac{1}{2} \quad \text{(for normal ordering)},$$

$$z = 1/(1 + \sin\xi) \quad \text{i.e.,} \quad \frac{1}{2} \le z \le 1 \quad \text{(for inverted ordering)} \quad .$$

$$\tan\omega = \frac{\sin\theta_{13}\cos\delta}{\tan(\epsilon - \theta_{12}^0)} \quad .$$

Real M'

1. Only the normal ordering of neutrino masses is allowed.
2. Only the first octant of θ_{23} is admissible.
3. Inclusion of Type-I see-saw corrections is incapable of making the TBM and GR mixing patterns consistent with the allowed ranges of the mixing angles.
4. NSM and BM alternatives can produce solutions in consonance with the neutrino masses and mixing. The allowed ranges of lightest neutrino mass is very narrow.



Complex M'

$$|\psi_3\rangle = \begin{pmatrix} \kappa_c[\frac{\sin\epsilon}{\cos\phi_1} f(\phi_1)\cos\theta^0_{12} - \frac{\cos\epsilon}{\cos\phi_2} f(\phi_2)\sin\theta^0_{12}]/m^+ \\ \frac{1}{\sqrt{2}}\{1 - \kappa_c[\frac{\sin\epsilon}{\cos\phi_1} f(\phi_1)\sin\theta^0_{12} + \frac{\cos\epsilon}{\cos\phi_2} f(\phi_2)\cos\theta^0_{12}]/m^+\} \\ \frac{1}{\sqrt{2}}\{1 + \kappa_c[\frac{\sin\epsilon}{\cos\phi_1} f(\phi_1)\sin\theta^0_{12} + \frac{\cos\epsilon}{\cos\phi_2} f(\phi_2)\cos\theta^0_{12}]/m^+\} \end{pmatrix} .$$

Mass matrices and Seesaw

- The $M_{\nu L}^{flavour}$ originates from $Type\ II\ Seesaw$ and is diagonalized by:

$$M^0 = M_{\nu L}^{mass} = U^{0T} M_{\nu L}^{flavour} U^0 = diag(m_1^{(0)}, m_1^{(0)}, m_3^{(0)}) \Rightarrow \Delta m_\odot^2 = 0$$

$$\text{with}\ \ U^0 = \begin{pmatrix} \cos\theta_{12}^0 & \sin\theta_{12}^0 & 0 \\ -\frac{\sin\theta_{12}^0}{\sqrt{2}} & \frac{\cos\theta_{12}^0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{\sin\theta_{12}^0}{\sqrt{2}} & -\frac{\cos\theta_{12}^0}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\Rightarrow$ **Corresponds to :** $\boxed{\theta_{12} = 0,\, \theta_{13} = 0,\, \theta_{23} = \pi/4.}$ $\Rightarrow$ **Mixing angles are either 0 or $\pi/4$ to start with.**

- Put $a = 2m_R x e^{-i\phi_1};\ \ b = 2m_R y e^{-i\phi_2}$ and $m_D = fu$.

  Type I Seesaw: $M'^{flavour} = \left[M_D^T (M_{\nu R}^{flavour})^{-1} M_D\right]$

$$M'^{mass} = U^{0T} M'^{flavour} U^0 = \frac{m_D^2}{\sqrt{2}\, x y m_R} \begin{pmatrix} 0 & y\, e^{i\phi_1} & y\, e^{i\phi_1} \\ y\, e^{i\phi_1} & \frac{x\ e^{i\phi_2}}{\sqrt{2}} & \frac{-x\ e^{i\phi_2}}{\sqrt{2}} \\ y\, e^{i\phi_1} & \frac{-x\ e^{i\phi_2}}{\sqrt{2}} & \frac{x\ e^{i\phi_2}}{\sqrt{2}} \end{pmatrix} .$$

x, $y \rightarrow$ dimensionless real constants of $\mathcal{O}(1)$.

$m_D,\ \ m_R\ \ \rightarrow$ Dirac and Right handed neutrino mass scales.

Popular mixings

$$U^0_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \;\; U^0_{BM} = \begin{pmatrix} \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\ \frac{1}{2} & -\frac{1}{2} & \sqrt{\frac{1}{2}} \end{pmatrix},$$

$$U^0_{GR} = \begin{pmatrix} \sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{\sqrt{5}\phi}} & 0 \\ -\frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & \frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{2}} \\ \frac{1}{\sqrt{2}}\sqrt{\frac{1}{\sqrt{5}\phi}} & -\frac{1}{\sqrt{2}}\sqrt{\frac{\phi}{\sqrt{5}}} & \sqrt{\frac{1}{2}} \end{pmatrix} \rightarrow \phi \equiv \frac{1+\sqrt{5}}{2}.$$

$$M^{mass}_{\nu R} = \frac{m_R}{2\sqrt{2}ab} \begin{pmatrix} 0 & b & b \\ b & \frac{a}{\sqrt{2}} & -\frac{a}{\sqrt{2}} \\ b & -\frac{a}{\sqrt{2}} & \frac{a}{\sqrt{2}} \end{pmatrix} .$$

with $a = ye^{-i\phi_2}$ and $b = xe^{-i\phi_1}$

Scalar Potential for A4

Total Potential $\Rightarrow$ $\boxed{V = V_{singlet} + V_{\mathscr{D}} + V_{\mathscr{T}}}$

with $V_{\mathscr{T}} = V_{triplet} + V_{ts}$ and $V_{\mathscr{D}} = V_{doublet} + V_{ds}$.

- $SU(2)_L$ Doublet Sector:

$$V_{doublet} = m_\eta^2 Q_\eta^{dd} + m_\Phi^2 O_1^{dd} + \frac{1}{2}\lambda_1^d \left[Q_\eta^{dd}\right]^2 + \frac{1}{2}\lambda_2^d \left\{[O_1^{dd}]^2 + \{O_2^{dd}\}^\dagger O_2^{dd} + O_1(T_s^{dd}, T_s^{dd\dagger})\right\} + \frac{1}{2}\lambda_3^d \left[Q_\eta^{dd} O_1^{dd}\right].$$

where, $O_1^{dd} \equiv O_1(\Phi^\dagger, \Phi);\ O_2^{dd} \equiv O_2(\Phi^\dagger, \Phi);\ T_s^{dd} \equiv T_s(\Phi, \Phi); Q_\eta^{dd} \equiv \eta^\dagger \eta$.

Minimization conditions:

Doublet sector:

$$\frac{\partial V_{\mathscr{D}}|_{min}}{\partial u^*} = 0 \Rightarrow u\left[2m_\eta^2 + 2\lambda_1^d u^* u + \lambda_3^d v^* v + \lambda_1^{sd} u_R^* u_R + 3\lambda_3^{sd} v_R^2\right] = 0.$$

and

$$\frac{\partial V_{\mathscr{D}}|_{min}}{\partial v_i^*} = 0 \Rightarrow \frac{v}{\sqrt{3}}\left[m_\Phi^2 + 4\lambda_2^d\left(\frac{v^* v}{3}\right) + \lambda_3^d u^* u + \frac{1}{2}\lambda_2^{sd} u_R^* u_R + \lambda_4^{sd}(u_R^* + u_R)v_R + \frac{5}{4}\lambda_5^{sd} v_R^2\right] = 0.$$

Scalar Potential Continued.. .

Total Potential $\Rightarrow \boxed{V = V_{singlet} + V_{\mathscr{D}} + V_{\mathscr{T}}}$

with $V_{\mathscr{T}} = V_{triplet} + V_{ts}$ and $V_{\mathscr{D}} = V_{doublet} + V_{ds}$.

- $SU(2)_L$ Singlet Sector:

$$\begin{aligned} V_{singlet} &= m^2_{\Delta_3^R} Q_3^{ss} + m^2_{\hat{\Delta}^R} O_1^{ss} + \frac{1}{2}\lambda_1^s \left[Q_3^{ss}\right]^2 \\ &+ \frac{1}{2}\lambda_2^s \left\{ [O_1^{ss}]^2 + (O_2^{ss})^\dagger O_2^{ss} + O_1(T_s^{ss}, T_s^{ss\dagger}) \right\} + \frac{1}{2}\lambda_3^s \left[Q_3^{ss} O_1^{ss}\right] \\ &+ \lambda_4^s \left[\mathscr{O}_2^{ss} \Delta_3^R + \text{ h.c.}\right] + \lambda_5^s \left[\widetilde{O}_3^{ss} \Delta_3^R \Delta_3^R + \text{ h.c.}\right] \quad . \end{aligned}$$

where; $O_1^{ss} \equiv O_1(\hat{\Delta}^{R\dagger}, \hat{\Delta}^R);\ O_2^{ss} \equiv O_2(\hat{\Delta}^{R\dagger}, \hat{\Delta}^R);\ T_s^{ss} \equiv T_s(\hat{\Delta}^R, \hat{\Delta}^R),$
$\widetilde{O}_3^{ss} \equiv O_3(\hat{\Delta}^{R\dagger}, \hat{\Delta}^{R\dagger})$ and $\widetilde{T}_s^{ss} \equiv T_s(\hat{\Delta}^R, \hat{\Delta}^{R\dagger}),\ \mathscr{O}_2^{ss} \equiv O_2(\hat{\Delta}^R, T_s^{ss\dagger}), Q_3^{ss} \equiv \Delta_3^{R\dagger}\Delta_3^R$.

- $SU(2)_L$ Doublet Sector:

$$\begin{aligned} V_{doublet} &= m^2_\eta Q^{dd}_\eta + m^2_\Phi O_1^{dd} + \frac{1}{2}\lambda_1^d \left[Q_\eta^{dd}\right]^2 + \frac{1}{2}\lambda_2^d \left\{ [O_1^{dd}]^2 + \{O_2^{dd}\}^\dagger O_2^{dd} \right. \\ &+ \left. O_1(T_s^{dd}, T_s^{dd\dagger}) \right\} + \frac{1}{2}\lambda_3^d \left[Q_\eta^{dd} O_1^{dd}\right]. \end{aligned}$$

where, $O_1^{dd} \equiv O_1(\Phi^\dagger, \Phi);\ O_2^{dd} \equiv O_2(\Phi^\dagger, \Phi);\ T_s^{dd} \equiv T_s(\Phi, \Phi); Q_\eta^{dd} \equiv \eta^\dagger \eta$.

Scalar Potential Continued... .

- $SU(2)_L$ Triplet Sector:

$$
\begin{aligned}
V_{triplet} &= \sum_{i=1}^{3} m^2_{\Delta_i^L} Q_i^{tt} + m^2_{\hat{\Delta}^L} O_1^{tt} + \frac{1}{2}\sum_{i=1}^{3}\lambda^t_{1_i}\left[Q_i^{tt}\right]^2 + \frac{1}{2}\sum_{k<j,\,k=1}^{2}\sum_{j=2}^{3}\lambda^t_{2jk}Q_j^{tt}Q_k^{tt} \\
&+ \frac{1}{2}\lambda_3^t\left\{[O_1^{tt}]^2 + \{O_2^{tt}\}^\dagger O_2^{tt} + O_1(T_s^{tt}, T_s^{tt\dagger})\right\} + \frac{1}{2}\sum_{i=1}^{3}\lambda^t_{4_i}\left[Q_i^{tt}O_1^{tt}\right] \\
&+ \lambda_5^t\left[\mathscr{O}_1^{tt}\Delta_1^L + \text{ h.c.}\right] + \lambda_6^t\left[\mathscr{O}_3^{tt}\Delta_2^L + \text{ h.c.}\right] + \lambda_7^t\left[\mathscr{O}_2^{tt}\Delta_3^L + \text{ h.c.}\right] \\
&+ \sum_{i=1}^{3}\lambda^t_{8_i}\left[\widetilde{O}_i^{tt}\Delta_i^L\Delta_i^L + \text{ h.c.}\right] + \left[\lambda^t_{9_1}\widetilde{O}_1^{tt}\Delta_2^L\Delta_3^L + \text{ h.c.} + \text{ cyclic}\right] .
\end{aligned}
$$

where; $O_1^{tt} \equiv O_1(\hat{\Delta}^{L\dagger}, \hat{\Delta}^L);\ O_2^{tt} \equiv O_2(\hat{\Delta}^{L\dagger}, \hat{\Delta}^L);\ T_s^{tt} \equiv T_s(\hat{\Delta}^L, \hat{\Delta}^L);$
$Q_i^{tt} \equiv \Delta_i^{L\dagger}\Delta_i^L;\ \mathscr{O}_i^{tt} \equiv O_i(\hat{\Delta}^L, T_s^{tt\dagger})\ \ (i = 1, 2, 3).$

- Singlet-Doublet Inter-sector terms:

$$
\begin{aligned}
V_{sd} &= \frac{1}{2}\lambda_1^{sd}\left[Q_3^{ss}Q_\eta^{dd}\right] + \frac{1}{2}\lambda_2^{sd}\left[Q_3^{ss}O_1^{dd}\right] + \frac{1}{2}\lambda_3^{sd}\left[Q_\eta^{dd}O_1^{ss}\right] + \lambda_4^{sd}\left[\{\mathscr{O}_3^{sd}\}^\dagger\Delta_3^R + h.c.\right] \\
&+ \frac{1}{2}\lambda_5^{sd}\left[O_1^{dd}O_1^{ss} + \{O_2^{ss}\}^\dagger O_2^{dd} + \{O_2^{dd}\}^\dagger O_2^{ss} + O_{1S}^{sd}\right].
\end{aligned}
$$

where; $\widetilde{T}_s^{ss} \equiv T_s(\hat{\Delta}^R, \hat{\Delta}^{R\dagger}),$ and $\widetilde{T}_s^{dd} \equiv T_s(\Phi, \Phi^\dagger); O_{1S}^{sd} \equiv O_1(\widetilde{T}_s^{dd}, \widetilde{T}_s^{ss});$
$\mathscr{O}_3^{sd} \equiv O_3(\hat{\Delta}^R, \widetilde{T}_s^{dd}); \widetilde{T}_a^{ss} \equiv T_a(\hat{\Delta}^R, \hat{\Delta}^{R\dagger})$ and $\widetilde{T}_a^{dd} \equiv T_a(\Phi, \Phi^\dagger).$

Scalar Potential Continued.... .

- Singlet-Triplet Inter-sector terms:

$$
\begin{aligned}
V_{ts} &= \frac{1}{2}\sum_{i=1}^{3}\lambda_{1_i}^{ts}\left[Q_3^{ss}Q_i^{tt}\right] + \frac{1}{2}\lambda_2^{ts}\left[Q_3^{ss}O_1^{tt}\right] + \frac{1}{2}\sum_{i=1}^{3}\lambda_{3_i}^{ts}\left[Q_i^{tt}O_1^{ss}\right] \\
&+ \frac{1}{2}\lambda_4^{ts}\left[O_1^{tt}O_1^{ss} + \{O_2^{ss}\}^\dagger O_2^{tt} + \{O_2^{tt}\}^\dagger O_2^{ss} + O_{1S}^{ts}\right] \\
&+ \sum_{i=1}^{3}\lambda_{5_i}^{ts}\left[\mathscr{O}_i^{ts}\Delta_i^{L\dagger} + h.c.\right] + \lambda_6^{ts}\left[\widetilde{\mathscr{O}}_3^{ts}\Delta_3^{R\dagger} + h.c.\right] \\
&+ \lambda_7^{ts}\left[O_1^{ts}\Delta_3^{L\dagger}\Delta_3^R + h.c.\right] + \lambda_8^{ts}\left[O_2^{ts}\Delta_1^{L\dagger}\Delta_3^R + h.c.\right] + \lambda_9^{ts}\left[O_3^{ts}\Delta_2^{L\dagger}\Delta_3^R + h.c.\right] \\
&+ \lambda_{10}^{ts}\left[\widetilde{O}_3^{ts}\Delta_3^R\Delta_1^L + h.c.\right] + \lambda_{11}^{ts}\left[\widetilde{O}_2^{ts}\Delta_3^R\Delta_3^L + h.c.\right] + \lambda_{12}^{ts}\left[\widetilde{O}_1^{ts}\Delta_3^R\Delta_2^L + h.c.\right] \quad .
\end{aligned}
$$

$$
\begin{aligned}
\text{where; } O_i^{ts} &\equiv O_i(\hat{\Delta}^{R\dagger}, \hat{\Delta}^L) \;\; (i = 1,2,3); O_{1S}^{ts} \equiv O_1(\widetilde{T}_s^{tt}, \widetilde{T}_s^{ss}); \\
\mathscr{O}_i^{ts} &\equiv O_i(\widetilde{T}_s^{ss}, \hat{\Delta}^L) \;\; ; \widetilde{O}_i^{ts} \equiv O_i(\hat{\Delta}^{R\dagger}, \hat{\Delta}^{L\dagger}) \;\; (i = 1,2,3); \widetilde{\mathscr{O}}_3^{ts} \equiv O_3(\widetilde{T}_s^{tt}, \hat{\Delta}^R).
\end{aligned}
$$

Minimization conditions:

Singlet sector:

$$
\frac{\partial V_{singlet}|_{min}}{\partial u_R^*} = 0 \Rightarrow u_R\left[m_{\Delta_3^R}^2 + \lambda_1^s u_R^* u_R + \frac{3}{2}\lambda_3^s v_R^2\right] + 3v_R^2\left[\lambda_4^s v_R + 2\lambda_5^s u_R^*\right] = 0 \;\; ,
$$

$$
\text{and } \frac{\partial V_{singlet}|_{min}}{\partial v_{Ri}^*} = 0 \Rightarrow v_R\left[m_{\hat{\Delta}^R}^2 + 4\lambda_2^s v_R^2 + \frac{\lambda_3^s}{2}u_R^* u_R + \lambda_4^s v_R(2u_R + u_R^*) + 2\lambda_5^s u_R^2\right] = 0
$$

Minimization conditions:

Doublet sector:

$$\frac{\partial V_{\mathscr{D}}|_{min}}{\partial u^*} = 0 \Rightarrow u\left[2m_\eta^2 + 2\lambda_1^d u^* u + \lambda_3^d v^* v + \lambda_1^{sd} u_R^* u_R + 3\lambda_3^{sd} v_R^2\right] = 0.$$

$$\frac{\partial V_{\mathscr{D}}|_{min}}{\partial v_i^*} = 0 \Rightarrow \frac{v}{\sqrt{3}}\left[m_\Phi^2 + 4\lambda_2^d\left(\frac{v^* v}{3}\right) + \lambda_3^d u^* u + \frac{1}{2}\lambda_2^{sd} u_R^* u_R + \lambda_4^{sd}(u_R^* + u_R) v_R + \frac{5}{4}\lambda_5^{sd} v_R^2\right] = 0$$

Triplet sector:

$$\begin{aligned}
\text{Choose } m_{\Delta_1^L} &= m_{\Delta_2^L} = m_{\Delta_3^L} = m_{\Delta^L} \;;\; \lambda^t_{1_1} = \lambda^t_{1_2} = \lambda^t_{1_3} = \lambda^t_a \;;\; \lambda^t_{4_1} = \lambda^t_{4_2} = \lambda^t_{4_3} = \lambda^t_b \\
\lambda^t_{221} &= \lambda^t_{232} = \lambda^t_{231} = \lambda^t_c \;;\; \lambda^t_{8_1} = \lambda^t_{8_2} = \lambda^t_{8_3} = \lambda^t_d \;;\; \lambda^t_{9_1} = \lambda^t_{9_2} = \lambda^t_{9_3} = \lambda^t_e \\
\lambda^{ts}_{1_1} &= \lambda^{ts}_{1_2} = \lambda^{ts}_{1_3} = \lambda^{ts}_a \;;\; \lambda^{ts}_{3_1} = \lambda^{ts}_{3_2} = \lambda^{ts}_{3_3} = \lambda^{ts}_b \;;\; \lambda^{ts}_{5_1} = \lambda^{ts}_{5_2} = \lambda^{ts}_{5_3} = \lambda^{ts}_c \\
\lambda^{ts}_{10} &= \lambda^{ts}_{11} = \lambda^{ts}_{12} = \lambda^{ts}_d \;;\; \lambda^{ts}_7 = \lambda^{ts}_8 = \lambda^{ts}_9 = \lambda^{ts}_f .
\end{aligned}$$

$$\begin{aligned}
\text{i)}\ \frac{\partial V_{\mathscr{T}}|_{min}}{\partial u_L^*} = 0 \quad &\Rightarrow \quad u_L\left[m^2_{\Delta^L} + (\lambda^t_a + \lambda^t_c) u_L^* u_L + \frac{1}{2}\lambda^t_b v_L^* v_L + \frac{1}{2}\lambda^{ts}_a u_R^* u_R + \frac{3}{2}\lambda^{ts}_b v_R^2\right] \\
&+ \quad 2v_L^2 u_L^*(\lambda^t_d + \lambda^t_e) + v_L v_R\left[-\frac{1}{2}\lambda^{ts}_c v_R + \lambda^{ts}_d u_R^* + \lambda^{ts}_f u_R\right] = 0. \\
\text{ii)}\ \frac{\partial V_{\mathscr{T}}|_{min}}{\partial v_{L1}^*} = 0 \quad &\Rightarrow \quad v_L\left[m^2_{\hat{\Delta}^L} + \frac{3}{2}\lambda^t_b u_L^* u_L + 2\lambda^t_3 v_L^* v_L + \frac{1}{2}\lambda^{ts}_2 u_R^* u_R + \frac{3}{2}\lambda^{ts}_4 v_R^2\right] \\
&+ \quad u_L\left[6u_L v_L^*(\lambda^t_d + \lambda^t_e) - \frac{3}{2}\lambda^{ts}_c v_R^2 + 3\lambda^{ts}_f u_R^* v_R + 3\lambda^{ts}_d u_R v_R\right] = 0.
\end{aligned}$$

$$\text{iii)}\ \frac{\partial V_{\mathscr{T}}|_{min}}{\partial v_{L2}^*} = \frac{\partial V_{\mathscr{T}}|_{min}}{\partial v_{L3}^*} = 0 \Rightarrow v_L v_R\left[-\frac{1}{4}\lambda^{ts}_4 v_R + \lambda^{ts}_6(u_R^* + u_R)\right] = 0.$$