Heavy hadron spectrum on lattice using NRQCD

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Introduction

- Lattice methods are powerful techniques in analyzing the spectrum of hadrons. However for hadrons containing heavy quarks particullarly bottom quark are difficult to analyze.
- For spectrum calculation it is necessary that $aM << 1$. For light quarks it is true but for charm quark $aM_c > 0.7$ and for bottom qurak $aM_b > 2$ with lattice spacing $a = 0.12fm$.
- However in hadrons containing heavy quarks the velocities of heavy quarks are non-relativistic. One can use effective theories like NRQCD. $M_{\Upsilon} = 9390$ MeV where as $2 \times M_b = 8360$ MeV ($\overline{MS}$ Scheme) and $M_{J/\psi} = 3096$ MeV where as $2 \times M_c = 2580$ MeV.

Foldy-Wouthuysen Transformation

- The Dirac equation $H\psi = i\frac{\partial\psi}{\partial t}$ where

$$H = \overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A}) + e\phi + m\beta$$

- Non-relativistic limit is reached by making the following transformation $\psi' = e^{iS}\psi$ where $S = -\frac{i}{2m}\beta\overrightarrow{\alpha}.(\overrightarrow{P} - e\overrightarrow{A})$.
- We get $i\frac{\partial\psi'}{\partial t} = H'\psi'$ where

$$\begin{aligned} H' &= e^{iS}He^{-iS} - ie^{iS}\frac{\partial e^{-iS}}{\partial t} \\ &= H + i[S,H] - \frac{1}{2}[S,[S,H]] - \frac{i}{6}[S,[S,[S,H]]] + \ldots\ldots. \\ &\quad -\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + \ldots\ldots\ldots \end{aligned}$$

- Defining $\theta = \vec{\alpha}.(\vec{P} - e\vec{A})$ we get (up to $O(v^4/c^4)$)

$$
\begin{aligned}
H' &= \beta(m + \frac{\theta^2}{2m} - \frac{\theta^4}{8m^3}) + e\phi - \frac{e}{8m^2}[\theta,[\theta,\phi]] - \frac{i}{8m^2}[\theta,\dot{\theta}] \\
&+ \frac{e\beta}{2m}[\theta,\phi] + i\beta\frac{\dot{\theta}}{2m} - \frac{\theta^3}{3m^2}
\end{aligned}
$$

- writting

$$
\psi' = \begin{pmatrix} u \\ v \end{pmatrix}
$$

$$
\begin{aligned}
i\frac{\partial u}{\partial t} &= [m - \frac{1}{2m}\sum_j D_j^2 - \frac{e}{2m}\sigma.B - \frac{1}{8m^3}(\sum_j D_j^2)^2 \\
&+ e\phi - \frac{e}{8m^2}\nabla.E - \frac{ie}{8m^2}\sigma.(\nabla \times E - E \times \nabla)]u
\end{aligned}
$$

NRQCD Lagrangian

- Similarly like QED we write NRQCD Lagrangian upto $O[(v/c)^6]$
  $\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}_{v^4} + \delta\mathcal{L}_{v^6}$

$$\mathcal{L}_0 = \psi(x)^\dagger (iD_0 + \frac{\vec{D}^2}{2m})\psi(x)$$

$$\begin{aligned}\delta\mathcal{L}_{v^4} &= c_1 \frac{1}{8m^3}\psi^\dagger D^4 \psi + c_2 \frac{g}{8m^2}\psi^\dagger(\vec{D}.\vec{E} - \vec{E}.\vec{D})\psi \\ &+ c_3 \frac{ie}{8m^2}\psi^\dagger \vec{\sigma}.(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\psi + c_4 \frac{g}{2m}\psi^\dagger\vec{\sigma}.\vec{B}\psi\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L}_{v^6} &= c_5 \frac{g}{m^3}\psi^\dagger\{\vec{D}^2, \vec{\sigma}.\vec{B}\}\psi + c_6 \frac{ig^2}{m^3}\psi^\dagger(\vec{\sigma}.\vec{E}\times\vec{E})\psi \\ &+ c_7 \frac{ig}{m^4}\psi^\dagger\{\vec{D}^2, \vec{\sigma}.(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})\}\psi\end{aligned}$$

- $\mathcal{L}_0$ merely gives us Schrodinger equation.
- $c_1, c_2, c_3, c_4 = 1$(tree level).

- To calculate c_7 let us consider the term $T_E = \bar{\psi}(q)\gamma^0 g\phi(q-p)\psi(p)$ with the positive energy spinor

$$\psi(p) = \left(\frac{E_p + m}{2E_p}\right)^{\frac{1}{2}} \begin{bmatrix} u \\ \frac{\sigma.p}{E_p+m} u \end{bmatrix}$$

$$\begin{aligned} T_E &= \sqrt{\frac{(E_p+m)(E_q+m)}{4E_pE_q}} \\ &\times u^\dagger [1 + \frac{p.q + i\sigma.q \times p}{(E_q+m)(E_p+m)}] g\phi(q-p)u \end{aligned}$$

- Term containg σ

$$V(p,q) = [\frac{i}{4m^2} - \frac{3i}{32m^4}(p^2+q^2)]\, u^\dagger \sigma.(q \times p) g\phi(q-p)u$$

- $c_7 = \frac{3}{64}$.
- c_5 can be calculated from $T_B(p,q) = -\bar{\psi}(q) g\gamma.A(q-p)\psi(p)$ and so on
- $c_5 = \frac{1}{8}, c_6 = -\frac{1}{8}$

Lattice NRQCD

- Replace continuum derivatives by lattice derivaties.For quark fields

$$
\begin{aligned}
a\Delta^{+}_{\mu}\psi(x) &= U_{\mu}(x)\psi(x+a\hat{\mu}) - \psi(x) \\
a\Delta^{-}_{\mu}\psi(x) &= \psi(x) - U^{\dagger}_{\mu}(x-a\hat{\mu})\psi(x-a\hat{\mu})
\end{aligned}
$$

  For gauge fields

$$
\begin{aligned}
a\Delta^{+}_{\rho}F_{\mu\nu}(x) &= U_{\rho}(x)F_{\mu\nu}(x+a\hat{\rho})U^{\dagger}_{\rho}(x) - F_{\mu\nu}(x) \\
a\Delta^{+}_{\rho}F_{\mu\nu}(x) &= F_{\mu\nu}(x) - U^{\dagger}_{\rho}(x-a\hat{\rho})F_{\mu\nu}(x)U_{\rho}(x-a\hat{\rho})
\end{aligned}
$$

  Here a is the lattice spacing and $U_{\mu}(x)$ is link variable.
- Symmetric derivatie

$$\Delta^{\pm} = \frac{1}{2}(\Delta^{+} + \Delta^{-})$$

- Laplacian

$$\Delta^{2} \equiv \sum_{i} \Delta^{+}_{i}\Delta^{-}_{i} = \sum_{i} \Delta^{-}_{i}\Delta^{+}_{i}$$

Green's Function

- The Lagrangian has the following form

$$\mathcal{L} = \psi^\dagger(x,t)D_4\psi(x,t) + \psi^\dagger(x,t)H\psi(x,t)$$

- H contains spatial derivaties only. E.O.M. corrsponding to $\psi^\dagger$

$$D_4\psi(x,t) + H\psi(x,t) = 0 \text{ after discretization}$$
$$U_t(x)\psi(x,t+1) - \psi(x,t) + aH\psi(x,t) = 0$$

Green's function obeys

$$U_t(x,t)G(x,t+1;0,0) - (1-aH)G(x,t;0,0) = \delta_{x,0}\delta_{t,0}$$
$$\Rightarrow \quad G(x,t+1;0,0) = U_t^\dagger(x,t)(1-aH)G(x,t;0,0)$$

- From renormalization considerations

$$G(x,t+1;0,0) = (1-\frac{aH_0}{2})(1-\frac{a\delta H}{2})U_t(x,t)^\dagger (1-\frac{a\delta H}{2})(1-\frac{aH_0}{2})G(x,t;0,0)$$

H_0 and δH are related as $H = H_0 + \delta H$. For stability purpose we modify

$$G(x,t+1;0,0) = (1-\frac{aH_0}{2n})^n(1-\frac{a\delta H}{2})U_t(x,t)^\dagger (1-\frac{a\delta H}{2})(1-\frac{aH_0}{2n})^n G(x,t;0,0)$$

with $G(x,t;0,0) = 0$ for $t < 0$ and $G(x,t;0,0) = \delta_{x,0}$ for $t = 0$. From the above equation it is evident that $n > \frac{3}{2m}$.

Relativistic fermions

- Action for free Dirac field

$$S[\psi,\bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - M)\psi(x)$$

Lattice version of the above action is

$$S = \sum_{n,m,\alpha,\beta} \bar{\hat{\psi}}_\alpha(n) K_{\alpha\beta}(n,m) \hat{\psi}_\beta(m)$$

where $K_{\alpha\beta}(n,m)$ is given by

$$K_{\alpha\beta}(n,m) = \sum_\mu \frac{1}{2}(\gamma_\mu)_{\alpha\beta}[\delta_{m,n+\hat{\mu}} - \delta_{m,n-\hat{\mu}}] + \hat{M}\delta_{\alpha\beta}\delta_{m,n}$$

- This action has doubling problem.

$$\delta_{n,m} = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} e^{i\hat{p}.(m-n)}$$

$$K_{\alpha\beta}(n,m) = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} [i\gamma_\mu \sin(\hat{p}_\mu) + \hat{M}] e^{i\hat{p}.(n-m)}$$

Staggered Fermions

- Reduce the BZ by distributing the fermion degrees of freedom over the lattice sites

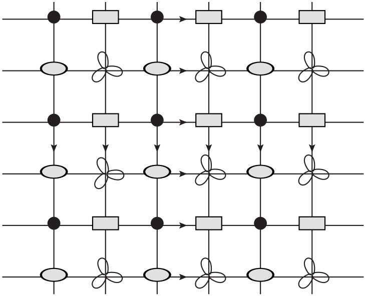

Figure: Distribution of fermionic degrees of freedom in two dimensional lattice

- In 4D we need 4 different Dirac fields

- Define new variable as

$$\hat{\psi}(n) = \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\chi(n); \hat{\bar{\psi}}(n) = \bar{\chi}(n)\gamma_4^{n_4}\gamma_3^{n_3}\gamma_2^{n_2}\gamma_1^{n_1}$$

- Action becomes

$$S = \frac{1}{2}\sum_{n,\mu}\eta_\mu(n)[\bar{\chi}(n)\chi(n+\hat{\mu}) - \bar{\chi}(n)\chi(n-\hat{\mu})] + \hat{M}\sum_n \bar{\chi}(n)\chi(n)$$

  where $\eta_1(n) = 1, \eta_2(n) = (-1)^{n_1}, \eta_3(n) = (-1)^{n_1+n_2}, \eta_4(n) = (-1)^{n_1+n_2+n_3}$
- Relabel the fields as $\chi(2N+s) \equiv \chi_s(N)$
- The inverse of the K matrix

$$K^{-1}(\hat{p}) = \frac{-i\sum_\mu \Gamma^\mu(\hat{p})\sin(\hat{p}_\mu/2) + \hat{M}}{\sum_\mu \sin^2(\hat{p}_\mu/2) + \hat{M}^2}$$

- HISQ action $S = \sum_x \bar{\psi}(x)(\gamma^\mu D_\mu^{HISQ} + M)\psi(x)$ where

$$D_\mu^{HISQ} = \Delta_\mu(W) - \frac{a^2}{6}(1+\epsilon)\Delta_\mu^3(X)$$

- $W_\mu(x) = F_\mu^{HISQ} U_\mu(x)$ and $X_\mu(x) = F_\mu U_\mu(x)$

Heavy-light correlator(light = hisq)

- $Q = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$, $\Gamma = \gamma_5$ or $\Gamma = \gamma_k$

$$\begin{aligned} C(\vec{p},t) &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q^\dagger(x)\Gamma^\dagger_{sk}(x)Q(x)Q^\dagger(0)\Gamma_{sc}(0)q(0)|0\rangle \\ &= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|q(0)q^\dagger(x)\Gamma Q(x)Q^\dagger(0)\Gamma|0\rangle \\ &= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[M(0,x)\gamma_4\Gamma G(x,0)\Gamma] \\ &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x,0)^\dagger\gamma_5\Gamma G(x,0)\Gamma] \\ &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x,0)^\dagger\gamma_5\Gamma S^\dagger G(x,0)S\Gamma](\text{Milc}) \end{aligned}$$

- $G(x,0)$ is now a 4×4 matrix in spinor space having vanishing lower components but it is in Dirac representation of gamma matrices. We can convert it to milc gamma representation by an unitary transformation $S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_y & \sigma_y \\ -\sigma_y & \sigma_y \end{pmatrix}$

B_c meson

- Plot for B_c meson correlators.

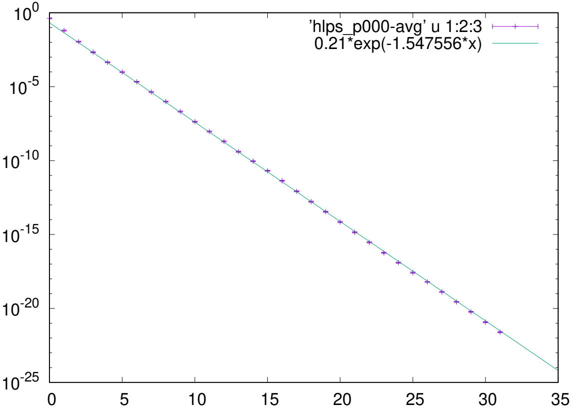

Figure: Heavy-light meson correlators obtained at zero momentum

- From fit $E_{B_c} = 1.547556\text{e} = 1.547556\text{e} \times 197.3/0.12 = 2544$ Mev

- As we used kinetic mass in tuning the bottomonium masses we had to use the following formula to calculate the mass of B_c.

$$M_{B_c} = E_{B_c} + \frac{1}{2}(M_{\eta_b} - E_{\eta_b})$$

Here E_{B_c}, E_{η_b} are the simulated masses and M_{η_b} is the pdg values.

- $M_{B_c} = 2544 + [(9445 - 2116)/2] = 6208$ MeV, error $= \pm$ 32 MeV.

Ω_{bbb} baryon

- Interpolator $(\mathcal{O}_k)_\alpha = \epsilon_{abc}({Q^a}^T C\gamma_k Q^b)Q^c_\alpha$ with $C = \gamma_4\gamma_4$

$$\begin{aligned} C_{ij\alpha\beta}(t) &= \sum_{\vec{x}} \langle 0|[\mathcal{O}_i(\vec{x},t)]_\alpha [\mathcal{O}_j^\dagger(\vec{0},0)]_\beta|0\rangle \\ &= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh} G^{ch}_{\alpha\beta}(x,0)\, Tr[C\gamma_i G^{bg}(x,0)\overline{C\gamma_j}{G^{af}}^T(x,0)] \end{aligned}$$

- The correlator has overlap with both spin 3/2 and spin 1/2 states

$$C_{ij}(t) = Z_{3/2}e^{-E_{3/2}t}\Pi P^{3/2}_{ij} + Z_{1/2}e^{-E_{1/2}t}\Pi P^{1/2}_{ij}$$

$\Pi = \frac{1}{2}(1+\gamma_4)$, $P^{3/2}_{ij} = \delta_{ij} - \frac{1}{3}\gamma_i\gamma_j$, $P^{1/2}_{ij} = \frac{1}{3}\gamma_i\gamma_j$ and $P^{3/2}_{ij}.P^{1/2}_{jk} = 0.$

- $P^{3/2}_{xx}.C_{xx} + P^{3/2}_{xy}.C_{yx} + P^{3/2}_{xz}.C_{zx} = \frac{2}{3}Z_{3/2}\Pi e^{-E_{3/2}t}$
- $P^{1/2}_{xx}.C_{xx} + P^{1/2}_{xy}.C_{yx} + P^{1/2}_{xz}.C_{zx} = \frac{1}{3}Z_{3/2}\Pi e^{-E_{1/2}t}$

- Plot for $\Omega_{bbb}(3/2)$ correlator

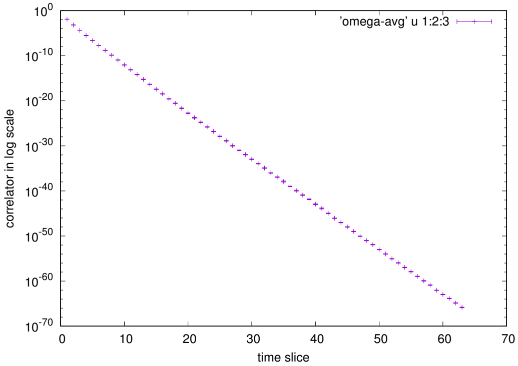

Figure: Omega 3/2

- $M_{\Omega_{bbb}(3/2)} = E_{\Omega_{bbb}(3/2)} + \frac{3}{2}(M_{Exp} - E_{Sim})$=14.355 GeV, error = $\pm$ 20 MeV.
- Splitting $M_{\Omega_{bbb}(3/2)} - M_{\Omega_{bbb}(1/2)}$= 21 MeV.

Ω_{bbc} baryon

- Interpolator $(\mathcal{O}_k)_\alpha = \epsilon_{abc}({Q^a}^T C\gamma_k q^b)Q^c_\alpha$

$$
\begin{aligned}
C_{jk\alpha\delta}(t) &= \sum_{\vec{x}} \langle 0|[\mathcal{O}_j(\vec{x},t)]_\alpha [\mathcal{O}_k^\dagger(\vec{0},0)]_\delta|0\rangle \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh} G^{ch}_{\alpha\delta}(x,0)\, Tr[C\gamma_j M^{bg}(x,0)\gamma_k\gamma_2 {G^{af}}^T(x,0)] \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh} G^{ch}_{\alpha\delta}\, Tr[\gamma_4\gamma_2\gamma_j M^{bg}\gamma_k\gamma_2 S^\dagger {G^{af}}^T S](\text{Milc})
\end{aligned}
$$

- Change $G(x,0)$ into milc gamma representation.
- $P^{3/2}_{xx}.C_{xx} + P^{3/2}_{xy}.C_{yx} + P^{3/2}_{xz}.C_{zx} = \frac{2}{3} Z_{3/2}\Pi e^{-E_{3/2}t}$
- $P^{1/2}_{xx}.C_{xx} + P^{1/2}_{xy}.C_{yx} + P^{1/2}_{xz}.C_{zx} = \frac{1}{3} Z_{3/2}\Pi e^{-E_{1/2}t}$

- Plot for Ω_{bbc} correlator

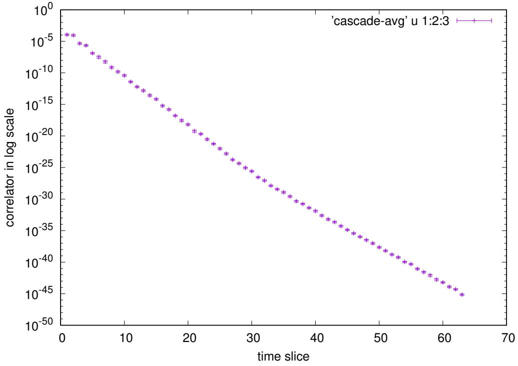

Figure: $Omega_{bbc}$ 3/2

- $M_{\Omega_{bbc}(3/2)} = E_{\Omega_{bbc}(3/2)} + (M_{Exp} - E_{Sim})$=11.06 GeV, error = $\pm$ 12 MeV.
- Splitting $M_{\Omega_{bbc}(3/2)} - M_{\Omega_{bbc}(1/2)}$= 22 MeV.

Ω_{bcc} baryon

- Interpolator $(\mathcal{O}_k)_\alpha = \epsilon_{abc}(Q^{a\,T} C\gamma_k q^b)q^c_\alpha$

$$
\begin{aligned}
C_{jk\alpha\delta}(t) &= \sum_{\vec{x}} \langle 0|[\mathcal{O}_j(\vec{x},t)]_\alpha [\mathcal{O}_k^\dagger(\vec{0},0)]_\delta|0\rangle \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh}[M^{ch}(x,0)\gamma_4]_{\alpha\delta}\, Tr[\gamma_4\gamma_2\gamma_j M^{bg}(x,0)\gamma_k\gamma_2 G^{af\,T}(x,0)] \\
&= \sum_{\vec{x}} \epsilon_{abc}\epsilon_{fgh}[S.M^{ch}\gamma_4.S^\dagger]_{\alpha\delta}\, Tr[\gamma_4\gamma_2\gamma_j M^{bg}\gamma_k\gamma_2 S^\dagger G^{af\,T} S](\text{Milc})
\end{aligned}
$$

- Change $G(x,0)$ into milc gamma representation.
- Change $M(x,0)$ into milc dirac representation.
- $P^{3/2}_{xx}.C_{xx} + P^{3/2}_{xy}.C_{yx} + P^{3/2}_{xz}.C_{zx} = \frac{2}{3}Z_{3/2}\Pi e^{-E_{3/2}t}$
- $P^{1/2}_{xx}.C_{xx} + P^{1/2}_{xy}.C_{yx} + P^{1/2}_{xz}.C_{zx} = \frac{1}{3}Z_{3/2}\Pi e^{-E_{1/2}t}$
- $M_{\Omega_{bcc}}(3/2) = E_{\Omega_{bcc}(3/2)} + \frac{1}{2}(M_{Exp} - E_{Sim})$=7.80 GeV, error $= \pm$ 12 MeV.
- Splitting $M_{\Omega_{bcc}(3/2)} - M_{\Omega_{bcc}(1/2)}$= 28 MeV.

THANK YOU

Backup

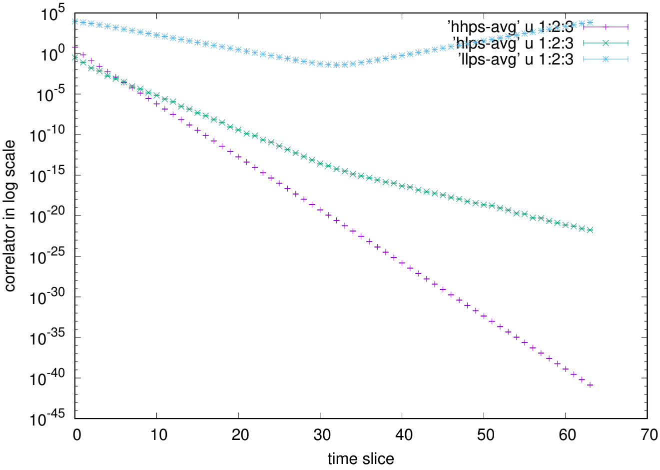

Figure: hh hl ll mesons

Tuning and mass calculation

- In NRQCD Lagrangian the rest mass term is not included.
- In order to tune b-quark we calculated 'kinetic mass' of η_b meson

$$\begin{aligned} E(p) - E(0) &= \sqrt{p^2 + M^2} - M \\ \Rightarrow \Delta E + M &= \sqrt{p^2 + M^2} \text{ where } \Delta E = E(p) - E(0) \\ \Rightarrow (\Delta E)^2 + 2M\Delta E &= p^2 \\ \Rightarrow M &= \frac{p^2 - (\Delta E)^2}{2\Delta E} \end{aligned}$$

$$\begin{aligned} E(p) &= E(0) + M(1 + \frac{p^2}{M^2})^{1/2} - M \\ &= E(0) + \frac{p^2}{2M} - \frac{p^4}{8M^3} \\ \Rightarrow E(p)^2 &= E(0)^2 + \frac{E(0)}{M}p^2 + \frac{p^4}{4M^2}(1 - \frac{E(0)}{M}) \end{aligned}$$

Green's Function for anti-quark

- Antiquarks transform as $\bar{3}$'s under color rotation i.e change $U_{x,\mu} \to U^*_{x,\mu}$. Replace ψ by $\tilde{\chi}$. To compare it with Dirac's theory we change the variable as $\chi = \tilde{\chi}^*$

$$\begin{aligned}
\tilde{\chi}(x,t)^\dagger U_t^*(x)\tilde{\chi}(x,t+1) &= (\chi^*(x,t))^\dagger U_t^*(x)\chi^*(x,t+1) \\
&= (\chi(x,t))^T (U_t^\dagger)^T (x)(\chi^\dagger(x,t+1))^T \\
&= -\chi^\dagger(x,t+1)U_t^\dagger(x)\chi(x,t)
\end{aligned}$$

We used the fact that it is 1×1 quantity so we can ignore the transpose sigh altogether and we put the minus sign because χ's are fermionic field they obey Grassmann algebra. So if the we write the quark action as $S_Q = \psi^\dagger K \psi$ then we for anti-quark we have $S_{\bar{Q}} = -\chi^\dagger K^\dagger \chi$.

Heavy-heavy correlator

- For mesons containing both heavy quarks let the heavy quark and anti-quark are created by two component spinor $\psi^\dagger$ and χ and their destruction operators are ψ and $\chi^\dagger$. As anti-quaks transform by $\bar{3}$ under color rotation so it is convinient to rename the anti-quark spinor.

$$\begin{aligned} C(\vec{p},t) &= \sum_x \langle 0|e^{i\vec{p}.\vec{x}} O(\vec{x},t)O^\dagger(\vec{0},0)|0\rangle \\ &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|\chi^\dagger(x)\Gamma_{sk}(x)\psi(x)\psi^\dagger(0)\Gamma^\dagger_{sc}(0)\chi(0)|0\rangle \\ &= -\sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0|\chi(0)\chi^\dagger(x)\Gamma_{sk}(x)\psi(x)\psi^\dagger(0)\Gamma^\dagger_{sc}(0)0\rangle \\ &= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[G^\dagger(x,0)\Gamma_{sk}(x)G(x,0)\Gamma^\dagger_{sc}(0)] \end{aligned}$$

- In the last line we have used $G^\dagger(x,0) = -[\chi(x)\chi^\dagger(0)]^\dagger$. Here $\Gamma(x) = \Omega\phi(x)$. ϕ is the smearing operator and Ω is a 2×2 matrix in spin space. $\Omega = I$ for pseudoscalar particles and $\Omega = \sigma_i$ for vector particles.

Improvement upto $O(a^4)$

- For $a = 0.12 fm$ it is desirable to correct operators upto order $O(a^4)$.
- Symmetric derivative

$$\begin{aligned}
\Delta_i^{\pm} f(x) &= \frac{1}{2a}[f(x+a\hat{i}) - f(x-a\hat{i})] \\
&= \partial_i f + \frac{a^2}{6}\partial_i^3 f \\
&= \partial_i f + \frac{a^2}{6}\Delta_i^{\pm}\Delta_i^{+}\Delta_i^{-} f \\
\partial_i f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f \\
\tilde{\Delta}_i^{\pm} f &= \Delta_i^{\pm} f - \frac{a^2}{6}\Delta_i^{+}\Delta_i^{\pm}\Delta_i^{-} f
\end{aligned}$$

- Laplacian

$$\tilde{\Delta}^2 = \Delta^2 - \frac{a^2}{12}\sum_i [\Delta_i^{+}\Delta_i^{-}]^2$$

- Gauge fields corrected upto $O(a^4)$ {using cloverleaf}

$$g\tilde{F}_{\mu\nu}(x) = gF_{\mu\nu}(x) - \frac{a^4}{6}[\Delta_\mu^{+}\Delta_\mu^{-} + \Delta_\nu^{+}\Delta_\nu^{-}]gF_{\mu\nu}(x)$$