Heavy hadron spectrum on lattice using NRQCD

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Lattice methods are powerful techniques in analyzing the spectrum of hadrons. However for hadrons containing heavy quarks particularly bottom quark are difficult to analyze.

For spectrum calculation it is necessary that $aM \ll 1$. For light quarks it is true but for charm quark $aM_c > 0.7$ and for bottom quark $aM_b > 2$ with lattice spacing $a = 0.12\, fm$.

However in hadrons containing heavy quarks the velocities of heavy quarks are non-relativistic. One can use effective theories like NRQCD. $M_\Upsilon = 9390\, \text{MeV}$ where as $2 \times M_b = 8360\, \text{MeV}$ ($\overline{\text{MS}}$ Scheme) and $M_{J/\psi} = 3096\, \text{MeV}$ where as $2 \times M_c = 2580\, \text{MeV}$.
The Dirac equation $H\psi = i\frac{\partial \psi}{\partial t}$ where 

$$H = \vec{\alpha} \cdot (\vec{P} - e\vec{A}) + e\phi + m\beta$$

Non-relativistic limit is reached by making the following transformation 

$$\psi' = e^{iS}\psi$$ 

where 

$$S = -\frac{i}{2m}\beta \vec{\alpha} \cdot (\vec{P} - e\vec{A})$$

We get $i\frac{\partial \psi'}{\partial t} = H'\psi'$ where 

$$H' = e^{iS}He^{-iS} - ie^{iS}\frac{\partial e^{-iS}}{\partial t}$$

$$= H + i[S, H] - \frac{1}{2}[S, [S, H]] - \frac{i}{6}[S, [S, [S, H]]] + .......$$

$$-\dot{S} - \frac{i}{2}[S, \dot{S}] + \frac{1}{6}[S, [S, \dot{S}]] + .........$$
Defining \( \theta = \overrightarrow{\alpha} \cdot (\overrightarrow{P} - e \overrightarrow{A}) \) we get (up to \( O(v^4/c^4) \))

\[
H' = \beta \left( m + \frac{\theta^2}{2m} - \frac{\theta^4}{8m^3} \right) + e\phi - \frac{e}{8m^2} [\theta, [\theta, \phi]] - \frac{i}{8m^2} [\theta, \dot{\theta}]
+ \frac{e\beta}{2m} [\theta, \phi] + i\beta \frac{\dot{\theta}}{2m} - \frac{\theta^3}{3m^2}
\]

writing

\[
\psi' = \begin{pmatrix} u \\ v \end{pmatrix}
\]

\[
i \frac{\partial u}{\partial t} = \left[ m - \frac{1}{2m} \sum_j D_j^2 - \frac{e}{2m} \sigma \cdot B - \frac{1}{8m^3} (\sum_j D_j^2)^2 \right.
+ e\phi - \frac{e}{8m^2} \nabla \cdot E - \frac{ie}{8m^2} \sigma \cdot (\nabla \times E - E \times \nabla)] u
\]
NRQCD Lagrangian

- Similarly like QED we write NRQCD Lagrangian upto $O[(\nu/c)^6]$

$$\mathcal{L} = \mathcal{L}_0 + \delta \mathcal{L}_{\nu^4} + \delta \mathcal{L}_{\nu^6}$$

$$\mathcal{L}_0 = \psi(x)\dagger(iD_0 + \frac{\vec{D}^2}{2m})\psi(x)$$

$$\delta \mathcal{L}_{\nu^4} = c_1 \frac{1}{8m^3} \psi\dagger D^4\psi + c_2 \frac{g}{8m^2} \psi\dagger (\vec{D}.\vec{E} - \vec{E}.\vec{D})\psi$$

$$+ c_3 \frac{ie}{8m^2} \psi\dagger \vec{\sigma} . (\vec{D} \times \vec{E} - \vec{E} \times \vec{D})\psi + c_4 \frac{g}{2m} \psi\dagger \vec{\sigma} . \vec{B} \psi$$

$$\delta \mathcal{L}_{\nu^6} = c_5 \frac{g}{m^3} \psi\dagger \{\vec{D}^2, \vec{\sigma} . \vec{B}\} \psi + c_6 \frac{ig^2}{m^3} \psi\dagger (\vec{\sigma} . \vec{E} \times \vec{E})\psi$$

$$+ c_7 \frac{ig}{m^4} \psi\dagger \{\vec{D}^2, \vec{\sigma} . (\vec{D} \times \vec{E} - \vec{E} \times \vec{D})\} \psi$$

- $\mathcal{L}_0$ merely gives us Schrodinger equation.

- $c_1, c_2, c_3, c_4 = 1$(tree level).
To calculate $c_7$ let us consider the term $T_E = \bar{\psi}(q)\gamma^0 g\phi(q - p)\psi(p)$ with the positive energy spinor

$$\psi(p) = \left(\frac{E_p + m}{2E_p}\right)^{\frac{1}{2}} \left[ \frac{u}{E_p + m} \right]$$

$$T_E = \sqrt{\frac{(E_p + m)(E_q + m)}{4E_pE_q}}$$

$$\times u^\dagger \left[ 1 + \frac{p.q + i\sigma.q \times p}{(E_q + m)(E_p + m)} \right] g\phi(q - p)u$$

Term containing $\sigma$

$$V(p, q) = \left[ \frac{i}{4m^2} - \frac{3i}{32m^4}(p^2 + q^2) \right] u^\dagger \sigma.(q \times p)g\phi(q - p)u$$

- $c_7 = \frac{3}{64}$.
- $c_5$ can be calculated from $T_B(p, q) = -\bar{\psi}(q)g\gamma.A(q - p)\psi(p)$ and so on
- $c_5 = \frac{1}{8}, c_6 = -\frac{1}{8}$
Replace continuum derivatives by lattice derivatives. For quark fields

\[ a\Delta^+_{\mu}\psi(x) = U^\mu(x)\psi(x + a\hat{\mu}) - \psi(x) \]
\[ a\Delta^-_{\mu}\psi(x) = \psi(x) - U^\dagger_\mu(x - a\hat{\mu})\psi(x - a\hat{\mu}) \]

For gauge fields

\[ a\Delta^+_{\rho}F_{\mu\nu}(x) = U^\rho(x)F_{\mu\nu}(x + a\hat{\rho})U^\dagger_\rho(x) - F_{\mu\nu}(x) \]
\[ a\Delta^+_{\rho}F_{\mu\nu}(x) = F_{\mu\nu}(x) - U^\dagger_\rho(x - a\hat{\rho})F_{\mu\nu}(x)U^\rho(x - a\hat{\rho}) \]

Here \( a \) is the lattice spacing and \( U^\mu(x) \) is link variable.

Symmetric derivative

\[ \Delta^\pm = \frac{1}{2}(\Delta^+ + \Delta^-) \]

Laplacian

\[ \Delta^2 = \sum_i \Delta^+_i \Delta^-_i = \sum_i \Delta^-_i \Delta^+_i \]
The Lagrangian has the following form
\[ \mathcal{L} = \psi^\dagger(x, t)D_4\psi(x, t) + \psi^\dagger(x, t)H\psi(x, t) \]

- \( H \) contains spatial derivatives only. E.O.M. corresponding to \( \psi^\dagger \)
\[ D_4\psi(x, t) + H\psi(x, t) = 0 \text{ after discretization} \]
\[ U_t(x)\psi(x, t + 1) - \psi(x, t) + aH\psi(x, t) = 0 \]

Green's function obeys
\[ U_t(x, t)G(x, t + 1; 0, 0) - (1 - aH)G(x, t; 0, 0) = \delta_{x,0}\delta_{t,0} \]
\[ \Rightarrow G(x, t + 1; 0, 0) = U_t^\dagger(x, t)(1 - aH)G(x, t; 0, 0) \]
From renormalization considerations

\[ G(x, t + 1; 0, 0) = (1 - \frac{aH_0}{2})(1 - \frac{a\delta H}{2})U_t(x, t)^\dagger \]

\[ (1 - \frac{a\delta H}{2})(1 - \frac{aH_0}{2})G(x, t; 0, 0) \]

\[ H_0 \text{ and } \delta H \text{ are related as } H = H_0 + \delta H. \text{ For stability purpose we modify} \]

\[ G(x, t + 1; 0, 0) = (1 - \frac{aH_0}{2n})^n(1 - \frac{a\delta H}{2})U_t(x, t)^\dagger \]

\[ (1 - \frac{a\delta H}{2})(1 - \frac{aH_0}{2n})^nG(x, t; 0, 0) \]

with \( G(x, t; 0, 0) = 0 \) for \( t < 0 \) and \( G(x, t; 0, 0) = \delta_{x,0} \) for \( t = 0 \). From the above equation it is evident that \( n > \frac{3}{2m} \).
Relativistic fermions

- Action for free Dirac field

\[ S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\gamma^\mu \partial_\mu - M)\psi(x) \]

Lattice version of the above action is

\[ S = \sum_{n, m, \alpha, \beta} \bar{\psi}_\alpha(n)K_{\alpha\beta}(n, m)\psi_\beta(m) \]

where \( K_{\alpha\beta}(n, m) \) is given by

\[ K_{\alpha\beta}(n, m) = \sum_\mu \frac{1}{2} (\gamma_\mu)_{\alpha\beta} [\delta_{m, n+\hat{\mu}} - \delta_{m, n-\hat{\mu}}] + \hat{M}\delta_{\alpha\beta}\delta_{m, n} \]

- This action has doubling problem.

\[ \delta_{n, m} = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} e^{i\hat{p}.(m-n)} \]

\[ K_{\alpha\beta}(n, m) = \int_{-\pi}^{\pi} \frac{d^4\hat{p}}{(2\pi)^4} [i\gamma_\mu \sin(\hat{p}_\mu) + \hat{M}]e^{i\hat{p}.(n-m)} \]
Staggered Fermions

- Reduce the BZ by distributing the fermion degrees of freedom over the lattice sites

**Figure:** Distribution of fermionic degrees of freedom in two dimensional lattice

- In 4D we need 4 different Dirac fields
Define new variable as
\[ \hat{\psi}(n) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4} \chi(n); \bar{\psi}(n) = \bar{\chi}(n) \gamma_4^{n_4} \gamma_3^{n_3} \gamma_2^{n_2} \gamma_1^{n_1} \]

Action becomes
\[
S = \frac{1}{2} \sum_{n, \mu} \eta_\mu(n) [\bar{\chi}(n) \chi(n + \hat{\mu}) - \bar{\chi}(n) \chi(n - \hat{\mu})] + \hat{M} \sum_n \bar{\chi}(n) \chi(n)
\]

where \( \eta_1(n) = 1, \eta_2(n) = (-1)^{n_1}, \eta_3(n) = (-1)^{n_1+n_2}, \eta_4(n) = (-1)^{n_1+n_2+n_3} \)

Relabel the fields as \( \chi(2N + s) \equiv \chi_s(N) \)

The inverse of the \( K \) matrix
\[
K^{-1}(\hat{\rho}) = \frac{-i \sum_\mu \Gamma^\mu(\hat{\rho}) \sin(\hat{\rho}_\mu / 2) + \hat{M}}{\sum_\mu \sin^2(\hat{\rho}_\mu / 2) + \hat{M}^2}
\]

HISQ action \( S = \sum_x \bar{\psi}(x) (\gamma^\mu D_{HISQ}^{\mu} + M) \psi(x) \) where
\[
D_{HISQ}^{\mu} = \Delta^{\mu}(W) - \frac{a^2}{6} (1 + \epsilon) \Delta^{3}(X)
\]

\( W_\mu(x) = F_{HISQ}^{\mu} U_\mu(x) \) and \( X_\mu(x) = F_{\mu} U_{\mu}(x) \)
Heavy-light correlator (light = hisq)

- $Q = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$, $\Gamma = \gamma_5$ or $\Gamma = \gamma_k$

$$C(\vec{p}, t) = \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0 | q(x)\Gamma_s k(x) Q(x) Q^\dagger(0) \Gamma_s c(0) q(0) | 0 \rangle$$

$$= - \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} \langle 0 | q(0) q^\dagger(x) \Gamma Q(x) Q^\dagger(0) \Gamma | 0 \rangle$$

$$= - \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[M(0, x) \gamma_4 \Gamma G(x, 0) \Gamma]$$

$$= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x, 0)^\dagger \gamma_5 \Gamma G(x, 0) \Gamma]$$

$$= \sum_{\vec{x}} e^{i\vec{p}.\vec{x}} Tr[\gamma_5 M(x, 0)^\dagger \gamma_5 \Gamma S^\dagger \Gamma] (\text{Milc})$$

- $G(x, 0)$ is now a $4 \times 4$ matrix in spinor space having vanishing lower components but it is in Dirac representation of gamma matrices. We can convert it to milc gamma representation by an unitary transformation $S = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_y & \sigma_y \\ -\sigma_y & \sigma_y \end{pmatrix}$
Heavy hadron spectrum on lattice using NRQCD

\( B_c \) meson

- Plot for \( B_c \) meson correlators.

![Plot for \( B_c \) meson correlators.](image)

**Figure**: Heavy-light meson correlators obtained at zero momentum

- From fit \( E_{B_c} = 1.547556 e = 1.547556 e \times \frac{197.3}{0.12} = 2544 \text{ Mev} \)
As we used kinetic mass in tuning the bottomonium masses we had to use the following formula to calculate the mass of $B_c$.

$$M_{B_c} = E_{B_c} + \frac{1}{2} (M_{\eta_b} - E_{\eta_b})$$

Here $E_{B_c}, E_{\eta_b}$ are the simulated masses and $M_{\eta_b}$ is the pdg values.

$M_{B_c} = 2544 + [(9445 - 2116)/2] = 6208$ MeV, error $= \pm 32$ MeV.
Heavy hadron spectrum on lattice using NRQCD

Baryons

$\Omega_{bbb}$ baryon

- Interpolator $(O_k)_\alpha = \epsilon_{abc} (Q^a C \gamma_k Q^b) Q^c_\alpha$ with $C = \gamma_4 \gamma_4$

$$C_{ij\alpha\beta}(t) = \sum_\vec{x} \langle 0 | [O_i(\vec{x}, t)]_\alpha [O_j^\dagger(0, 0)]_\beta | 0 \rangle = \sum_\vec{x} \epsilon_{abc} \epsilon_{fgh} G^{ch}_{\alpha\beta}(x, 0) \text{Tr} [C \gamma_i G^{bg}(x, 0) \overline{C} \gamma_j G^{af}^T(x, 0)]$$

- The correlator has overlap with both spin $3/2$ and spin $1/2$ states

$$C_{ij}(t) = Z_{3/2} e^{-E_{3/2} t} \Pi P_{ij}^{3/2} + Z_{1/2} e^{-E_{1/2} t} \Pi P_{ij}^{1/2}$$

$$\Pi = \frac{1}{2} (1 + \gamma_4), \quad P_{ij}^{3/2} = \delta_{ij} - \frac{1}{3} \gamma_i \gamma_j, \quad P_{ij}^{1/2} = \frac{1}{3} \gamma_i \gamma_j$$ and $P_{ij}^{3/2} P_{jk}^{1/2} = 0$.

- $P_{xx}^{3/2} C_{xx} + P_{xy}^{3/2} C_{yx} + P_{xz}^{3/2} C_{zx} = \frac{2}{3} Z_{3/2} \Pi e^{-E_{3/2} t}$

- $P_{xx}^{1/2} C_{xx} + P_{xy}^{1/2} C_{yx} + P_{xz}^{1/2} C_{zx} = \frac{1}{3} Z_{3/2} \Pi e^{-E_{1/2} t}$
Plot for $\Omega_{bbb}(3/2)$ correlator

Figure: Omega 3/2

- $M_{\Omega_{bbb}(3/2)} = E_{\Omega_{bbb}(3/2)} + \frac{3}{2}(M_{Exp} - E_{Sim}) = 14.355$ GeV, error = ± 20 MeV.
- Splitting $M_{\Omega_{bbb}(3/2)} - M_{\Omega_{bbb}(1/2)} = 21$ MeV.
Heavy hadron spectrum on lattice using NRQCD

**Baryons**

\[ \Omega_{bbc} \]

\[ \Omega_{bbc} \text{ baryon} \]

**Interpolator**

\[ \mathcal{O}_k(\alpha) = \epsilon_{abc} (Q^{aT} C \gamma_k q^b) Q^c \]

\[ C_{jk\alpha\delta}(t) = \sum \langle 0 | [\mathcal{O}_j(\vec{x}, t)]_{\alpha} [\mathcal{O}_k^\dagger(0, 0)]_{\delta} | 0 \rangle \]

\[ = \sum \epsilon_{abc} \epsilon_{fgh} G^{ch}_{\alpha\delta}(x, 0) \text{Tr}[C \gamma_j M^{bg}(x, 0) \gamma_k \gamma_2 G^{afT}(x, 0)] \]

\[ = \sum \epsilon_{abc} \epsilon_{fgh} G^{ch}_{\alpha\delta} \text{Tr}[\gamma_4 \gamma_2 \gamma_j M^{bg} \gamma_k \gamma_2 S^\dagger G^{afT} S](\text{Milc}) \]

- Change \( G(x, 0) \) into milc gamma representation.

- \( P_{xx}^{3/2} . C_{xx} + P_{xy}^{3/2} . C_{yx} + P_{xz}^{3/2} . C_{zx} = \frac{2}{3} Z_{3/2 \Pi} e^{-E_{3/2} t} \)

- \( P_{xx}^{1/2} . C_{xx} + P_{xy}^{1/2} . C_{yx} + P_{xz}^{1/2} . C_{zx} = \frac{1}{3} Z_{3/2 \Pi} e^{-E_{1/2} t} \)
Plot for $\Omega_{bbc}$ correlator

$M_{\Omega_{bbc}(3/2)} = E_{\Omega_{bbc}(3/2)} + (M_{Exp} - E_{Sim}) = 11.06$ GeV, error = ± 12 MeV.

Splitting $M_{\Omega_{bbc}(3/2)} - M_{\Omega_{bbc}(1/2)} = 22$ MeV.

Figure: Omega_{bbc} 3/2
Heavy hadron spectrum on lattice using NRQCD

Baryons

\[ \Omega_{bcc} \]

\[ \Omega_{bcc} \text{ baryon} \]

- Interpolator \( (O_k)_\alpha = \epsilon_{abc}(Q^a T C \gamma_k q^b)q^c \)

\[
C_{jk\alpha\delta}(t) = \sum_{\vec{x}} \langle 0 | [O_j(\vec{x}, t)]_\alpha [O_k^\dagger(\vec{0}, 0)]_\delta | 0 \rangle
\]

\[
= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} [M^{ch}(x, 0) \gamma_4]_{\alpha\delta} \text{Tr}[\gamma_4 \gamma_2 \gamma_j M^{bg}(x, 0) \gamma_k \gamma_2 G^{af T}(x, 0)]
\]

\[
= \sum_{\vec{x}} \epsilon_{abc} \epsilon_{fgh} [S \cdot M^{ch} \gamma_4 \cdot S^\dagger]_{\alpha\delta} \text{Tr}[\gamma_4 \gamma_2 \gamma_j M^{bg} \gamma_k \gamma_2 S^\dagger G^{af T} S](\text{Milc})
\]

- Change \( G(x, 0) \) into milc gamma representation.
- Change \( M(x, 0) \) into milc dirac representation.

\[
P_{xx}^{3/2} \cdot C_{xx} + P_{xy}^{3/2} \cdot C_{yx} + P_{xz}^{3/2} \cdot C_{zx} = \frac{2}{3} Z_{3/2} \Pi e^{-E_{3/2} t}
\]

\[
P_{xx}^{1/2} \cdot C_{xx} + P_{xy}^{1/2} \cdot C_{yx} + P_{xz}^{1/2} \cdot C_{zx} = \frac{1}{3} Z_{3/2} \Pi e^{-E_{1/2} t}
\]

\[ M_{\Omega_{bcc}}(3/2) = E_{\Omega_{bcc}}(3/2) + \frac{1}{2} (M_{\text{Exp}} - E_{\text{Sim}}) = 7.80 \text{ GeV}, \text{ error} = \pm 12 \text{ MeV}. \]

- Splitting \( M_{\Omega_{bcc}}(3/2) - M_{\Omega_{bcc}}(1/2) = 28 \text{ MeV}. \)
THANK YOU
Figure: hh hl ll mesons
In NRQCD Lagrangian the rest mass term is not included.

In order to tune b-quark we calculated 'kinetic mass' of $\eta_b$ meson

\[
E(p) - E(0) = \sqrt{p^2 + M^2} - M
\]

\[\Rightarrow \Delta E + M = \sqrt{p^2 + M^2} \text{ where } \Delta E = E(p) - E(0)\]

\[\Rightarrow (\Delta E)^2 + 2M\Delta E = p^2\]

\[\Rightarrow M = \frac{p^2 - (\Delta E)^2}{2\Delta E}\]

\[
E(p) = E(0) + M(1 + \frac{p^2}{M^2})^{1/2} - M
\]

\[= E(0) + \frac{p^2}{2M} - \frac{p^4}{8M^3}\]

\[\Rightarrow E(p)^2 = E(0)^2 + \frac{E(0)}{M}p^2 + \frac{p^4}{4M^2}(1 - \frac{E(0)}{M})\]
Antiquarks transform as \( \bar{3} \)'s under color rotation i.e change \( U_{x,\mu} \rightarrow U_{x,\mu}^* \).

Replace \( \psi \) by \( \tilde{\chi} \). To compare it with Dirac's theory we change the variable as \( \chi = \tilde{\chi}^* \):

\[
\tilde{\chi}(x, t)^\dagger U_t^*(x) \tilde{\chi}(x, t + 1) = (\chi^*(x, t))^\dagger U_t^*(x) \chi^*(x, t + 1) = (\chi(x, t))^T (U_t^\dagger)^T(x) (\chi^\dagger(x, t + 1))^T = -\chi^\dagger(x, t + 1) U_t^\dagger(x) \chi(x, t)
\]

We used the fact that it is \( 1 \times 1 \) quantity so we can ignore the transpose sigh altogether and we put the minus sign because \( \chi \)'s are fermionic field they obey Grassmann algebra. So if the we write the quark action as \( S_Q = \psi^\dagger K \psi \) then we for anti-quark we have \( S_{\bar{Q}} = -\chi^\dagger K^\dagger \chi \).
Heavy hadron spectrum on lattice using NRQCD

Baryons

$\Omega_{bcc}$

**Heavy-heavy correlator**

- For mesons containing both heavy quarks let the heavy quark and anti-quark are created by two component spinor $\psi^\dagger$ and $\chi$ and their destruction operators are $\psi$ and $\chi^\dagger$. As anti-quarks transform by $\bar{3}$ under color rotation so it is convienent to rename the anti-quark spinor.

\[
C(\vec{p}, t) = \sum_x \langle 0 | e^{i \vec{p} \cdot \vec{x}} O(\vec{x}, t) O^\dagger(\vec{0}, 0) | 0 \rangle
\]

\[
= \sum_{\vec{x}} e^{i \vec{p} \cdot \vec{x}} \langle 0 | \chi^\dagger(x) \Gamma_{sk}(x) \psi(x) \psi^\dagger(0) \Gamma_{sc}^\dagger(0) \chi(0) | 0 \rangle
\]

\[
= - \sum_{\vec{x}} e^{i \vec{p} \cdot \vec{x}} \langle 0 | \chi(0) \chi^\dagger(x) \Gamma_{sk}(x) \psi(x) \psi^\dagger(0) \Gamma_{sc}^\dagger(0) 0 \rangle
\]

\[
= \sum_{\vec{x}} e^{i \vec{p} \cdot \vec{x}} \text{Tr}[G^\dagger(x, 0) \Gamma_{sk}(x) G(x, 0) \Gamma_{sc}^\dagger(0)]
\]

- In the last line we have used $G^\dagger(x, 0) = -[\chi(x) \chi^\dagger(0)]^\dagger$. Here $\Gamma(x) = \Omega \phi(x)$. $\phi$ is the smearing operator and $\Omega$ is a $2 \times 2$ matrix in spin space. $\Omega = I$ for pseudoscalar particles and $\Omega = \sigma_i$ for vector particles.
Improvement upto $O(a^4)$

- For $a = 0.12 fm$ it is desirable to correct operators upto order $O(a^4)$.
- Symmetric derivative

$$\Delta_i^\pm f(x) = \frac{1}{2a} [f(x + a\hat{i}) - f(x - a\hat{i})]$$

$$= \partial_i f + \frac{a^2}{6} \partial_i^3 f$$

$$= \partial_i f + \frac{a^2}{6} \Delta_i^\pm \Delta_i^+ \Delta_i^- f$$

$$\partial_i f = \Delta_i^\pm f - \frac{a^2}{6} \Delta_i^+ \Delta_i^\pm \Delta_i^- f$$

$$\tilde{\Delta}_i^\pm f = \Delta_i^\pm f - \frac{a^2}{6} \Delta_i^+ \Delta_i^\pm \Delta_i^- f$$

- Laplacian

$$\tilde{\Delta}^2 = \Delta^2 - \frac{a^2}{12} \sum_i [\Delta_i^+ \Delta_i^-]^2$$

- Gauge fields corrected upto $O(a^4)$ \{using cloverleaf\}

$$g\tilde{F}_{\mu\nu}(x) = gF_{\mu\nu}(x) - \frac{a^4}{6} [\Delta^+_{\mu} \Delta^-_{\mu} + \Delta^+_{\nu} \Delta^-_{\nu}] gF_{\mu\nu}(x)$$