

Relativistic hydrodynamics with spin

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recent works with **F. Becattini, B. Friman, A. Jaiswal, A. Kumar, R. Ryblewski, and E. Speranza**
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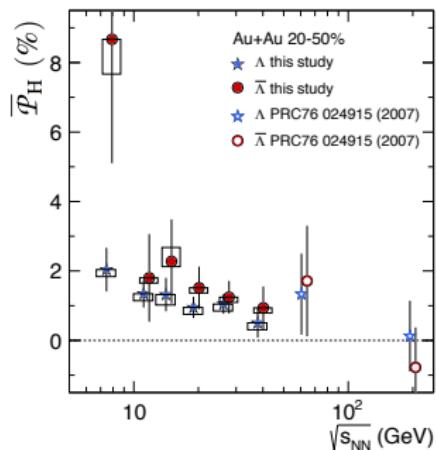
18. ZIMANYI SCHOOL & WINTER WORKSHOP ON HEAVY ION PHYSICS
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- Non-central heavy-ion collisions create fireballs with large global angular momenta which may generate a spin polarization of the hot and dense matter in a way similar to the Einstein-de Haas and Barnett effects
- Much effort has recently been invested in studies of polarization and spin dynamics of particles produced in high-energy nuclear collisions, both from the experimental and theoretical point of view

L. Adamczyk et al. (STAR), (2017), **Nature 548 (2017) 62-65**

Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid
www.sciencenews.org/article/smashing-gold-ions-creates-most-swirly-fluid-ever



PERFECT-FLUID HYDRODYNAMICS = LOCAL EQUILIBRIUM + CONSERVATION LAWS
USUALLY ENERGY, LINEAR MOMENTUM, AND CONSERVED CHARGES ARE INCLUDED

**FOR PARTICLES WITH SPIN, THE CONSERVATION OF ANGULAR MOMENTUM IS NOT TRIVIAL —
IT REQUIRES INTRODUCTION OF ADDITIONAL LAGRANGE MULTIPLIERS
THAT PLAY A ROLE OF NEW HYDRODYNAMIC VARIABLES**

Canonical (quantization) version

$$e^{-P\beta + \xi} \longrightarrow \widehat{\rho}_{\text{LEG}} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\widehat{T}_{\text{can}}^{\mu\nu} \beta_{\nu} - \frac{1}{2} \omega_{\lambda\nu} \widehat{J}_{\text{can}}^{\mu,\lambda\nu} - \xi \widehat{j}^{\mu} \right) \right] \quad (1)$$

Σ is a space-like hypersurface, for example, corresponding to a constant LAB time t , in this case $\widehat{\rho}_{\text{LEG}} = \widehat{\rho}_{\text{LEG}}(t)$

β_{ν} and ξ are the Lagrange multiplier functions, whose meaning is the ratio between the local four-velocity u^{μ} and temperature T (a four-temperature vector) and the ratio between local chemical potential μ and T

one has to introduce an antisymmetric tensor field $\omega_{\lambda\nu}$, it is dubbed the **spin chemical potential** or the **spin polarization tensor**

Canonical tensors obtained by the Noether Theorem:

$$\partial_\mu \widehat{T}_{\text{can}}^{\mu\nu} = 0, \quad \partial_\mu \widehat{J}_{\text{can}}^{\mu,\lambda\nu} = 0. \quad (2)$$

Angular momentum has orbital and spin parts

$$\widehat{J}_{\text{can}}^{\mu,\lambda\nu} = x^\lambda \widehat{T}_{\text{can}}^{\mu\nu} - x^\nu \widehat{T}_{\text{can}}^{\mu\lambda} + \widehat{S}_{\text{can}}^{\mu,\lambda\nu} \equiv \widehat{L}_{\text{can}}^{\mu,\lambda\nu} + \widehat{S}_{\text{can}}^{\mu,\lambda\nu}, \quad (3)$$

$$\partial_\mu \widehat{J}_{\text{can}}^{\mu,\lambda\nu} = \widehat{T}_{\text{can}}^{\lambda\nu} - \widehat{T}_{\text{can}}^{\nu\lambda} + \partial_\mu \widehat{S}_{\text{can}}^{\mu,\lambda\nu} = 0, \quad \partial_\mu \widehat{S}_{\text{can}}^{\mu,\lambda\nu} = \widehat{T}_{\text{can}}^{\nu\lambda} - \widehat{T}_{\text{can}}^{\lambda\nu}. \quad (4)$$

Pseudo-gauge transformation

(different localization of energy density and angular momentum, global charges not changed)

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2} \partial_\lambda (\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}), \quad (5)$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}. \quad (6)$$

Belinfante's construction: superpotential defined as $\widehat{\Phi} = \widehat{S}_{\text{can}}^{\lambda,\mu\nu}$

$$\widehat{T}_{\text{Bel}}^{\mu\nu} = \widehat{T}_{\text{can}}^{\mu\nu} + \frac{1}{2} \partial_\lambda (\widehat{S}_{\text{can}}^{\lambda,\mu\nu} - \widehat{S}_{\text{can}}^{\mu,\lambda\nu} - \widehat{S}_{\text{can}}^{\nu,\lambda\mu}), \quad \widehat{S}_{\text{Bel}}^{\lambda,\mu\nu} = 0. \quad (7)$$

Locally, angular momentum (and consequently spin) can be eliminated locally by a translation that sets $x = 0$.

Canonical version

$$\widehat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\widehat{T}_{\text{can}}^{\mu\nu} \beta_{\nu} - \frac{1}{2} \omega_{\lambda\nu} \widehat{S}_{\text{can}}^{\mu,\lambda\nu} - \xi \widehat{j}^{\mu} \right) \right] \quad (8)$$

equivalent to

$$\widehat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\widehat{T}_{\text{Bel}}^{\mu\nu} \beta_{\nu} - \frac{1}{2} (\omega_{\lambda\nu} - \bar{\omega}_{\lambda\nu}) \widehat{S}_{\text{can}}^{\mu,\lambda\nu} - \xi \widehat{j}^{\mu} \right) \right] \quad (9)$$

with thermal vorticity (F. Becattini)

$$\omega_{\lambda\nu} = -\frac{1}{2} (\partial_{\lambda} \beta_{\nu} - \partial_{\nu} \beta_{\lambda}) \quad (10)$$

but generally different from

Belinfante's version

$$\widehat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_{\mu} \left(\widehat{T}_{\text{Bel}}^{\mu\nu} \beta_{\nu} - \xi \widehat{j}^{\mu} \right) \right] \quad (11)$$

the canonical and Belinfante approaches agree if $\omega_{\lambda\nu} = \bar{\omega}_{\lambda\nu}$

one can argue that in global equilibrium, where $\widehat{\rho}_{\text{LEQ}}$ is strictly constant, indeed $\omega_{\lambda\nu} = \bar{\omega}_{\lambda\nu} = \text{const.}$

based on the works by F. Becattini and collaborators

also in local equilibrium $\omega_{\lambda\nu}(x) = \bar{\omega}_{\lambda\nu}(x)$

- 1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin
- 2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface
(defined often by the condition $T=\text{const}$)
- 3) Calculate thermal vorticity $\omega_{\alpha\beta}(x) \neq \text{const}$
- 4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$
- 5) Make predictions about spin polarization

**SUCH A METHOD WORKS WELL,
DESCRIBES MOST OF THE DATA, BUT...CAN WE TAKE IT FOR GRANTED?**

in local equilibrium $\widehat{\rho}_{\text{LEQ}}$ is approximately constant (with dissipation effects neglected)

$$T^{\mu\nu} = \text{tr}(\widehat{\rho}_{\text{LEQ}} \widehat{T}^{\mu\nu}), \quad S^{\mu,\lambda\nu} = \text{tr}(\widehat{\rho}_{\text{LEQ}} \widehat{S}^{\mu,\lambda\nu}), \quad j^\mu = \text{tr}(\widehat{\rho}_{\text{LEQ}} \widehat{j}^\mu). \quad (12)$$

these tensors are all functions of the hydrodynamic variables β_μ , $\omega_{\mu\nu}$, and ξ

$$T^{\mu\nu} = T^{\mu\nu}[\beta, \omega, \xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta, \omega, \xi], \quad j^\mu = j^\mu[\beta, \omega, \xi], \quad (13)$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^\mu = 0 \quad (14)$$

Spin dependent phase-space distribution functions 1

standard scalar functions $f(x, p)$ are generalized to 2×2 Hermitean matrices in spin space for each value of the space-time position x and four-momentum p

F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Annals Phys. 338 (2013) 32

$$[f^+(x, p)]_{rs} \equiv f_{rs}^+(x, p) = \bar{u}_r(p) X^+ u_s(p), \quad (15)$$

$$[f^-(x, p)]_{rs} \equiv f_{rs}^-(x, p) = -\bar{v}_s(p) X^- v_r(p). \quad (16)$$

$$X^\pm = \exp [\pm \xi(x) - \beta_\mu(x) p^\mu] M^\pm, \quad M^\pm = \exp \left[\pm \frac{1}{2} \omega_{\mu\nu}(x) \Sigma^{\mu\nu} \right]. \quad (17)$$

here $\Sigma^{\mu\nu}$ is the Dirac spin operator, electric- and magnetic-like three-vectors

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}. \quad (18)$$

special case in this talk

$$M^\pm = 1 \pm \frac{1}{2} \omega_{\mu\nu} \Sigma^{\mu\nu}. \quad (19)$$

Spin dependent phase-space distribution functions 2

The spin observables are represented by the Pauli matrices σ and the expectation values of σ provide information on the polarization of spin-1/2 particles in their rest frame

$$f^\pm(x, p) = e^{\pm\xi - p \cdot \beta} \left[1 - \frac{1}{2} \mathbf{P} \cdot \boldsymbol{\sigma} \right], \quad (20)$$

average polarization per particle

$$\mathbf{P} = \frac{1}{m} \left[E_p \mathbf{b} - \mathbf{p} \times \mathbf{e} - \frac{\mathbf{p} \cdot \mathbf{b}}{E_p + m} \mathbf{p} \right] = \mathbf{b}_* \quad (\mathbf{b} \text{ field in the particle rest frame}) \quad (21)$$

$$\langle \mathbf{P}(x, p) \rangle = \frac{1}{2} \frac{\text{tr}_2 [(f^+ + f^-) \boldsymbol{\sigma}]}{\text{tr}_2 [f^+ + f^-]} = -\frac{1}{4} \mathbf{P}. \quad (22)$$

Equilibrium Wigner functions

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*

GLW framework

$$\mathcal{W}_{\text{eq}}^+(x, k) = \frac{1}{2} \sum_{r,s=1}^2 \int dP \delta^{(4)}(k - p) u^r(p) \bar{u}^s(p) f_{rs}^+(x, p), \quad (23)$$

$$\mathcal{W}_{\text{eq}}^-(x, k) = -\frac{1}{2} \sum_{r,s=1}^2 \int dP \delta^{(4)}(k + p) v^s(p) \bar{v}^r(p) f_{rs}^-(x, p). \quad (24)$$

Clifford-algebra expansion

(used in many early works on QED and QGP plasma, e.g., H.T. Elze, M. Gyulassy, D. Vasak, Phys.Lett. B177 (1986) 402)
for the equilibrium

$$\mathcal{W}_{\text{eq}}^\pm(x, k) = \frac{1}{4} [\mathcal{F}_{\text{eq}}^\pm(x, k) + i\gamma_5 \mathcal{P}_{\text{eq}}^\pm(x, k) + \gamma^\mu \mathcal{V}_{\text{eq},\mu}^\pm(x, k) + \gamma_5 \gamma^\mu \mathcal{A}_{\text{eq},\mu}^\pm(x, k) + \Sigma^{\mu\nu} \mathcal{S}_{\text{eq},\mu\nu}^\pm(x, k)].$$

and any other Wigner function

$$\mathcal{W}^\pm(x, k) = \frac{1}{4} [\mathcal{F}^\pm(x, k) + i\gamma_5 \mathcal{P}^\pm(x, k) + \gamma^\mu \mathcal{V}_\mu^\pm(x, k) + \gamma_5 \gamma^\mu \mathcal{A}_\mu^\pm(x, k) + \Sigma^{\mu\nu} \mathcal{S}_{\mu\nu}^\pm(x, k)].$$

Global-equilibrium Wigner function

WF, A. Kumar, R. Ryblewski, Phys. Rev. C98 (2018) 044906

$$(\gamma_\mu K^\mu - m) \mathcal{W}(x, k) = C[\mathcal{W}(x, k)]. \quad (25)$$

Here K^μ is the operator defined by the expression

$$K^\mu = k^\mu + \frac{i\hbar}{2} \partial^\mu, \quad (26)$$

In the case of global equilibrium, with the vanishing collision term, the Wigner function $\mathcal{W}(x, k)$ exactly satisfies the equation

$$(\gamma_\mu K^\mu - m) \mathcal{W}(x, k) = 0. \quad (27)$$

the leading order terms in \hbar can be taken from $\mathcal{W}_{\text{eq}}^\pm(x, k)$

$$\mathcal{F}^{(0)} = \mathcal{F}_{\text{eq}}, \quad (28)$$

$$\mathcal{P}^{(0)} = 0, \quad (29)$$

$$\mathcal{V}_\mu^{(0)} = \mathcal{V}_{\text{eq},\mu}, \quad (30)$$

$$\mathcal{A}_\mu^{(0)} = \mathcal{A}_{\text{eq},\mu}, \quad (31)$$

$$\mathcal{S}_{\mu\nu}^{(0)} = \mathcal{S}_{\text{eq},\mu\nu}. \quad (32)$$

in the NLO in \hbar we get the kinetic equation (well known in the literature)

$$k^\mu \partial_\mu \mathcal{F}_{\text{eq}}(x, k) = 0, \quad (33)$$

$$k^\mu \partial_\mu \mathcal{A}_{\text{eq}}^v(x, k) = 0, \quad k_v \mathcal{A}_{\text{eq}}^v(x, k) = 0, \quad (34)$$

Global equilibrium — Eq. (33) and Eq. (34) are exactly fulfilled, what about local equilibrium

Local equilibrium — only moments of Eq. (33) and Eq. (34) are satisfied

$$\partial_\alpha N_{\text{eq}}^\alpha(x) = 0, \quad \partial_\alpha T_{\text{GLW}}^{\alpha\beta}(x) = 0, \quad \partial_\lambda S_{\text{GLW}}^{\lambda,\mu\nu}(x) = 0. \quad (35)$$

GLW — forms proposed in the textbook on the kinetic theory by de Groot - van Leeuwen - van Weert

charge current

$$N_{\text{GLW}}^\alpha = n u^\alpha, \quad n = 4 \sinh(\xi) n_{(0)}(T) \quad (36)$$

energy-momentum tensor (with a perfect-fluid form)

$$T_{\text{GLW}}^{\mu\nu}(x) = (\varepsilon + P) u^\mu u^\nu - P g^{\mu\nu}, \quad (37)$$

$$\varepsilon = 4 \cosh(\xi) \varepsilon_{(0)}(T), \quad P = 4 \cosh(\xi) P_{(0)}(T), \quad (38)$$

$n_{(0)}(T), \varepsilon_{(0)}(T), P_{(0)}(T)$ — particle density, energy density, and pressure of classical particles at the temperature T

spin tensor

$$S_{\text{GLW}}^{\lambda,\mu\nu} = \frac{\hbar \cosh(\xi)}{m^2} \int dP e^{-\beta \cdot P} p^\lambda \left(m^2 \omega^{\mu\nu} + 2 p^\alpha p^{[\mu} \omega^{\nu]}_\alpha \right) = S_{\text{ph}}^{\lambda,\mu\nu} + S_{\Delta}^{\lambda,\mu\nu}. \quad (39)$$

only $S_{\text{ph}}^{\lambda,\mu\nu}$ was used in WF, B. Friman, A. Jaiswal, E.Speranza, Phys.Rev. C97 (2018) 041901

NLO corrections in \hbar again

LO generates corrections in the NLO

$$\mathcal{P}^{(1)} = -\frac{1}{2m} \partial^\mu \mathcal{A}_{\text{eq},\mu}, \quad (40)$$

$$\mathcal{V}_\mu^{(1)} = -\frac{1}{2m} \partial^\nu \mathcal{S}_{\text{eq},\nu\mu}, \quad (41)$$

$$\mathcal{S}_{\mu\nu}^{(1)} = \frac{1}{2m} (\partial_\mu \mathcal{V}_{\text{eq},\nu} - \partial_\nu \mathcal{V}_{\text{eq},\mu}). \quad (42)$$

IMPORTANT IF the canonical formalism is used

$$T_{\text{GLW}}^{\mu\nu}(x) = \frac{1}{m} \text{tr}_4 \int d^4k k^\mu k^\nu W(x, k) = \frac{1}{m} \int d^4k k^\mu k^\nu \mathcal{F}(x, k). \quad (43)$$

$$T_{\text{can}}^{\mu\nu}(x) = \int d^4k k^\nu \mathcal{V}^\mu(x, k) \quad (44)$$

quantum corrections induce asymmetry $T_{\text{can}}^{\mu\nu}(x) \neq T_{\text{can}}^{\nu\mu}(x)$

From canonical to GLW case 1

Including the components of $\mathcal{V}^\mu(x, k)$ up to the first order in the equilibrium case we obtain

$$T_{\text{can}}^{\mu\nu}(x) = T_{\text{GLW}}^{\mu\nu}(x) + \delta T_{\text{can}}^{\mu\nu}(x) \quad (45)$$

where

$$\delta T_{\text{can}}^{\mu\nu}(x) = -\frac{\hbar}{2m} \int d^4k k^\nu \partial_\lambda S_{\text{eq}}^{\lambda\mu}(x, k) = -\partial_\lambda S_{\text{GLW}}^{\nu,\lambda\mu}(x). \quad (46)$$

The canonical energy-momentum tensor is conserved

$$\partial_\alpha T_{\text{can}}^{\alpha\beta}(x) = 0. \quad (47)$$

$$\begin{aligned} S_{\text{can}}^{\lambda,\mu\nu} &= \frac{\hbar \sinh(\zeta) \cosh(\xi)}{\zeta} \int dP e^{-\beta P} (\omega^{\mu\nu} p^\lambda + \omega^{\nu\lambda} p^\mu + \omega^{\lambda\mu} p^\nu) \\ &\equiv \frac{\hbar w}{4\zeta} (u^\lambda \omega^{\mu\nu} + u^\mu \omega^{\nu\lambda} + u^\nu \omega^{\lambda\mu}) = S_{\text{GLW}}^{\lambda,\mu\nu} + S_{\text{GLW}}^{\mu,\nu\lambda} + S_{\text{GLW}}^{\nu,\lambda\mu}. \end{aligned} \quad (48)$$

The canonical spin tensor is not conserved

$$\partial_\lambda S_{\text{can}}^{\lambda,\mu\nu}(x) = T_{\text{can}}^{\nu\mu} - T_{\text{can}}^{\mu\nu} = -\partial_\lambda S_{\text{GLW}}^{\mu,\lambda\nu}(x) + \partial_\lambda S_{\text{GLW}}^{\nu,\lambda\mu}(x). \quad (49)$$

if we introduce the tensor $\Phi_{\text{can}}^{\lambda,\mu\nu}$ defined by the relation

$$\Phi_{\text{can}}^{\lambda,\mu\nu} \equiv S_{\text{GLW}}^{\mu,\lambda\nu} - S_{\text{GLW}}^{\nu,\lambda\mu}, \quad (50)$$

we can write

$$S_{\text{can}}^{\lambda,\mu\nu} = S_{\text{GLW}}^{\lambda,\mu\nu} - \Phi_{\text{can}}^{\lambda,\mu\nu} \quad (51)$$

and

$$T_{\text{can}}^{\mu\nu} = T_{\text{GLW}}^{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi_{\text{can}}^{\lambda,\mu\nu} + \Phi_{\text{can}}^{\mu,\nu\lambda} + \Phi_{\text{can}}^{\nu,\mu\lambda}). \quad (52)$$

The canonical and GLW frameworks are connected by a pseudo-gauge transformation. Similarly to Belinfante, it leads to a symmetric energy-momentum tensor, however, does not eliminate the spin degrees of freedom.

The FFJS approach employs the energy-momentum tensor in the GLW version and the phenomenological spin tensor of the form

$$S_{\text{ph}}^{\lambda,\mu\nu} = \frac{\hbar w}{4\zeta} U^\lambda \omega^{\mu\nu}. \quad (53)$$

It can be related to the canonical spin tensor through the following relation

$$S_{\text{can}}^{\lambda,\mu\nu} = S_{\text{ph}}^{\lambda,\mu\nu} + S_{\text{ph}}^{\mu,\nu\lambda} + S_{\text{ph}}^{\nu,\lambda\mu} \quad (54)$$

This we can interpret as the pseudo-gauge transformation

$$S_{\text{ph}}^{\lambda,\mu\nu} = S_{\text{can}}^{\lambda,\mu\nu} - \Phi_{\text{ph}}^{\lambda,\mu\nu}, \quad \Phi_{\text{ph}}^{\lambda,\mu\nu} = S_{\text{ph}}^{\mu,\nu\lambda} + S_{\text{ph}}^{\nu,\lambda\mu}. \quad (55)$$

The correct energy-momentum tensor is

$$T_{\text{ph}}^{\mu\nu} = T_{\text{can}}^{\mu\nu} + \frac{1}{2} \partial_\lambda (\Phi_{\text{ph}}^{\lambda,\mu\nu} + \Phi_{\text{ph}}^{\mu,\nu\lambda} + \Phi_{\text{ph}}^{\nu,\lambda\mu}) = T_{\text{GLW}}^{\mu\nu} - \partial_\lambda S_{\Delta}^{\nu,\lambda\mu}. \quad (56)$$

and the conservation of the angular momentum gives

$$\partial_\lambda S_{\text{ph}}^{\lambda,\mu\nu} = T_{\text{ph}}^{\nu\mu} - T_{\text{ph}}^{\mu\nu} = -\partial_\lambda S_{\Delta}^{\lambda,\mu\nu}. \quad (57)$$

This is equivalent to the conservation of the GLW spin tensor

we introduce the phase-space density $\Delta\Pi_\mu$ of the **Pauli-Lubański vector**

$$E_p \frac{d\Delta\Pi_\mu(x, p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} \Delta\Sigma_\lambda(x) E_p \frac{dJ^{\lambda,\nu\alpha}(x, p)}{d^3p} \frac{p^\beta}{m}. \quad (58)$$

only the spin-part contributes here, the results are the same for the canonical and GLW versions

$$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} E_p \frac{dS^{\lambda,\nu\alpha}(x, p)}{d^3p} = \frac{\hbar \cosh(\xi)}{(2\pi)^3} e^{-p\cdot\beta} p^\lambda \tilde{\omega}_{\mu\beta}. \quad (59)$$

dividing by the total density of particles and antiparticles, we find

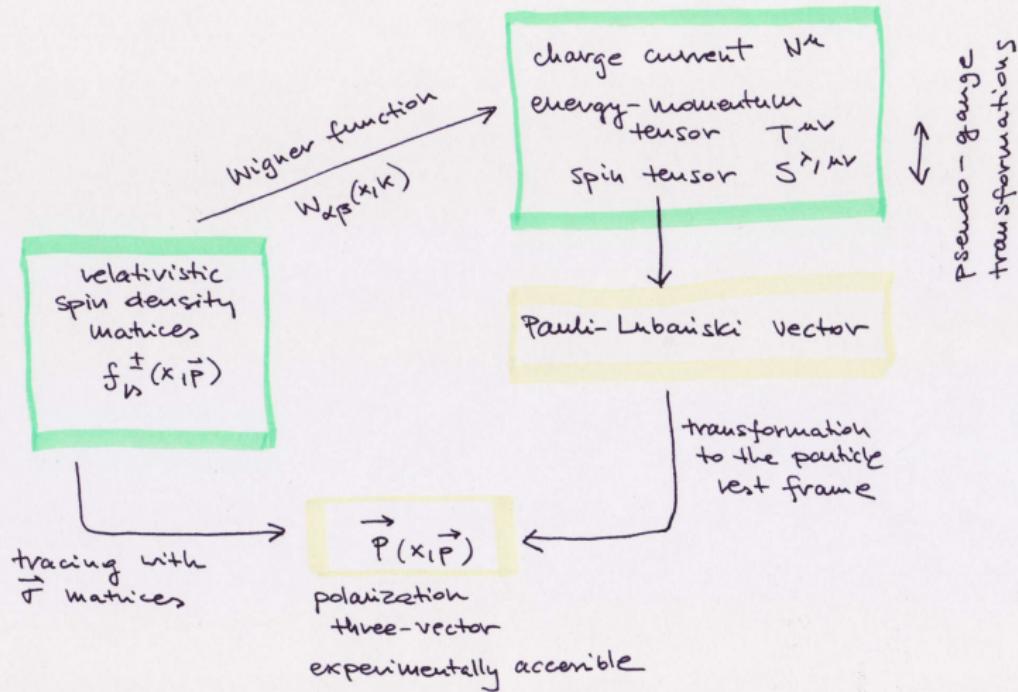
$$\pi_\mu(x, p) \equiv \frac{\Delta\Pi_\mu(x, p)}{\Delta N(x, p)} = -\frac{\hbar}{4m} \tilde{\omega}_{\mu\beta} p^\beta. \quad (60)$$

in PRF

$$\pi_*^0 = 0, \quad \pi_* = -\frac{\hbar}{4} \mathbf{P}. \quad (61)$$

This is an important result showing that the space part of the PL vector in PRF agrees with the mean polarization three-vector

Summary 1



Classical treatment of spin 1

internal angular momentum tensor $s^{\alpha\beta}$

M. Matthison, Neue mechanik materieller systemes, Acta Phys. Polon. 6 (1937) 163.

$$s^{\alpha\beta} = \frac{1}{m} \epsilon^{\alpha\beta\gamma\delta} p_\gamma s_\delta. \quad (62)$$

$$\mathbf{s} \cdot \mathbf{p} = 0, \quad s^\alpha = \frac{1}{2m} \epsilon^{\alpha\beta\gamma\delta} p_\beta s_\gamma \quad (63)$$

A straightforward generalization of the phase-space distribution function $f(\mathbf{x}, \mathbf{p})$ is a spin dependent distribution $f(\mathbf{x}, \mathbf{p}, \mathbf{s})$

$$\int dS \dots = \frac{m}{\pi \mathfrak{s}} \int d^4s \delta(s \cdot s + \mathfrak{s}^2) \delta(p \cdot s) \dots \quad (64)$$

$$\mathfrak{s}^2 = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \frac{3}{4} \quad (65)$$

$$\int dS = \frac{m}{\pi \mathfrak{s}} \int d^4s \delta(s \cdot s + \mathfrak{s}^2) \delta(p \cdot s) = 2 \quad (66)$$

equilibrium distribution functions for particles and antiparticles in the form

$$f_{\text{eq}}^{\pm}(x, p, s) = \exp\left(-p \cdot \beta(x) \pm \xi(x) + \frac{1}{2}\omega_{\alpha\beta}(x)s^{\alpha\beta}\right). \quad (67)$$

conserved "currents"

$$N_{\text{eq}}^{\mu} = \int dP \int dS p^{\mu} [f_{\text{eq}}^{+}(x, p, s) - f_{\text{eq}}^{-}(x, p, s)], \quad (68)$$

$$T_{\text{eq}}^{\mu\nu} = \int dP \int dS p^{\mu} p^{\nu} [f_{\text{eq}}^{+}(x, p, s) + f_{\text{eq}}^{-}(x, p, s)] \quad (69)$$

$$S_{\text{eq}}^{\lambda,\mu\nu} = \int dP \int dS p^{\lambda} s^{\mu\nu} [f_{\text{eq}}^{+}(x, p, s) + f_{\text{eq}}^{-}(x, p, s)] \quad (70)$$

For $|\omega_{\mu\nu}| < 1$ we obtain the formalism that agrees with that based on the quantum description of spin (in the GLW version)

PL vector can be expressed by the simple expression

$$\pi_\mu = -\mathbf{g} \frac{\tilde{\omega}_{\mu\beta}}{P} \frac{p^\beta}{m} L(P\mathbf{g}), \quad (71)$$

where $L(x)$ is the Langevin function defined by the formula

$$L(x) = \coth(x) - \frac{1}{x}. \quad (72)$$

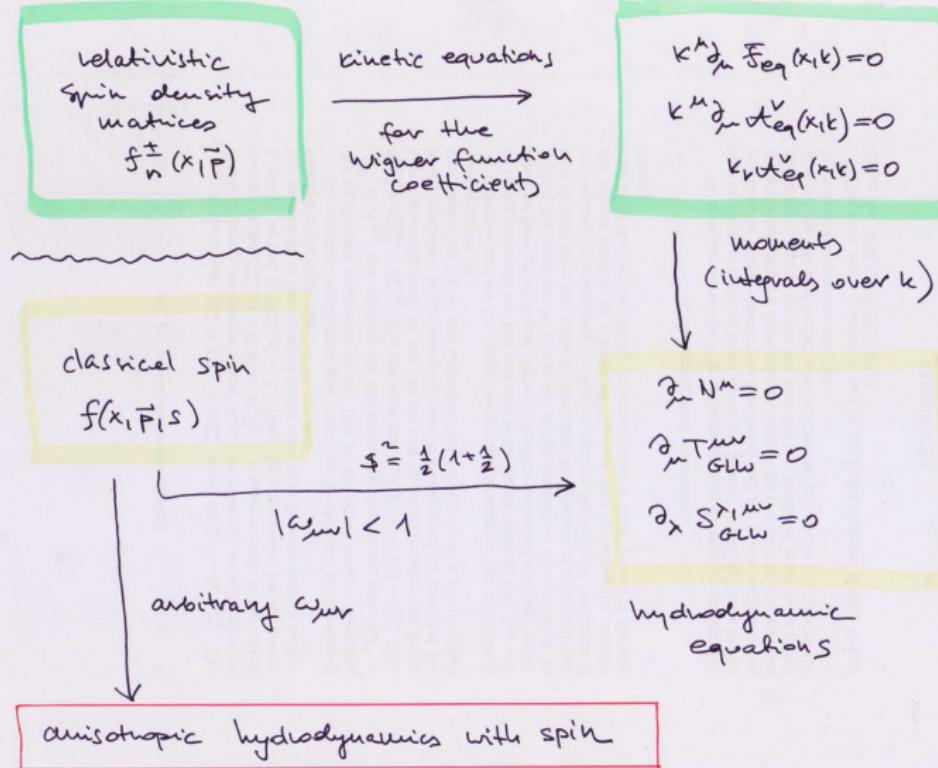
in PRF the direction of the PL vector agrees with that of the polarization vector \mathbf{P} . For small and large P we obtain two important results:

$$\pi_* = -\mathbf{g} \frac{\mathbf{P}}{P}, \quad |\pi_*| = \mathbf{g} = \sqrt{\frac{3}{4}}, \quad \text{if } P \gg 1 \quad (73)$$

and

$$\pi_* = -\mathbf{g}^2 \frac{\mathbf{P}}{3}, \quad |\pi_*| = \mathbf{g}^2 \frac{P}{3} = \frac{P}{4}, \quad \text{if } P \ll 1. \quad (74)$$

Summary 2



1. The arguments collected in our works suggest using **the de Groot - van Leeuwen - van Weert (GLW) forms of the energy-momentum and spin tensors**, together with their conservation laws, as the building blocks for construction of hydrodynamics with spin.
2. **The GLW framework can be connected with the canonical one (obtained with the help of the Noether theorem from the underlying Lagrangian) through a pseudo-gauge transformation** that has been explicitly constructed. Both, the GLW and canonical frameworks include spin degrees of freedom, hence can be used at the same footing to describe spin polarization phenomena.
3. **The pseudo-gauge transformation from the canonical to the Belinfante forms neglects the spin degrees of freedom and leads to a formalism that is not satisfactory for description of the polarization phenomena** — the total angular momentum in the Belinfante approach has the form of the orbital angular momentum which can be always set equal to zero by a Lorentz transformation.
4. **Using the classical concept of spin one can formulate a consistent framework of hydrodynamics with spin**, which for small values of the polarization agrees with the approach using relativistic spin-density matrices. The classical-spin approach is free from the problems connected with normalization of the polarization three-vector and indicates that the hydrodynamic system becomes anisotropic if the spin densities are large. The classical approach allows also for the explicit definition of a conserved entropy current.

more in arXiv:1811.04409

Quark Matter 2021 in Kraków, Oct. 4-9, 2021, ICE Congress Center