# Calculation for Non-global Logarithms with Neural Networks

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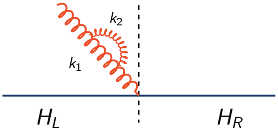
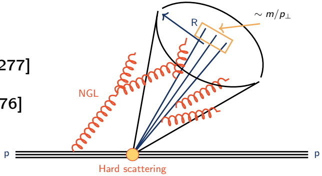
International Workshop on Boosted Object Phenomenology
July 18, 2020

# Introduction: history of NGLs

- Non-global observable: miscancellation of real and virtual emissions of gluons, for observable with boundaries in phase space.




Cancellation between the real production and virtual loop contribution for gluon 2




- Dasgupta-Salam: Monte-Carlo simulation [hep-ph/0104277]
- Banfi-Marchesini-Smye (BMS) equation: [hep-ph/0206076]
- More recent progress:
  - Fixed-order up to 5-loop [Schwartz and Zhu, 1403.4949]
  - Dressed-gluon expansion [Larkoski, Moult, Neill 1501.04596]
  - Renormalization-group [Balsigera, Becher and Shao 1901.09038]

# Outline

1. Symmetries of the BMS equation

2. Calculation for NGL expansions

3. Preliminary NN results and Outlook

# Symmetries of the BMS: stereographic projection

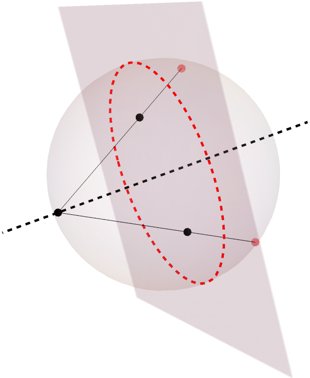
Recall the BMS equation under PSL(2, R)
[Schwartz and Zhu, 1403.4949], [Hatta and Ueda, 0909.0056]

$$\partial_L g_{ab}(L) = \int d\Omega_j W_{ab}^j$$
$$\times \left\{ \left[ \frac{1 + \langle ab \rangle}{(1 + \langle aj \rangle)(1 + \langle jb \rangle)} \right]^{L/2} g_{aj}(L) g_{jb}(L) - g_{ab}(L) \right\}$$

with

$$g_{ab}(L) = g(\langle ab \rangle, L) = g\left( \frac{1 - cos\theta_{ab}}{2cos\theta_i cos\theta_j}, L \right)$$

Computation time: $n_L n_j^3 n_\phi^2$



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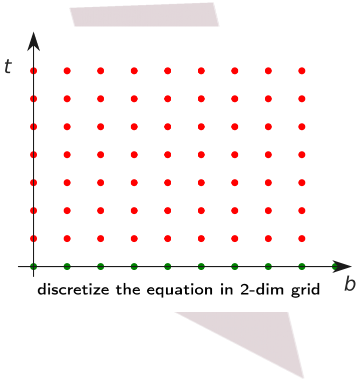


# Outline

# Linearized BMS equation

Re-organize the BMS equation according to [Banfi, Marchesini and Smye, hep-ph/0206076]

$$\partial g_{ab}(L) = \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} g_{ab}(L) \left[U_{abj}(L) - 1\right] + \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U_{abj} \left[g_{aj}(L) g_{jb}(L) - g_{ab}(L)\right]$$

Then expand it order by order up to 4-loop

$$\partial g^{(2)}_{ab}(L) = \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U^{(1)}_{abj}(L)$$

$$\partial g^{(3)}_{ab}(L) = \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U^{(2)}_{abj}(L) + \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} \left[g^{(2)}_{aj}(L) + g^{(2)}_{jb}(L) - g^{(2)}_{ab}(L)\right]$$

$$\begin{aligned}\partial g^{(4)}_{ab}(L) = & \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U^{(3)}_{abj}(L) + \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U^{(1)}_{abj}(L)\, g^{(2)}_{ab}(L) \\ & + \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} U^{(1)}_{abj}(L) \left[g^{(2)}_{aj}(L) + g^{(2)}_{jb}(L) - g^{(2)}_{ab}(L)\right] \\ & + \int_L \frac{d\Omega_j}{4\pi} W^j_{ab} \left[g^{(3)}_{aj}(L) + g^{(3)}_{jb}(L) - g^{(3)}_{ab}(L)\right]\end{aligned}$$

Relation with BFKL evolution and dressed gluon approximation

# Check with the fixed-order expansion

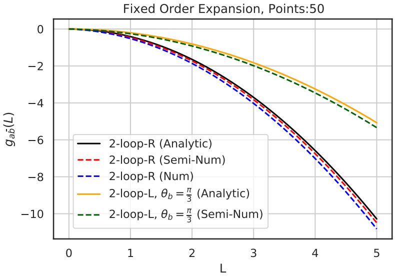
- Fixed-order result for $g_{n\bar{n}}$: [Schwartz and Zhu, 1403.4949]

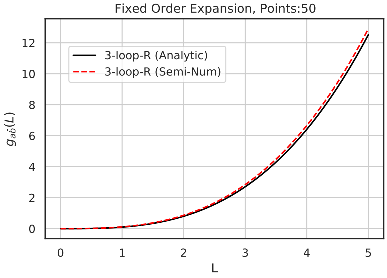
$$g_{n\bar{n}}(L) = 1 - \frac{\pi^2}{24}L^2 + \frac{\zeta(3)}{12}L^3 + \frac{\pi^4}{34560}L^4 + \left(\frac{17\zeta(5)}{480} - \frac{\pi^2\zeta(3)}{360}\right)L^5 + \cdots$$

$$L = \frac{\alpha_s}{\pi}N_c log\frac{m_L}{m_R}$$

- a and b in left hemisphere:

$$\frac{1}{L^2}g_{ab}^{(2)}(x) = \frac{1}{4}logxlog(1+x) - \frac{1}{8}log^2(1+x) + \frac{1}{4}Li_2(-x)$$



# Partially resummed result of BMS

For the first term

$$\partial g_{ab}^{(2R)}(L) = \frac{1}{4\pi}\int_0^1 dcos\theta_j \int_0^{2\pi} d\phi_j W_{ab}^{j} g_{ab}^{(2R)}(L)\left[U_{abj}(L) - 1\right]$$

With the dipole radiator and resummation factor, for only a is left

$$W_{ab}^{j} = \frac{(ab)}{(aj)(jb)}$$

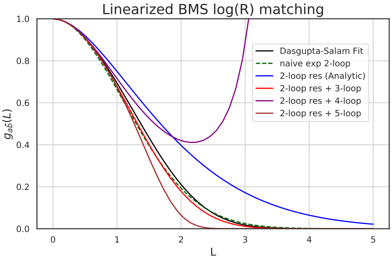
$$U_{abj}(L) = 2^{L/2} cos^{L}\theta_j \left\{\frac{(an)}{[aj](jn)}\right\}^{L/2}$$

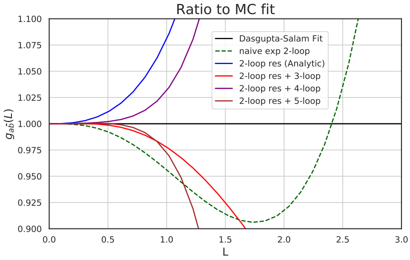
The analytical result is

$$\begin{aligned} g_{n\bar{n}}^{(2R)}(L) &= exp^{-\frac{1}{2}(\gamma_E L + log\Gamma(1+L))} \\ &= 1 - \frac{\pi^2}{24}L^2 + \frac{\zeta(3)}{6}L^3 - \frac{\pi^4}{1920}L^4 + O\left(L^5\right) \end{aligned}$$

For 3-loop result, from the second part

$$\int_L \frac{d\Omega_j}{4\pi} W_{n\bar{n}}^{j}\left[g_{nj}^{(2)}(L) + g_{j\bar{n}}^{(2)}(L) - g_{n\bar{n}}^{(2)}(L)\right] = \int_L \frac{d\Omega_j}{4\pi} W_{\bar{n}n}^{j} g_{\bar{n}j}^{(2)}(L) = -\frac{\zeta(3)}{4}L^2$$

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$$\int_L \frac{d\Omega_j}{4\pi} W_{n\bar{n}}^j \left[g_{nj}^{(2)}(L) + g_{j\bar{n}}^{(2)}(L) - g_{n\bar{n}}^{(2)}(L)\right] = \int_L \frac{d\Omega_j}{4\pi} W_{\bar{n}n}^j g_{\bar{n}j}^{(2)}(L) = -\frac{\zeta(3)}{4}L^2$$

# Comparison with dressed-gluon

Dressed gluon from BMS expansion

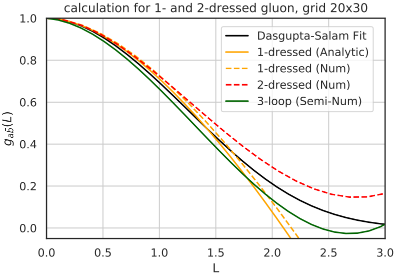
$$g_{ab} = 1 + g_{ab}^{(1)} + g_{ab}^{(2)} + \dots$$

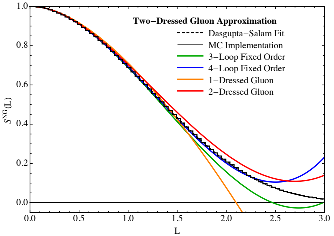
For one and two dressed gluon:

$$\partial_L g_{ab}^{(1)}(L) = \int_L \frac{d\Omega_j}{4\pi} W_{ab}^j \left(U_{abj}(L) - 1\right)$$

$$\partial_L g_{ab}^{(2)}(L) = \int_L \frac{d\Omega_j}{4\pi} W_{ab}^j U_{abj}(L) \left[g_{aj}^{(1)}(L) + g_{jb}^{(1)}(L)\right] - \int_L \frac{d\Omega_j}{4\pi} W_{ab}^j g_{ab}^{(1)}(L)$$

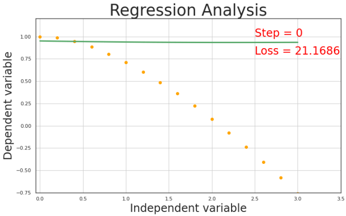
Compare with the original article: solve the 1-dressed gluon twice, and insert to the formula of 2-dressed gluon



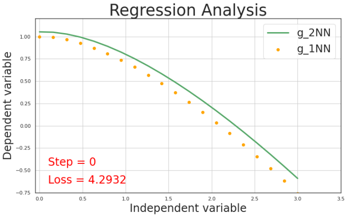




[Larkoski, Moult, and Neill, 1501.04596]

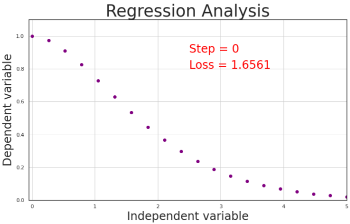
# Outline




Calculation for 1-dressed gluon



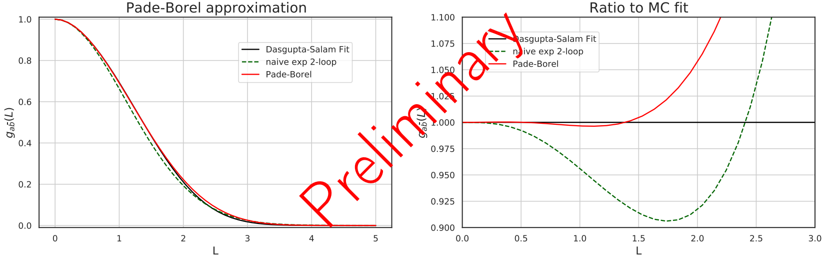

Calculation for 2-dressed gluon




Calculation for partial resummation

# Padé approximation and outlook

Finally, we want recover the large order behaviour:




Next Step:

- fixed-order result up to 7-loop (arbirary hemisphere) and 8-loop (back to back)
- full solution of BMS equation via NN
- compare with approximate resummation via dressed gluon

Future:

- apply to a handful of integro-differential evolution equations/ renormalization-group running

**Thank you for your attention**

# Extra Slides

# Perturbative expansion of the BMS equation

Leading NGLs arise from the strongly ordered gluon emissions

$$E_1 \gg E_2 \gg E_3 \gg \cdots \gg E_n$$

The multi-gluon emission amplitude is simplified, and at large $N_c$ limit:

$$\begin{aligned} \left|\mathcal{M}_{ab}^{1\cdots m}\right|^2 &= \left|\langle p_1 \cdots p_m \left| Y_a^\dagger Y_b \right| 0\rangle\right|^2 \\ &= N_c^m g^{2m} \sum_{\text{perms of } 1\cdots m} \frac{(p_a \cdot p_b)}{(p_a \cdot p_1)(p_1 \cdot p_2)\cdots(p_m \cdot p_b)} \end{aligned}$$

And the expansion of BMS at 3-loop reads:

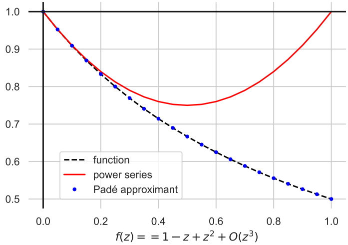
$$\begin{aligned} \partial g_{ab}^{(3)}(L) = \frac{L^2}{2} \int_\Omega 1_L 2_R 3_R W_{ab}^1 \left(W_{ab}^2 - W_{a1}^2 - W_{1b}^2\right)\left(W_{ab}^3 - W_{a1}^3 - W_{1b}^3\right) \\ + \int_\Omega 1_L W_{ab}^1 \left[g_{a1}^{(2)}(L) + g_{1b}^{(2)}(L) - g_{ab}^{(2)}(L)\right] \end{aligned}$$

# Padé approximation

The Padé approximant $Z_{[M/N]}(t)$ of order $[M/N]$ is defined by the condition:

$$P_M(t) - Q_N(t)\, Z_{[M/N]}(t) = O\left(t^{M+N+1}\right)$$

Typically the symmetric $[N/N]$ Padé approximant features the fastest approximation for increasing N although this depends on the function, for a example, $f(z) = \frac{1}{1+z}$, is shown below




Plot of truncated power series expansion and its Padé approximants [1/1]

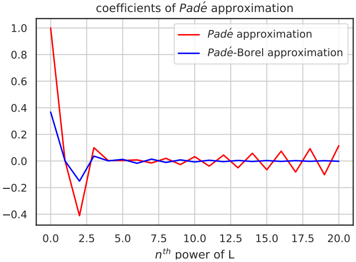
# Padé-Borel approximation

A common technique, if the large order asymptotics of the series coefficients are known, is Borel resummation, which the Borel sum is defined by

$$B(g) = \sum_{n=0}^{\infty} B_n g^n, \; with \; B_n = \frac{f_n}{n!}$$

In practice, the coefficients $B_n$ might not be known analytically, therefore, instead of explicitly computing the Borel sum, we simply approximate it using Padé approximation

$$f(t) = \int_0^{\infty} dg e^{-g} B^{[m/n]}(gt)$$




coefficients of Padé and Padé-Borel approximation for $g_{a\bar{b}}(L)$