Hard Scattering

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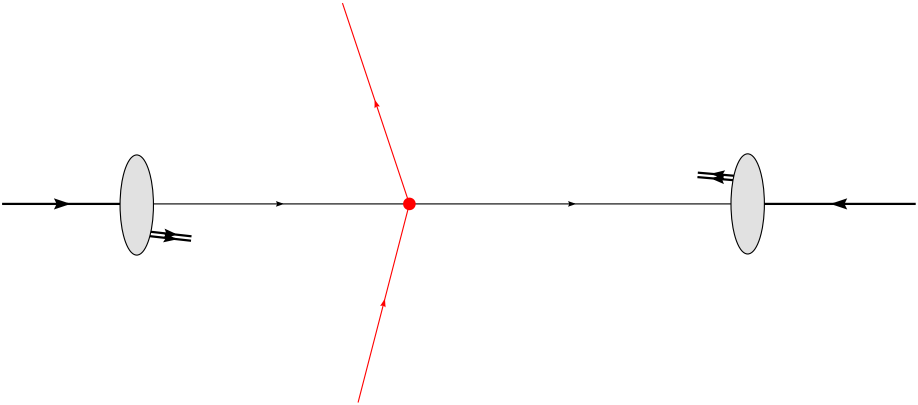
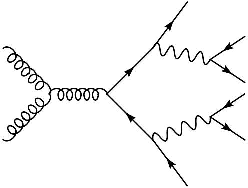


Matrix elements

- Perturbation theory/Feynman diagrams give us (fairly accurate) final states for a few number of legs ($O(1)$).



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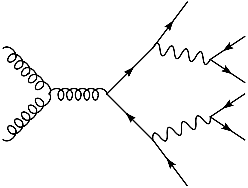

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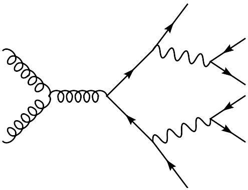

- OK for very inclusive observables.
- Starting point for further simulation.
- Want exclusive final state at the LHC ($O(100)$).
- Want arbitrary cuts.
- $\rightarrow$ use Monte Carlo methods.

Matrix elements

Where do we get (LO) $|M|^2$ from?

- Most/important simple processes (SM and BSM) are 'built in'.
- Calculate yourself (≤ 3 particles in final state).
- Matrix element generators:
  - MadGraph/MadEvent.
  - Comix/AMEGIC (part of Sherpa).
  - HELAC/PHEGAS.
  - Whizard.
  - CalcHEP/CompHEP.

  generate code or event files that can be further processed.
- $\rightarrow$ FeynRules interface to ME generators.

Also NLO mostly automatically available.
See "Matching and Merging".

Cross section formula

From Matrix element, we calculate

$$\sigma = \int f_i(x_1,\mu^2) f_j(x_2,\mu^2) \frac{1}{F} \overline{\sum} |M|^2 \qquad \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n \ ,$$

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now,

$$\frac{1}{F} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}\Phi_n = J(\vec{x}) \prod_{i=1}^{3n-2} \mathrm{d}x_i \qquad \left(\mathrm{d}\Phi_n = (2\pi)^4 \delta^{(4)}(\ldots) \prod_{i=1}^{n} \frac{\mathrm{d}^3 \vec{p}}{(2\pi)^3 2E_i} \right)$$

such that

$$\sigma = \int g(\vec{x}) \, \mathrm{d}^{3n-2} \vec{x} \; , \qquad \left(g(\vec{x}) = J(\vec{x}) f_i f_j \overline{\sum} |M|^2 \Theta(\text{cuts}) \right)$$

$$= \frac{1}{N} \sum_{i=1}^{N} \frac{g(\vec{x}_i)}{p(\vec{x}_i)} = \frac{1}{N} \sum_{i=1}^{N} w_i \; .$$

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We generate events $\vec{x}_i$ with weights w_i.

Mini event generator

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where $w_{\max}$ has to be chosen sensibly.
$\to$ reweighting, when $\max(w_i) = \bar{w}_{\max} > w_{\max}$, as

$$P_i = \frac{w_i}{\bar{w}_{\max}} = \frac{w_i}{w_{\max}} \cdot \frac{w_{\max}}{\bar{w}_{\max}} \ ,$$

i.e. reject events with probability $(w_{\max}/\bar{w}_{\max})$ afterwards.

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Matrix elements

Some comments:

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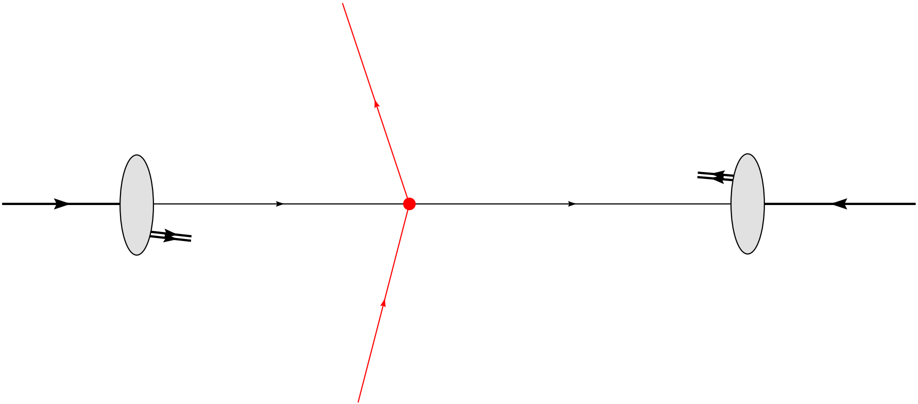
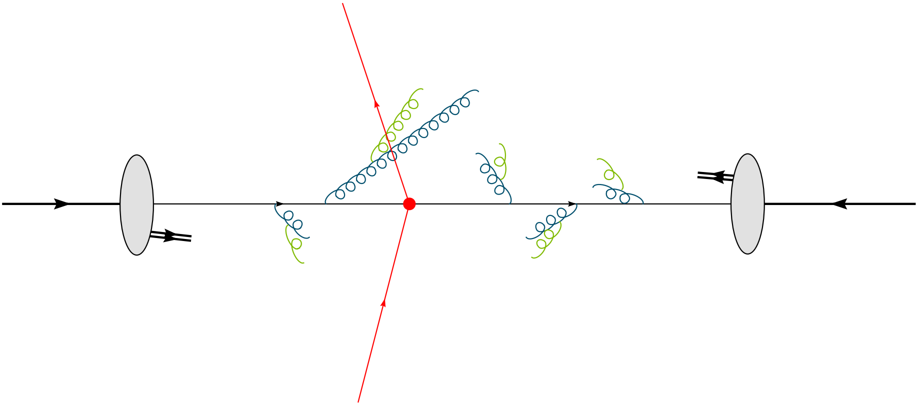
- Use common Monte Carlo techniques to generate events efficiently. Goal: small variance in w_i distribution!
- Efficient generation closely tied to knowledge of $f(\vec{x}_i)$, *i.e.* the matrix element's propagator structure.
  $\rightarrow$ build phase space generator already while generating ME's automatically.

Parton Showers

Hard matrix element



Hard matrix element → parton showers



Parton showers

Quarks and gluons in final state, pointlike.

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- Know short distance (short time) fluctuations from matrix element/Feynman diagrams: $Q \sim$ few GeV to O(TeV).

- Measure hadronic final states, long distance effects, $Q_0 \sim 1\,\text{GeV}$.

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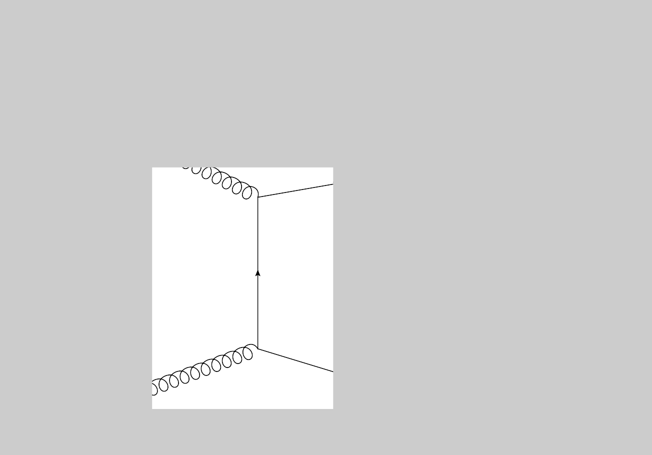
Generated from emissions *ordered* in Q.
Soft and/or collinear emissions.

ME approximated by parton cascade

Evolution in scale, typically $Q \sim 1\,\text{TeV}$ down to $Q \sim 1\,\text{GeV}$.



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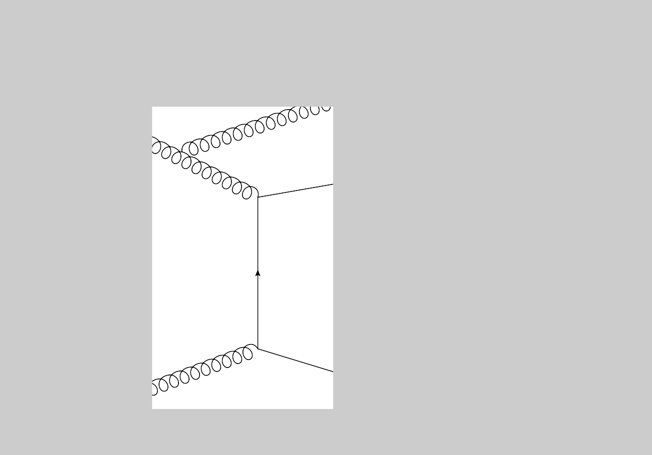

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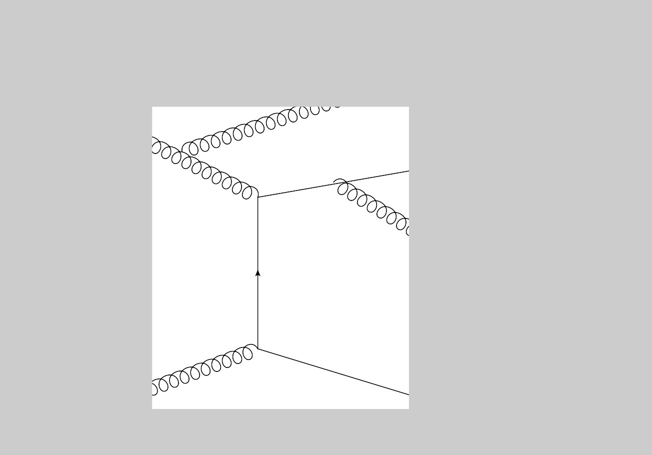

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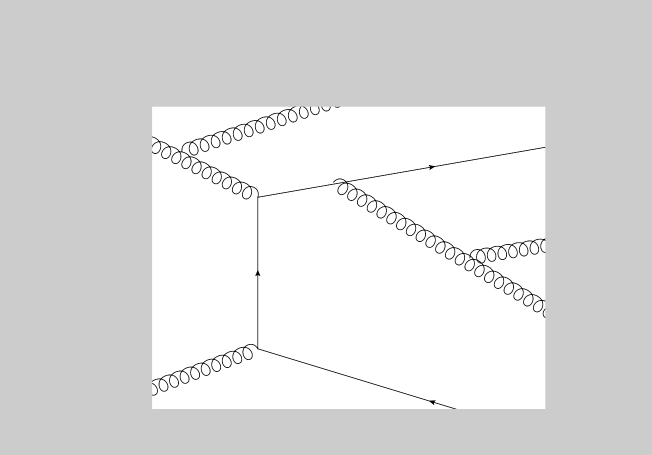

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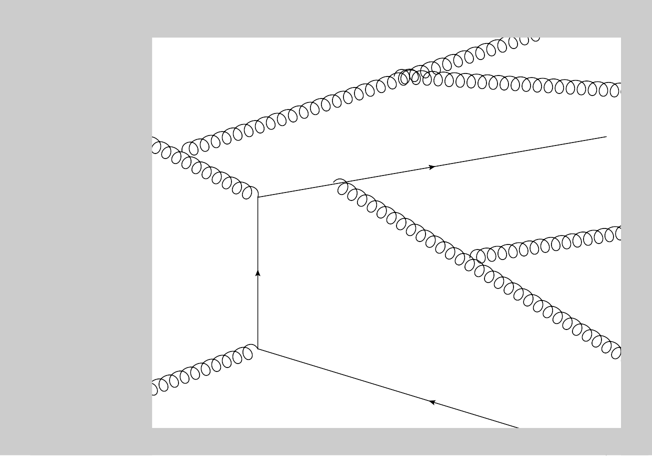


e^+e^- annihilation

Good starting point: $e^+e^- \to q\bar{q}g$:

Final state momenta in one plane (orientation usually averaged).
Write momenta in terms of

$$x_i = \frac{2p_i \cdot q}{Q^2} \quad (i = 1,2,3) \ ,$$

$$0 \le x_i \le 1 \ , x_1 + x_2 + x_3 = 2 \ ,$$

$$q = (Q,0,0,0) \ ,$$

$$Q \equiv E_{cm} \ .$$

$(x_1, x_2) = (x_q, x_{\bar{q}})$ –plane:

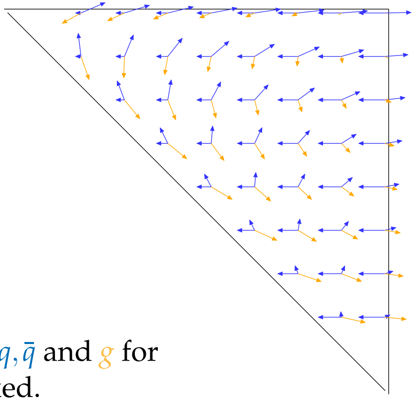


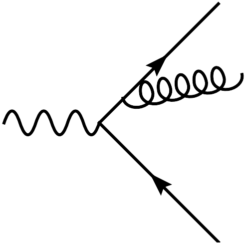
Fig: momentum configuration of $q, \bar{q}$ and g for given point (x_1, x_2), $\bar{q}$ direction fixed.

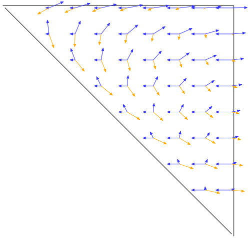
e^+e^- annihilation

Differential cross section:

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \frac{x_1 + x_2}{(1-x_1)(1-x_2)}$$

Collinear singularities: $x_1 \to 1$ or $x_2 \to 1$. Soft singularity: $x_1, x_2 \to 1$.





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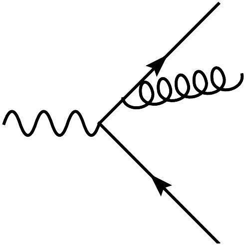

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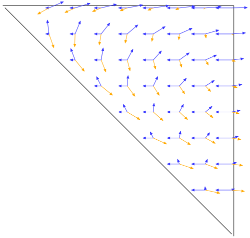

Rewrite in terms of x_3 and $\theta = \angle(q, g)$:

$$\frac{d\sigma}{d\cos\theta \, dx_3} = \sigma_0 \frac{C_F \alpha_S}{2\pi} \left[\frac{2}{\sin^2\theta} \frac{1 + (1-x_3)^2}{x_3} - x_3 \right]$$

Singular as $\theta \to 0$ and $x_3 \to 0$.

e^+e^- annihilation

Can separate into two jets as

$$\frac{2\mathrm{d}\cos\theta}{\sin^2\theta} = \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\theta}{1+\cos\theta}$$
$$= \frac{\mathrm{d}\cos\theta}{1-\cos\theta} + \frac{\mathrm{d}\cos\bar{\theta}}{1-\cos\bar{\theta}}$$
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So, we rewrite $\mathrm{d}\sigma$ in collinear limit as

$$\mathrm{d}\sigma = \sigma_0 \sum_{\text{jets}} \frac{\mathrm{d}\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} C_F \frac{1+(1-z)^2}{z^2} \mathrm{d}z$$

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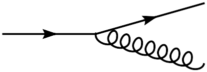
$$= \sigma_0 \sum_{\text{jets}} \frac{\mathrm{d}\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} P(z) \mathrm{d}z$$

with DGLAP splitting function $P(z)$.

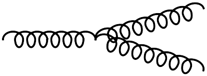
Collinear limit

Universal DGLAP splitting kernels for collinear limit:

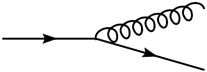
$$d\sigma = \sigma_0 \sum_{\text{jets}} \frac{d\theta^2}{\theta^2} \frac{\alpha_S}{2\pi} P(z) dz$$



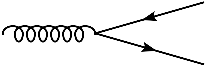
$$P_{q\to qg}(z) = C_F \frac{1+z^2}{1-z}$$



$$P_{g\to gg}(z) = C_A \frac{(1-z(1-z))^2}{z(1-z)}$$



$$P_{q\to gq}(z) = C_F \frac{1+(1-z)^2}{z}$$



$$P_{g\to qq}(z) = T_R(1-2z(1-z))$$

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Note: Other variables may equally well characterize the collinear limit:

$$\frac{\mathrm{d}\theta^2}{\theta^2} \sim \frac{\mathrm{d}Q^2}{Q^2} \sim \frac{\mathrm{d}p_\perp^2}{p_\perp^2} \sim \frac{\mathrm{d}\tilde{q}^2}{\tilde{q}^2} \sim \frac{\mathrm{d}t}{t}$$

whenever $Q^2, p_\perp^2, t \to 0$ means "collinear".

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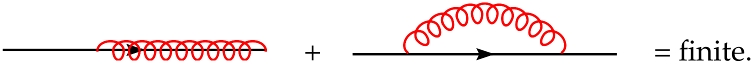
- θ: HERWIG
- Q^2: PYTHIA ≤ 6.3, SHERPA.
- $p_\perp$: PYTHIA ≥ 6.4, ARIADNE, Catani–Seymour showers.
- $\tilde{q}$: Herwig++.

Resolution

Need to introduce resolution t_0, e.g. a cutoff in $p_\perp$. Prevent us from the singularity at $\theta \to 0$.

Emissions below t_0 are unresolvable.

Finite result due to virtual corrections:



unresolvable + virtual emissions are included in Sudakov form factor via unitarity (see below!).

Towards multiple emissions

Starting point: factorisation in collinear limit, single emission.

$$\sigma_{2+1}(t_0) = \sigma_2(t_0) \int_{t_0}^{t} \frac{\mathrm{d}t'}{t'} \int_{z_-}^{z_+} \mathrm{d}z \, \frac{\alpha_S}{2\pi} \hat{P}(z) \; = \sigma_2(t_0) \int_{t_0}^{t} \mathrm{d}t \, W(t) \; .$$

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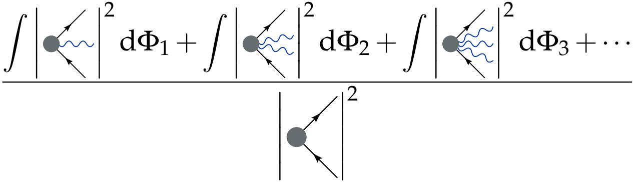
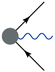
Simple example:
Multiple photon emissions, strongly ordered in t.
We want

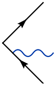
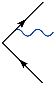
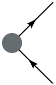
$$W_{\mathrm{sum}} = \sum_{n=1} W_{2+n} =$$



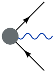
for any number of emissions.

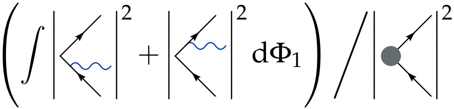
Towards multiple emissions

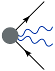
$(n = 1)$ 

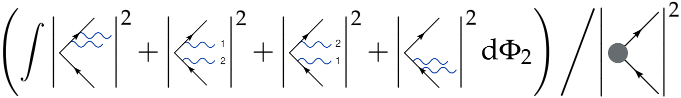
$W_{2+1} = \Big(\int \Big|$  $\Big|^2 + \Big|$  $\Big|^2 \, \mathrm{d}\Phi_1 \Big) \Big/ \Big|$  $\Big|^2 = \frac{2}{1!} \int_{t_0}^{t} \mathrm{d}t \, W(t) \, .$

Towards multiple emissions

$(n=1)$ 

$$W_{2+1} =$$  $$= \frac{2}{1!}\int_{t_0}^{t} \mathrm{d}t\, W(t)\ .$$

$(n=2)$ 

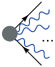
$$W_{2+2} =$$ 

$$= 2^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\, W(t')W(t'') = \frac{2^2}{2!}\left(\int_{t_0}^{t} \mathrm{d}t\, W(t)\right)^2\ .$$

We used

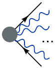
$$\int_{t_0}^{t} \mathrm{d}t_1 \ldots \int_{t_0}^{t_{n-1}} \mathrm{d}t_n\ W(t_1)\ldots W(t_n) = \frac{1}{n!}\left(\int_{t_0}^{t} \mathrm{d}t\, W(t)\right)^n\ .$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

$$W_{2+n} = \frac{2^n}{n!}\left(\int_{t_0}^{t} \mathrm{d}t\, W(t)\right)^n$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

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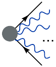
So, in total we get

$$\sigma_{>2}(t_0) = \sigma_2(t_0) \sum_{k=1}^{\infty} \frac{2^k}{k!} \left(\int_{t_0}^{t} \mathrm{d}t \, W(t) \right)^k = \sigma_2(t_0) \left(\mathrm{e}^{2 \int_{t_0}^{t} \mathrm{d}t \, W(t)} - 1 \right)$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

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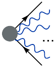
$$= \sigma_2(t_0) \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right)$$

Sudakov Form Factor

$$\Delta(t_0, t) = \exp\left[- \int_{t_0}^{t} \mathrm{d}t\, W(t) \right]$$

Towards multiple emissions

Easily generalized to n emissions  by induction. *i.e.*

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$$= \sigma_2(t_0) \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right)$$

Sudakov Form Factor in QCD

$$\Delta(t_0, t) = \exp\left[- \int_{t_0}^{t} \mathrm{d}t\, W(t) \right] = \exp\left[- \int_{t_0}^{t} \frac{\mathrm{d}t}{t} \int_{z_-}^{z_+} \frac{\alpha_S(z,t)}{2\pi} \hat{P}(z,t) \mathrm{d}z \right]$$

Sudakov form factor

Note that

$$\sigma_{\text{all}} = \sigma_2 + \sigma_{>2} = \sigma_2 + \sigma_2 \left(\frac{1}{\Delta^2(t_0, t)} - 1 \right) ,$$
$$\Rightarrow \Delta^2(t_0, t) = \frac{\sigma_2}{\sigma_{\text{all}}} .$$

$$\text{Two jet rate} = \Delta^2 = P^2(\text{No emission in the range } t \to t_0) .$$

$$\text{Sudakov form factor} = \text{No emission probability} .$$

Often $\Delta(t_0, t) \equiv \Delta(t)$.

- Hard scale t, typically CM energy or $p_\perp$ of hard process.
- Resolution t_0, two partons are resolved as two entities if inv mass or relative $p_\perp$ above t_0.
- P^2 (not P), as we have two legs that evolve independently.

Sudakov form factor from Markov property

Unitarity

$$P(\text{"some emission"}) + P(\text{"no emission"}) = P(0 < t \le T) + \bar{P}(0 < t \le T) = 1\ .$$

Multiplication law (no memory)

$$\bar{P}(0 < t \le T) = \bar{P}(0 < t \le t_1)\bar{P}(t_1 < t \le T)$$

Sudakov form factor from Markov property

Unitarity

$$P(\text{“some emission”}) + P(\text{“no emission”}) \\ = P(0 < t \le T) + \bar{P}(0 < t \le T) = 1\ .$$

Multiplication law (no memory)

$$\bar{P}(0 < t \le T) = \bar{P}(0 < t \le t_1)\bar{P}(t_1 < t \le T)$$

Then subdivide into n pieces: $t_i = \frac{i}{n}T, 0 \le i \le n$.

$$\bar{P}(0 < t \le T) = \lim_{n\to\infty}\prod_{i=0}^{n-1}\bar{P}(t_i < t \le t_{i+1}) = \lim_{n\to\infty}\prod_{i=0}^{n-1}\big(1 - P(t_i < t \le t_{i+1})\big)$$

$$= \exp\left(-\lim_{n\to\infty}\sum_{i=0}^{n-1}P(t_i < t \le t_{i+1})\right) = \exp\left(-\int_0^T \frac{\mathrm{d}P(t)}{\mathrm{d}t}\mathrm{d}t\right)\ .$$

Sudakov form factor

Again, no–emission probability!

$$\bar{P}(0 < t \leq T) = \exp\left(-\int_0^T \frac{\mathrm{d}P(t)}{\mathrm{d}t}\mathrm{d}t\right)$$

So,

$$\begin{aligned}\mathrm{d}P(\text{first emission at } T) &= \mathrm{d}P(T)\bar{P}(0 < t \leq T) \\ &= \mathrm{d}P(T)\exp\left(-\int_0^T \frac{\mathrm{d}P(t)}{\mathrm{d}t}\mathrm{d}t\right)\end{aligned}$$

That's what we need for our parton shower! Probability density for next emission at t:

$$\mathrm{d}P(\text{next emission at } t) =$$
$$\frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\ \exp\left[-\int_{t_0}^{t}\frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\right]$$

Parton shower Monte Carlo

Probability density:

$$\mathrm{d}P(\text{next emission at } t) = \frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\ \exp\left[-\int_{t_0}^{t}\frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\right]$$

Conveniently, the probability distribution is $\Delta(t)$ itself.

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Conveniently, the probability distribution is $\Delta(t)$ itself.
Hence, parton shower very roughly from (HERWIG):

1. Choose flat random number $0 \le \rho \le 1$.
2. If $\rho < \Delta(t_{\max})$: no resolbable emission, stop this branch.
3. Else solve $\rho = \Delta(t_{\max})/\Delta(t)$
(= no emission between $t_{\max}$ and t) for t.
Reset $t_{\max} = t$ and goto 1.

Determine z essentially according to integrand in front of exp.

Parton shower Monte Carlo

Probability density:

$$\mathrm{d}P(\text{next emission at } t) = \frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\ \exp\left[-\int_{t_0}^{t}\frac{\mathrm{d}t}{t}\int_{z_-}^{z_+}\frac{\alpha_S(z,t)}{2\pi}\hat{P}(z,t)\mathrm{d}z\right]$$

Conveniently, the probability distribution is $\Delta(t)$ itself.

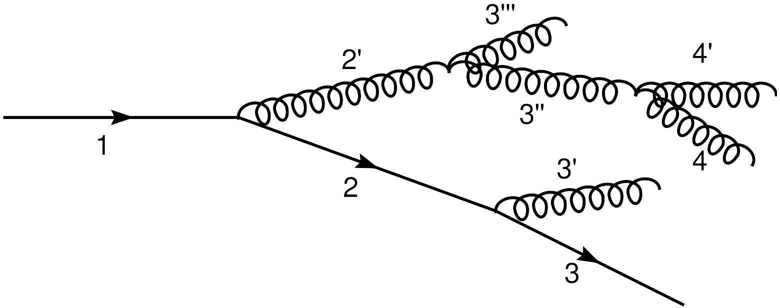
- That was old HERWIG variant. Relies on (numerical) integration/tabulation for $\Delta(t)$.
- Pythia, now also Herwig++, use the Veto Algorithm.
- Method to sample x from distribution of the type

$$\mathrm{d}P = F(x)\exp\left[-\int^{x}\mathrm{d}x'\,F(x')\right]\mathrm{d}x\ .$$

Simpler, more flexible, but slightly slower.

Parton cascade

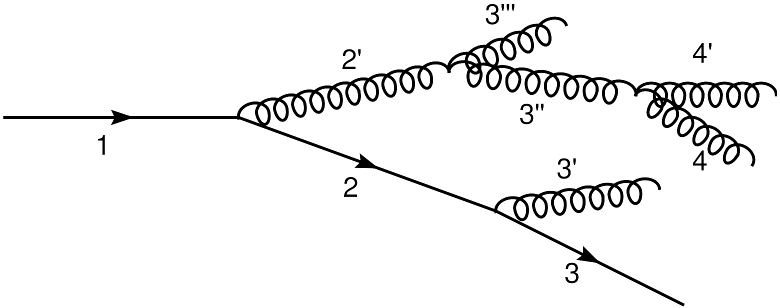
Get tree structure, ordered in evolution variable t:



Here: $t_1 > t_2 > t_3$; $t_2 > t_{3'}$ etc.
Construct four momenta from (t_i, z_i) and (random) azimuth ϕ.

Parton cascade

Get tree structure, ordered in evolution variable t:

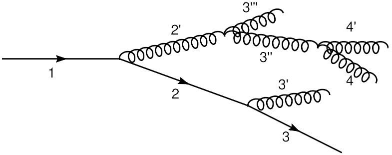


Here: $t_1 > t_2 > t_3$; $t_2 > t_{3'}$ etc.
Construct four momenta from (t_i, z_i) and (random) azimuth ϕ.

Not at all unique!
Many (more or less clever) choices still to be made.

Parton cascade

Get tree structure, ordered in evolution variable t:



- t can be θ, Q^2, $p_\perp$, ...
- Choice of hard scale $t_{\max}$ not fixed. "Some hard scale".
- z can be light cone momentum fraction, energy fraction, ...
- Available parton shower phase space.
- Integration limits.
- Regularisation of soft singularities.
- ...

Good choices needed here to describe wealth of data!

Soft emissions

- Only *collinear* emissions so far.
- Including *collinear+soft*.
- *Large angle+soft* also important.

Soft emissions

- Only *collinear* emissions so far.
- Including *collinear+soft*.
- *Large angle+soft* also important.

Soft emission: consider *eikonal factors*,
here for $q(p+q) \to q(p)g(q)$, soft g:

$$u(p)\ \not{\varepsilon}\frac{\not{p}+\not{q}+m}{(p+q)^2-m^2} \longrightarrow u(p)\frac{p\cdot\varepsilon}{p\cdot q}$$

soft factorisation. Universal, *i.e.* independent of emitter.
In general:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega}\frac{d\Omega}{2\pi}\frac{\alpha_S}{2\pi}\sum_{ij} C_{ij} W_{ij} \quad \text{(“QCD–Antenna”)}$$

with

$$W_{ij} = \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{iq})(1-\cos\theta_{qj})}\ .$$

Soft emissions

We define

$$W_{ij} = \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{iq})(1-\cos\theta_{qj})} \equiv W_{ij}^{(i)} + W_{ij}^{(j)}$$

with

$$W_{ij}^{(i)} = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{qj}}\right) .$$

$W_{ij}^{(i)}$ is only collinear divergent if $q \| i$ etc .

Soft emissions

We define

$$W_{ij} = \frac{1 - \cos\theta_{ij}}{(1 - \cos\theta_{iq})(1 - \cos\theta_{qj})} \equiv W_{ij}^{(i)} + W_{ij}^{(j)}$$

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$$W_{ij}^{(i)} = \frac{1}{2}\left(W_{ij} + \frac{1}{1 - \cos\theta_{iq}} - \frac{1}{1 - \cos\theta_{qj}}\right) .$$

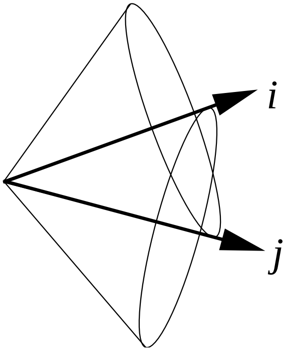
$W_{ij}^{(i)}$ is only collinear divergent if $q \| i$ etc .
After integrating out the azimuthal angles, we find

$$\int \frac{d\phi_{iq}}{2\pi} W_{ij}^{(i)} = \begin{cases} \dfrac{1}{1 - \cos\theta_{iq}} & (\theta_{iq} < \theta_{ij}) \\ 0 & \text{otherwise} \end{cases}$$

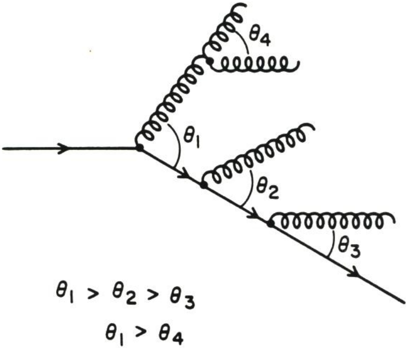
That's angular ordering.

Angular ordering

Radiation from parton *i* is bound to a cone, given by the colour partner parton *j*.



Results in angular ordered parton shower and suppresses soft gluons viz. hadrons in a jet.



Colour coherence from CDF

Events with 2 hard (> 100 GeV) jets and a soft 3rd jet (∼ 10 GeV)

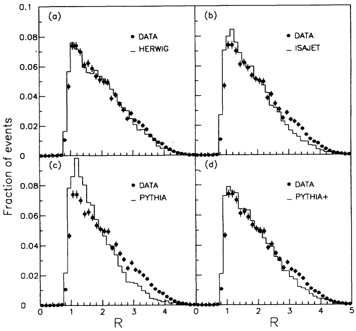


FIG. 14. Observed R distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

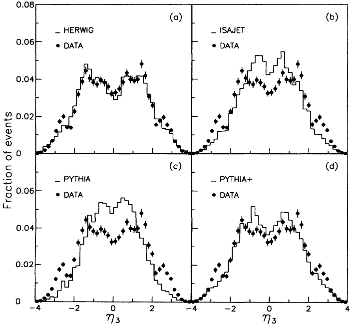


FIG. 13. Observed η_3 distribution compared to the predictions of (a) HERWIG; (b) ISAJET; (c) PYTHIA; (d) PYTHIA+.

F. Abe *et al.* [CDF Collaboration], Phys. Rev. D **50** (1994) 5562.

Best description with angular ordering.

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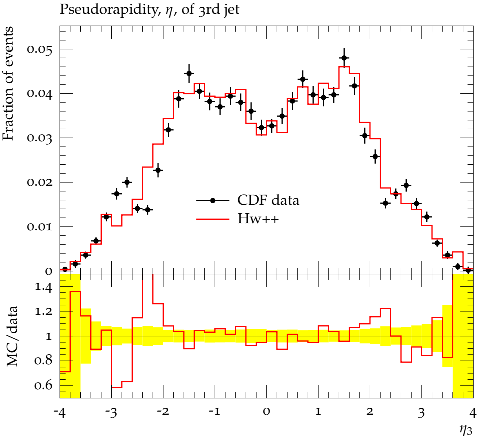

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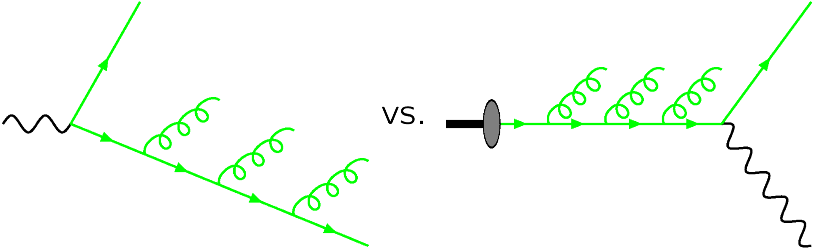
Best description with angular ordering.

Initial state radiation



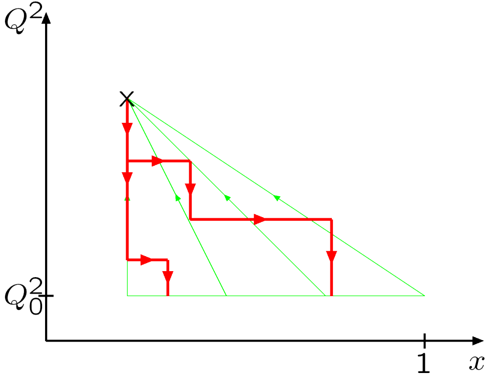
Similar to final state radiation. Sudakov form factor ($x' = x/z$)

$$\Delta(t, t_{\max}) = \exp\left[-\sum_b \int_t^{t_{\max}} \frac{\mathrm{d}t}{t} \int_{z_-}^{z_+} \mathrm{d}z \frac{\alpha_S(z,t)}{2\pi} \frac{x' f_b(x',t)}{x f_a(x,t)} \hat{P}_{ba}(z,t)\right]$$

Have to divide out the pdfs.

Initial state radiation

Evolve backwards from hard scale Q^2 *down* towards cutoff scale Q_0^2. Thereby increase x.



With parton shower we *undo* the DGLAP evolution of the pdfs.

Dipoles

Exact kinematics when recoil is taken by spectator(s).

- Dipole showers.
- Ariadne.
- Recoils in Pythia.
- New dipole showers, based on
  - Catani Seymour dipoles.
  - QCD Antennae.
  - Herwig, Sherpa, Vincia, Dire, …
  - Goal: matching with NLO.
- Generalized to IS–IS, IS–FS.