New thermodynamic identities for 5d vacuum black holes

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Outline

1. Black hole Mechanics
2. Stationary biaxisymmetric vacuum black holes
3. Harmonic map formulation and conserved currents
4. New identities

Asymptotically flat black holes

- Stationary AF solutions $(\mathcal{M}, g)$ represent isolated gravitational systems in equilibrium. At large spatial distance g is flat with certain decay.

- We may associate to an AF solution a mass M and angular momentum J. These are **geometric invariants**. For physically realistic matter, $M \geq 0$ (positive mass theorem).

- A spacetime containing a black hole is characterized by an event horizon $\mathcal{N} = \mathbb{R} \times H$. We associate to a black hole its surface gravity κ and A. These are also geometric invariants.

Black hole mechanics

- A well known result of classical GR are the **laws of black hole mechanics**. These are mathematical results (Bardeen, Carter, Hawking)
- Simplest setting: equilibrium (stationary) and axisymmetric asymptotically flat vacuum black holes. The Smarr relation is

$$\boxed{M = \frac{\kappa A}{2\pi} + 2\Omega_H J}$$

- Apply Stokes' theorem using the stationary Killing field k: The volume term vanishes using $\mathrm{Ric}(g) = 0$.

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star \mathrm{d}k = -\frac{1}{8\pi}\int_H \star \mathrm{d}k - \frac{1}{8\pi}\int_\Sigma \mathrm{d}\star \mathrm{d}k$$

since $\mathrm{d}\star\mathrm{d}k = -2\star\mathrm{Ric}(k) = 0$.

General Relativity in $D > 4$

- High-energy physics (string theory) predicts the existence of additional spatial dimensions. Low-energy regime governed by GR and its extensions (e.g. supergravity, higher order corrections).

- Black holes arise naturally in this context, and indeed string theory has produced a microscopic derivation of their entropy. The $D = 5$ case is best understood.

- AdS/CFT correspondence - the bulk theory is best understood for $D = 5$.

- Intrinsic mathematical problem (geometric inequalities, singularity theorems, classification of solutions of Einstein's equations).

Stationary biaxisymmetric black holes $(\mathcal{M}, g)$

Rigidity theorem: a stationary rotating black hole is axisymmetric

(Hollands,Ishibashi,Wald)

Consider $\mathbb{R} \times U(1) \times U(1)$ invariant asymptotically flat black holes:

$$\mathrm{Ric}(g) = 0$$

Smarr relation:

$$M = \frac{3\kappa A}{16\pi} + \frac{3}{2}\Omega_i J_i$$

- Two explicit 3-parameter (M, J_1, J_2) families:
  1. Myers-Perry black hole: $H \cong S^3$ (Myers-Perry 86)
  2. Black Ring $H \cong S^1 \times S^2$ (Emparan, Reall 01; Pomeransky, Sen'kov 07)
- Failure of no-hair theorem: MP and black rings may carry the same (M, J_1, J_2).

Main Results (HK, Lucietti 18)

Smarr relation cannot distinguish between different black holes.

- Purpose of work: derive new identities for thermodynamic variables of AF black holes in this class.
- Our identities depend on the topology and include contributions from purely gravitational topological charges. They give new nonlinear relations $M = f(\kappa, A, J_i, \mathcal{G})$.
- They arise from the $SL(3, \mathbb{R})$ hidden symmetry of the vacuum Einstein equations.

Orbit space

A great deal is known about BH solutions in this class. (Hollands, Yzadjiev 08)

- Key idea: Reduce problem to 2d **orbit space** $\mathcal{B} \equiv \mathcal{M} \setminus (\mathbb{R} \times U(1)^2)$.
- $\mathcal{B}$ is diffeomorphic to upper half plane. $\partial\mathcal{B}$ corresponds to axes of symmetry and horizons in $\mathcal{M}$.
- Analysis shows $H = S^3, S^1 \times S^2$, or $L(p,q)$.
- Exterior region may contain 'bubbles' (2-cycles) supported by rotation. No vacuum solutions of this kind known.

Rod Structure

- Let $\xi_I = (\partial_t, m_i)$ generate $\mathbb{R} \times U(1)^2$. Define

$$\rho \equiv \sqrt{-\det g(\xi_I, \xi_J)} \in [0, \infty)$$

  Field equations $\Rightarrow$ isometry group is surface-orthogonal and $\Delta\rho = 0$ on $\mathcal{B}$.

- $\rho = 0$ represents symmetry 'axis' where g_{IJ} is degenerate. Axis divides into 'rods' I_a on which

$$g(\mathbf{v}_a, \mathbf{v}_a) = 0 \qquad \mathbf{v}_a = v_a{}^i \mathbf{m}_i, \; v_a{}^j \in \mathbb{Z}$$

  Horizon rod corresponds to where $\xi = \partial_t + \Omega_i \mathbf{m}_i$ is null.

Existence and Uniqueness

Specification of rod structure determines the topology of $\mathcal{M}$.

Uniqueness theorem (Hollands, Yzadjiev 08)

An AF vacuum black hole with $\mathbb{R} \times U(1)^2$ isometry is *uniquely* specified by its interval data and angular momenta J_i.

Existence theorem (Khuri, Weinstein, Yamada 17)

There exists a smooth vacuum black hole spacetime (apart from possible conical singularities on axes) given a fixed specification of rod structure and J_i satisfying appropriate compatibility conditions.

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Harmonic Map formulation

- Well known that vacuum $D = 5$ spacetimes with *two* commuting Killing fields ζ_μ, $\mu = 0, 1$ = 3d gravity theory coupled to a **harmonic map** with Riemannian target space $SL(3, \mathbb{R}) \setminus SO(3)$.

$$\mathbf{g} = \beta_{\mu\nu}(\mathrm{d}x^\mu + B^\mu \mathrm{d}w)(\mathrm{d}x^\nu + B^\nu \mathrm{d}w) + \frac{\rho^2 \mathrm{d}w^2}{|\beta|} + g_2$$

- In our case we have 1 more symmetry $\Rightarrow$ whole system is

$$S_H = \int_{\mathcal{B}} |\mathrm{d}\Phi|_h^2 \, \mathrm{dVol}(g_2)$$

where $\Phi : (\mathcal{B}, g_2) \to (SL(3, \mathbb{R}) \setminus SO(3), h)$ is a 3×3 matrix built from $\beta_{\mu\nu}$ and B^μ (5 functions)

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Hidden symmetries and conserved currents

$$\Phi^A{}_B = \begin{pmatrix} \beta_{\mu\nu} + \beta^{-1} U_\mu U_\nu & -\beta^{-1} U_\mu \\ -\beta^{-1} U_\nu & \beta^{-1} \end{pmatrix}, \qquad \mathsf{d}U_\mu = \star(k \wedge \xi \wedge \mathsf{d}\zeta_\mu)$$

Field equations invariant under $SL(3, \mathbb{R})$ symmetry of target space:

$$\mathsf{d}(\rho \star_2 \mathsf{d}\Phi) = 0$$

Can be written as conserved currents:

$$\mathsf{d}\mathcal{J}^\mu_\nu = 0, \qquad \mathsf{d}\mathcal{J}^\mu = 0, \qquad \mathcal{J}^A_B = \rho \star_2 \mathsf{d}\Phi^A{}_B$$

$$\mathcal{J}^\mu_\nu \equiv \rho \beta^{\mu\alpha} \star_2 \mathsf{d}\beta_{\alpha\nu} + B^\mu \mathsf{d}U_\nu + \mathsf{d}C^\mu_\nu$$

$C^\mu_\nu =$ are arbitrary smooth fields.

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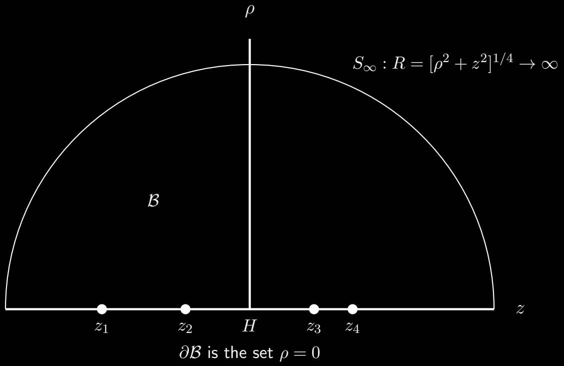
$\mathcal{B}$ is upper half plane $(\rho, z)|\rho > 0, z \in \mathbb{R}$

Integrate over $\mathcal{B}$ and apply Stokes' to derive identities:

$$0 = \int_{\mathcal{B}} \mathrm{d}\mathcal{J}^{\mu}_{\nu} = \int_{\partial\mathcal{B}} \mathcal{J}^{\mu}_{\nu} + \int_{S_\infty} \mathcal{J}^{\mu}_{\nu}$$

The boundary of the orbit space consists of $\partial\mathcal{B} \cup S_\infty$



Behaviour of Φ on the boundary and asymptotic end

- Need to evaluate $\mathcal{J}^{\mu}_{\nu}, \mathcal{J}^{\mu}$ on $\partial\mathcal{B}$ - requires careful analysis of the $\rho \to 0$ behaviour of $\beta_{\mu\nu}, U_\mu$ on axis rods and H.
- B^μ jumps discontinuously over corner points
- For S_∞, use asymptotic expansion of g:

$$g \sim -\left(1 - \frac{8M}{3\pi R^2}\right) \mathrm{d}t^2 - \frac{8}{\pi R^2} J_1 \sin^2\theta \mathrm{d}t\mathrm{d}\phi_1 - \frac{8}{\pi R^2} J_2 \cos^2\theta \mathrm{d}t\mathrm{d}\phi_2$$
$$\left(1 + \frac{4M}{3\pi R^2}\right) R^2 \sin^2\theta \mathrm{d}\phi_1^2 + \left(1 + \frac{4M}{3\pi R^2}\right) R^2 \cos^2\theta \mathrm{d}\phi_2^2$$
$$\frac{16\zeta \sin^2\theta \cos^2\theta}{R^2} \mathrm{d}\phi_1 \mathrm{d}\phi_2 + \mathrm{d}R^2 + R^2 \mathrm{d}\theta^2$$

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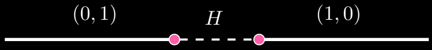
Reductions and Strategy

- Derive identities for each choice of 2 ζ_μ from $\mathbb{R} \times U(1)^2$. Natural choice: $\zeta_\mu = (m_1, m_2)$ - leads to Smarr relation plus determine subleading terms in asymptotic expansion of g.
- $\zeta_\mu = (k, \xi)$ where $k = \partial_t$ and $\xi = \partial_t + \Omega_i m_i$ is null generator of H - leads to new nonlinear identities relating $(M, J_i, \kappa, A, \Omega_i)$.
- Reduction requires $\beta = \det \beta_{\mu\nu} < 0$ in $\mathcal{M}$. We can prove this must be true everywhere except possibly inside the ergoregion. **Conjecture:** $\beta < 0$ in black hole exterior except non-generically on axes.

$$\beta_{\mu\nu} = \begin{pmatrix} k \cdot k & k \cdot \xi \\ k \cdot \xi & \xi \cdot \xi \end{pmatrix}$$

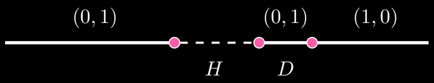
- Myers-Perry $H = S^3$



$$\Omega_1 J_2 + \Omega_2 J_1 - \frac{16 M \Omega_1 \Omega_2}{9\pi}\left(M + \frac{3\kappa A}{16\pi}\right) = 0$$

Examples

- Black ring $H = S^1 \times S^2$



$$\Omega_1 J_2 + \Omega_2 J_1 - \frac{16 M \Omega_1 \Omega_2}{9\pi}\left(M + \frac{3\kappa A}{16\pi}\right) = \frac{2}{3} M \Omega_1 \mathcal{G}_\xi[D] + \frac{\kappa \Omega_1 A}{8\pi} \mathcal{G}_k[D]$$

- where we have defined a purely gravitational flux

$$\mathcal{G}_\zeta[D] = \frac{1}{2\pi} \int_{[D]} \star(k \wedge \mathsf{d}\zeta)$$

where $\zeta = k$ or ξ.

Summary of Results

- We have derived families of new identities relating thermodynamic variables and the rod structure. They are finer grained than Smarr, i.e. they detect different topologies.
- identified new purely gravitational fluxes carried by 2-cycles and discs
- Works for arbitrary rod structures, so yields identities for yet-to-be explicitly found black hole solutions.

Further directions

- $D = 4$ get Smarr + formulae for M, J in terms of rod data.
- Einstein-Maxwell theory, supergravity theories admit non-linear σ model formulations with more complicated target spaces
  $\Rightarrow$ richer families of divergence identities which could be useful to rule out solutions / yield quantum selection rules
- Variational (first law) identities?