

Light-cones and quantum caustics in quenched spin chains

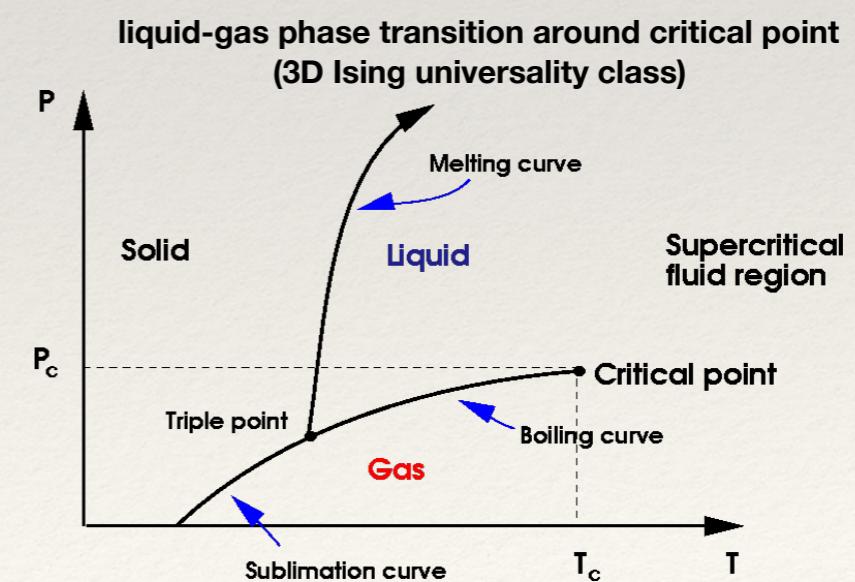
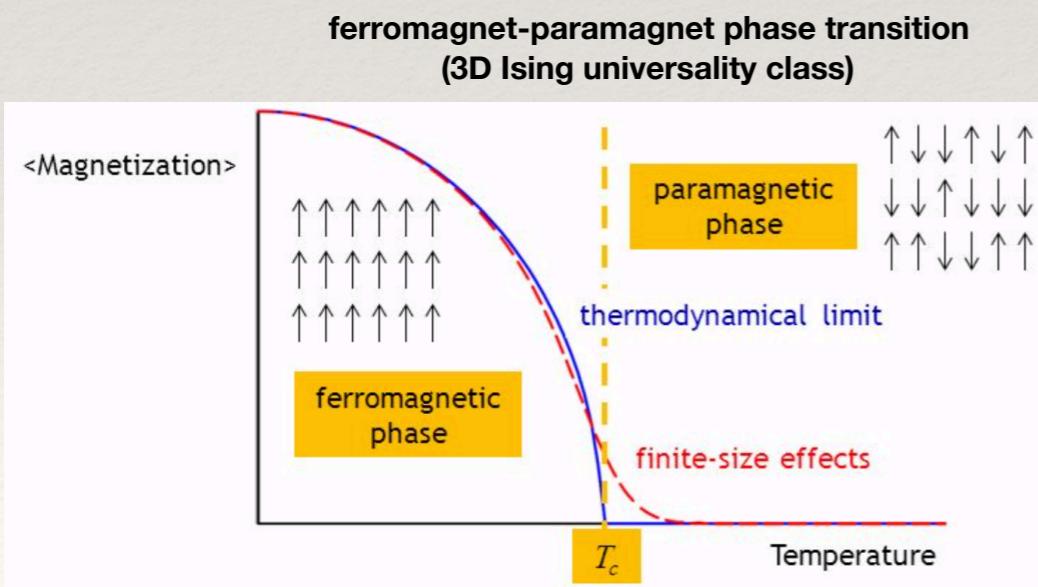
CAP 2019 Simon Fraser University

Duncan O'Dell & Wyatt Kirkby

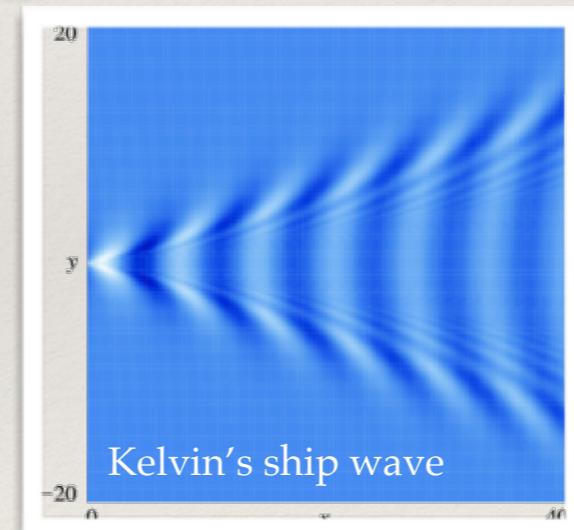
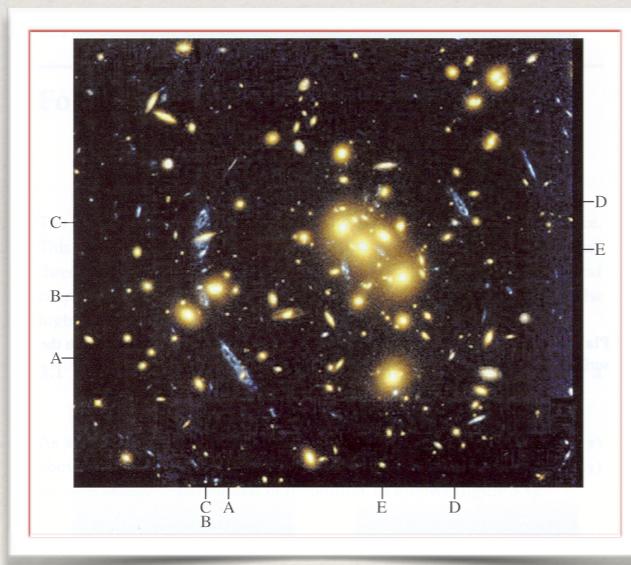
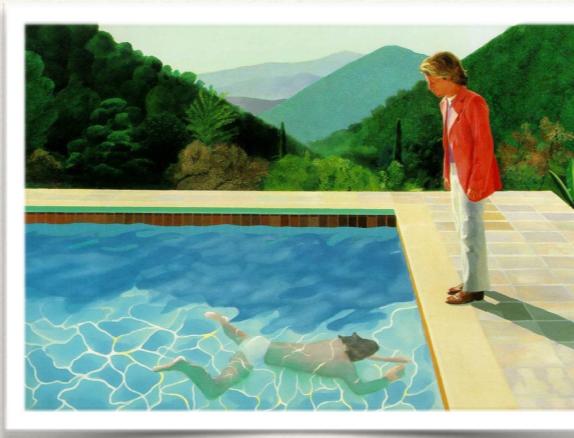
McMaster University

Universality at equilibrium: phase transitions

- non-analytic change in ground state properties as a control parameter is varied
- criticality / divergence $C_V \sim |T - T_c|^{-\alpha}$
- scale invariance
- universality classes usually depend on dimension

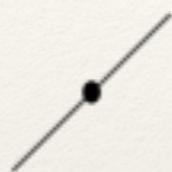


Universality in dynamics?

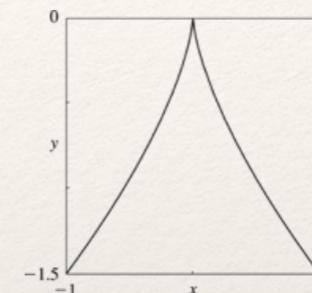


singularities in wave dynamics (“natural focusing”)

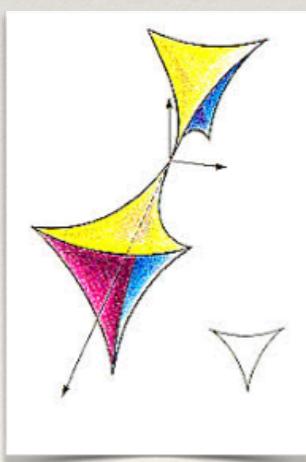
Catastrophe theory: structurally stable singularities



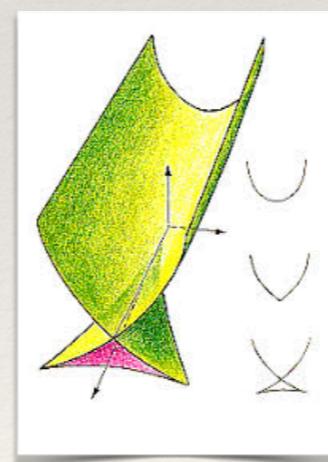
1D: point



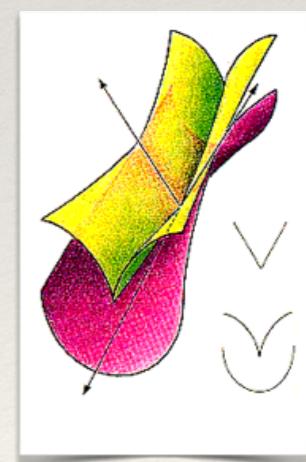
2D: cusp



3D: elliptic umbilic



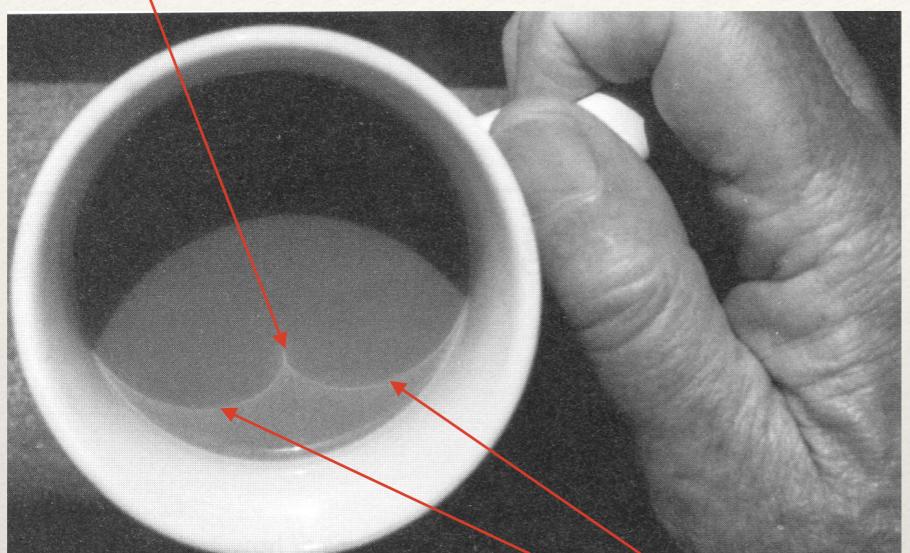
3D: swallowtail



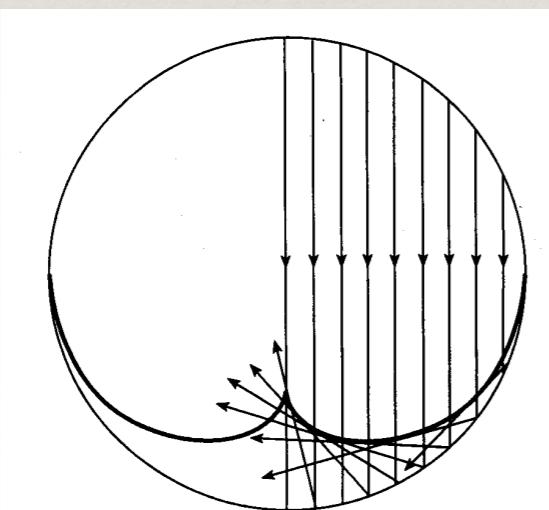
3D: hyperbolic umbilic

Cusp caustic

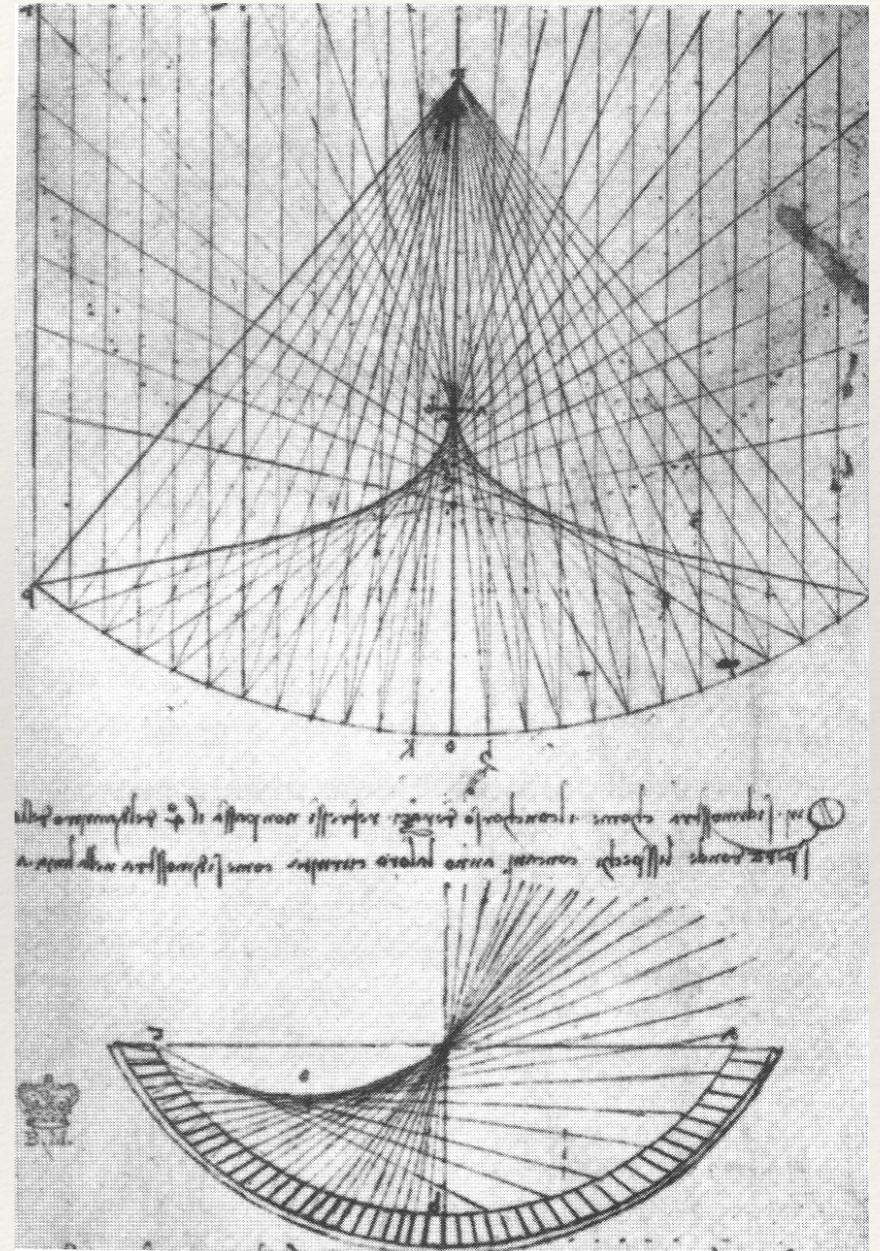
cusp point



fold lines



- Structurally stable
- Generic
- Singular...light intensity diverges (in geometric theory)



Leonardo de Vinci c. 1508

Structurally stable caustics and their generating functions with $K \leq 4$

name	codimension K	$\phi(s; C)$ [generating function]
Fold	1	$s^3/3 + Cs$
Cusp	2	$s^4/4 + C_2 s^2/2 + C_1 s$
Swallowtail	3	$s^5/5 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
Elliptic umbilic	3	$s_1^3 - 3s_1 s_2^2 - C_3(s_1^2 + s_2^2) - C_2 s_2 - C_1 s_1$
Hyperbolic umbilic	3	$s_1^3 + s_2^3 - C_3 s_1 s_2 - C_2 s_2 - C_1 s_1$
Butterfly	4	$s^6/6 + C_4 s^4/4 + C_3 s^3/3 + C_2 s^2/2 + C_1 s$
Parabolic umbilic	4	$s_1^4 + s_1 s_2^2 + C_4 s_2^2 + C_3 s_1^2 + C_2 s_2 + C_1 s_1$

R. Thom *Structural Stability and Morphogenesis* (Benjamin, 1975); V.I. Arnol'd, Russ. Math. Survs. 30 (5) (1975) p.1

Caustics occur where $\frac{\partial \phi}{\partial s_i} = 0$ (Fermat's principle) AND $\frac{\partial^2 \phi}{\partial s_i^2} = 0$

Example: the cusp

$$\phi(s; C) = s^4/4 + C_2 s^2/2 + C_1 s$$

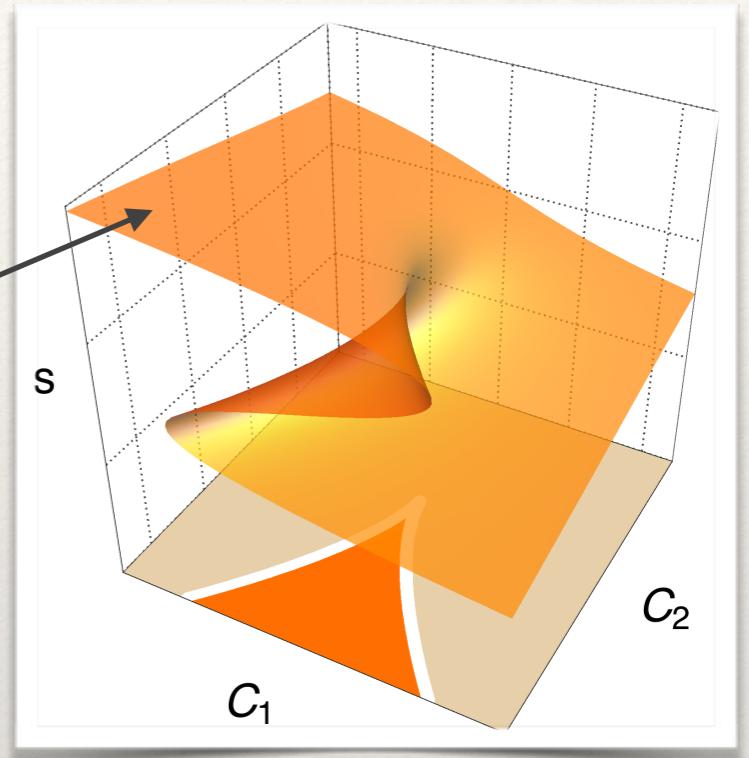
Ray equation (Fermat's principle) :

$$\frac{\partial \phi}{\partial s} = s^3 + C_2 s + C_1 = 0$$

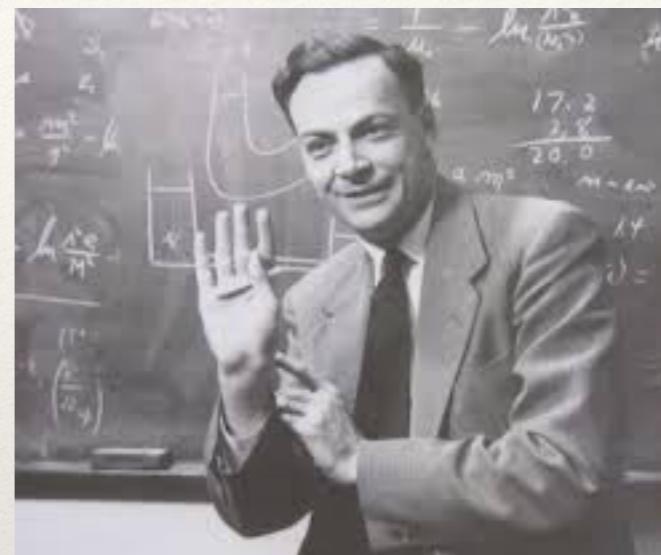
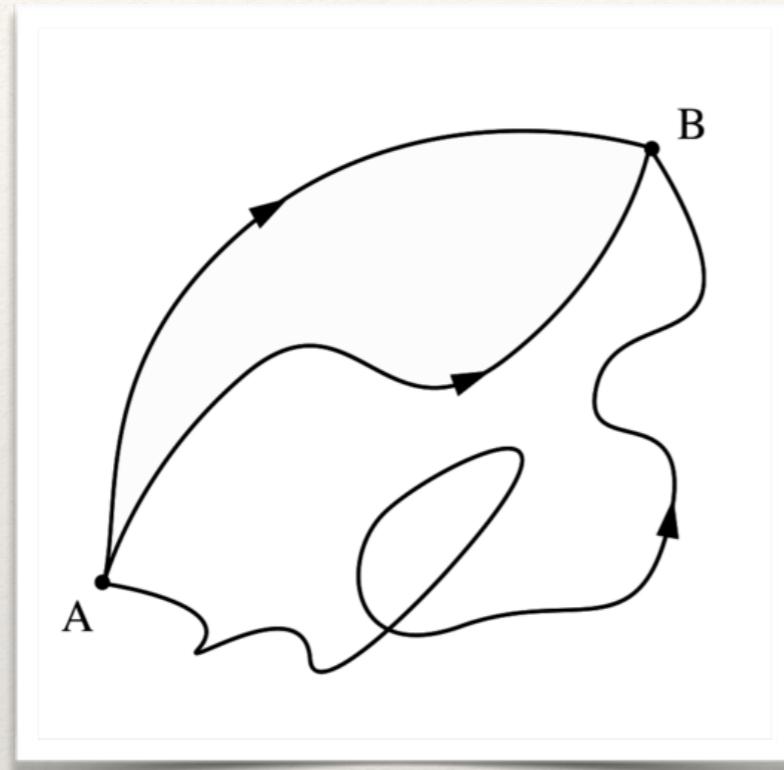
Caustic equation : $\frac{\partial^2 \phi}{\partial s^2} = 3s^2 + C_2 = 0$

Eliminate s :

$$C_1 = \pm \sqrt{\frac{4}{27}} (-C_2)^{3/2}$$



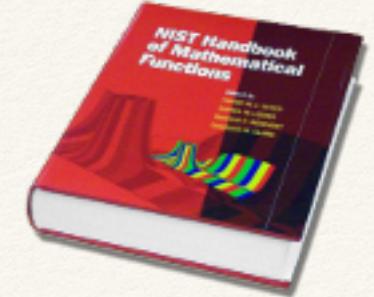
Wave theory: Feynman path integral



Richard Feynman

$$\Psi(B) = \mathcal{N} \sum_{\text{paths } j} e^{iS_j/\hbar}$$

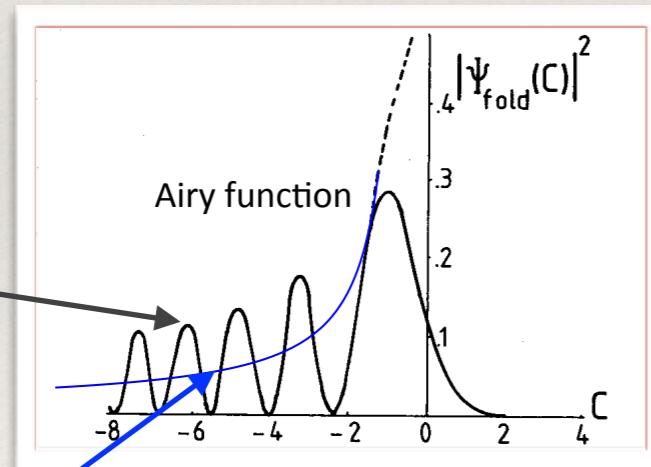
Wave catastrophes



$$\begin{aligned}\Psi_{\text{fold}}(C) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^3/3 + Cs)} ds \\ &= \sqrt{2\pi} \text{Ai}[C]\end{aligned}$$

G.B. Airy, Trans. Camb. Phil. Soc. **6**, 379 (1838)

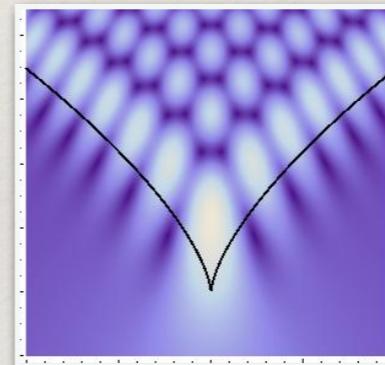
WKB theory



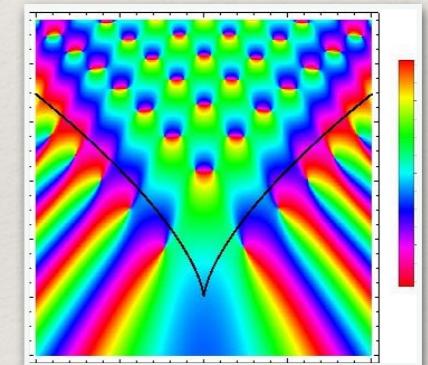
ray theory

$$\Psi_{\text{cusp}}(C_1, C_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^4/4 + C_2 s^2/2 + C_1 s)} ds$$

T. Pearcey, Phil. Mag. **37**, 311 (1946)



amplitude



phase

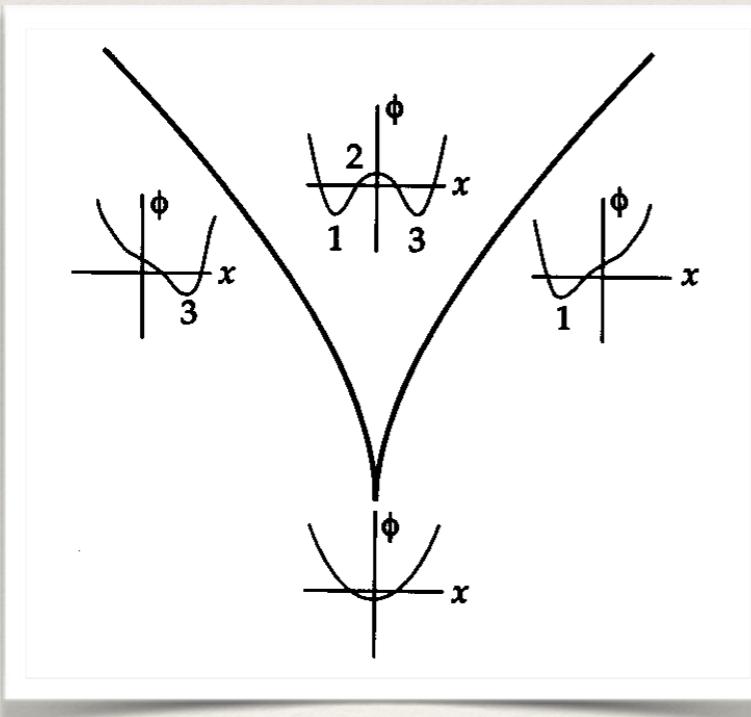
Singularities are removed by interference

The Pearcey function



$$\Psi_{\text{cusp}}(C_1, C_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^4/4 + C_2 s^2/2 + C_1 s)} ds$$

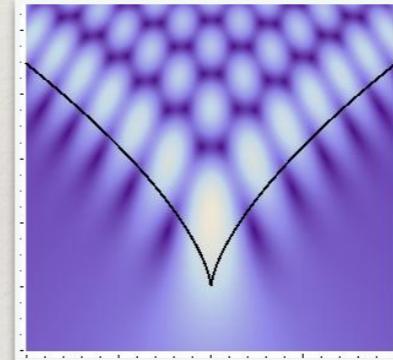
T. Pearcey, Phil. Mag. 37, 311 (1946)



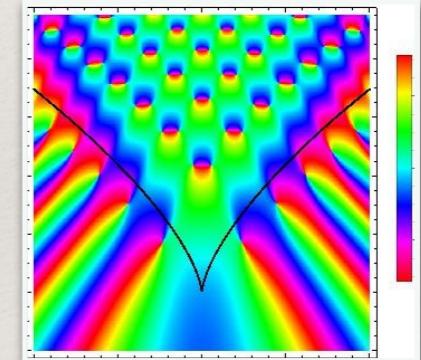
C_2

C_1

There are three classical solutions inside the cusp and one outside



amplitude



phase

“Light” cones

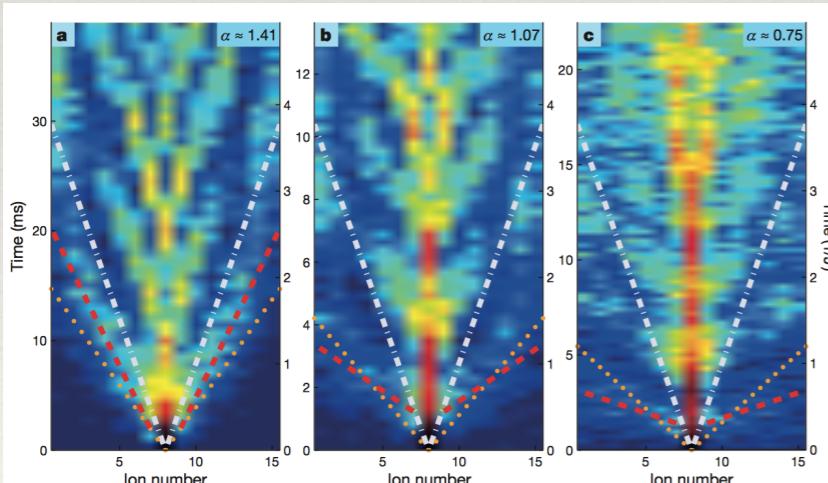
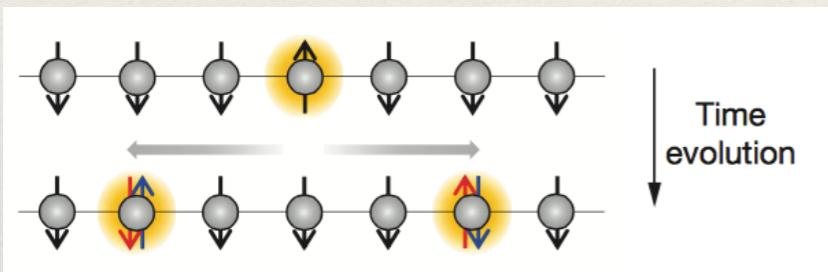
Spin chain

LETTER

doi:10.1038/nature13461

Quasiparticle engineering and entanglement propagation in a quantum many-body system

P. Jurcevic^{1,2*}, B. P. Lanyon^{1,2*}, P. Hauke^{1,3}, C. Hempel^{1,2}, P. Zoller^{1,3}, R. Blatt^{1,2} & C. F. Roos^{1,2}



Measured magnetization $\langle \sigma_i^z(t) \rangle$

Atoms in optical lattice (Bose Hubbard)

LETTER

doi:10.1038/nature10748

Light-cone-like spreading of correlations in a quantum many-body system

Marc Cheneau¹, Peter Barmettler², Dario Poletti², Manuel Endres¹, Peter Schauß¹, Takeshi Fukuhara¹, Christian Gross¹, Immanuel Bloch^{1,3}, Corinna Kollath^{2,4} & Stefan Kuhr^{1,5}

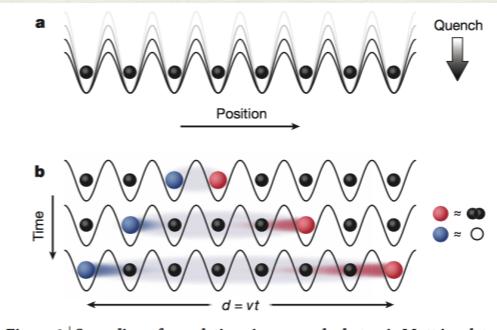


Figure 1 | Spreading of correlations in a quenched atomic Mott insulator.

$$C_d(t) = \langle \hat{s}_j(t) \hat{s}_{j+d}(t) \rangle - \langle \hat{s}_j(t) \rangle \langle \hat{s}_{j+d}(t) \rangle$$

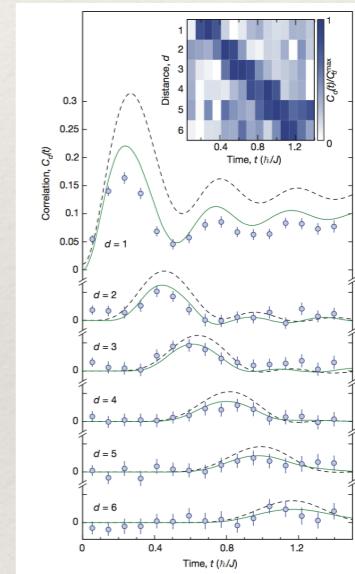


Figure 2 | Time evolution of the two-point parity correlations. After the

Other experiments:

- T. Fukuhara *et al* Nature 502, 76 (2013).
- T. Langen *et al* Nat. Physics 9, 640 (2013).
- P. Richerme *et al* Nature 511, 198 (2014).

Lieb-Robinson bound

E. H. Lieb and D. W. Robinson, “The finite group velocity of quantum spin systems”
Comm. Math. Phys. 28, 251 (1972).

Local quench at $t=0$ $|\psi(t = 0)\rangle = \hat{\sigma}_l^x |\psi_0\rangle$

Change in observable \hat{O} :

$$| \langle \psi(t) | \hat{O} | \psi(t) \rangle - \langle \psi_0 | \hat{O} | \psi_0 \rangle | \leq \| [\hat{O}(t), \sigma_l^x(0)] \|$$

If interactions between spins have exponential spatial dependence then:

$$\| [\hat{O}(t), \sigma_l^x(0)] \| \leq \| \hat{O} \| F(d, t), \quad F(d, t) = C e^{\mu(v_{LR}|t|-d)}$$

C, μ = +ve constants

d = distance from initial excitation

v_{LR} = Lieb-Robinson velocity

Quasiparticle model

P. Calabrese and J. Cardy, Phys. Rev. Lett. 96, 136801 (2006).

Lieb-Robinson bound arises from finite **group velocity** of quasiparticles

$$|\Psi(t)\rangle = e^{-iHt/\hbar} b_{x=0}^\dagger |0\rangle_b = \frac{1}{\sqrt{N}} \sum_k e^{-i\epsilon_k t/\hbar} |k\rangle$$

creates quasiparticle at x=0

quasiparticle dispersion

Continuum approximation:

$$\Psi_{\text{CA}}(x, t) = \frac{\sqrt{a}}{2\pi} \int_{-\pi/a}^{\pi/a} dk e^{i\Phi(k; x, t)}$$

$$\boxed{\Phi(k; x, t) = kx - \epsilon_k t / \hbar}$$

$$v_g = (1/\hbar) |\mathrm{d}\epsilon_k / \mathrm{d}k|$$

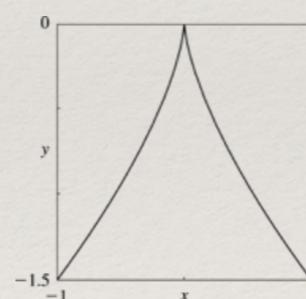
$$v_{\text{LR}} = \max_k |\mathrm{d}\epsilon_k / \mathrm{d}k|$$



$$\left. \begin{aligned} \partial\Phi/\partial k &= 0 \\ \partial^2\Phi/\partial k^2 &= 0 \end{aligned} \right\} \begin{aligned} &\text{light cone equations} \\ &\text{match caustic} \\ &\text{conditions!} \end{aligned}$$

Prediction of catastrophe theory

structurally stable catastrophes in 2D
(x,t) plane are fold lines and cusps.....



Transverse Field Ising model (TFIM)

$$H = -J \sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - h \sum_i \sigma_i^z$$

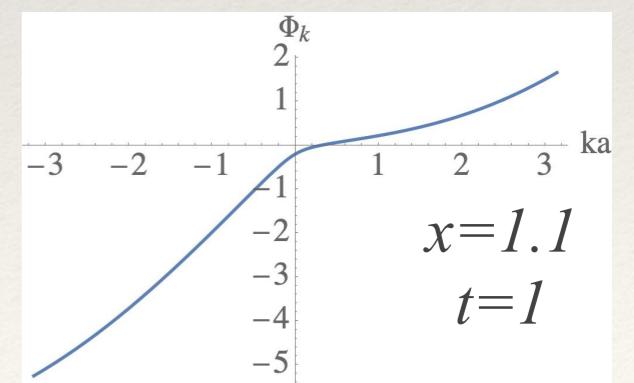
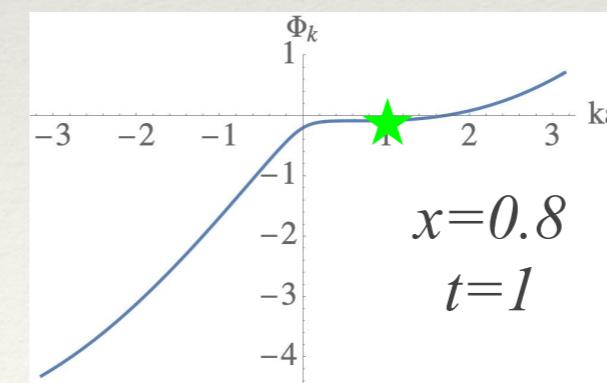
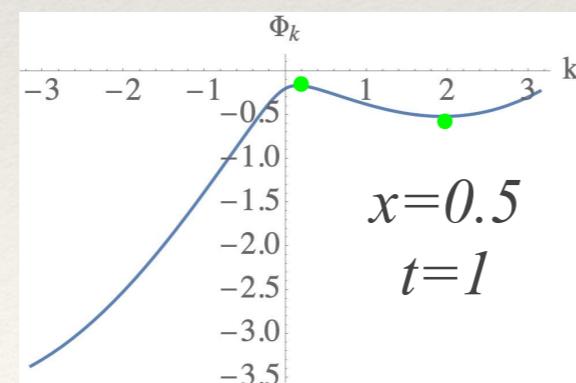
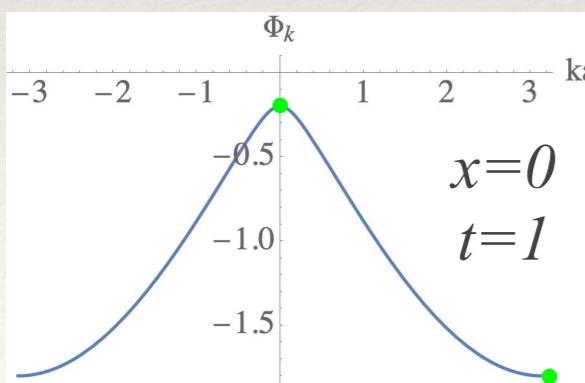
- Can be exactly diagonalized to give fermionic quasiparticles with **dispersion relation**:

$$\epsilon_k = 2J \sqrt{(\cos(ka) - g)^2 + \sin^2(ka)} \quad g = h/J$$

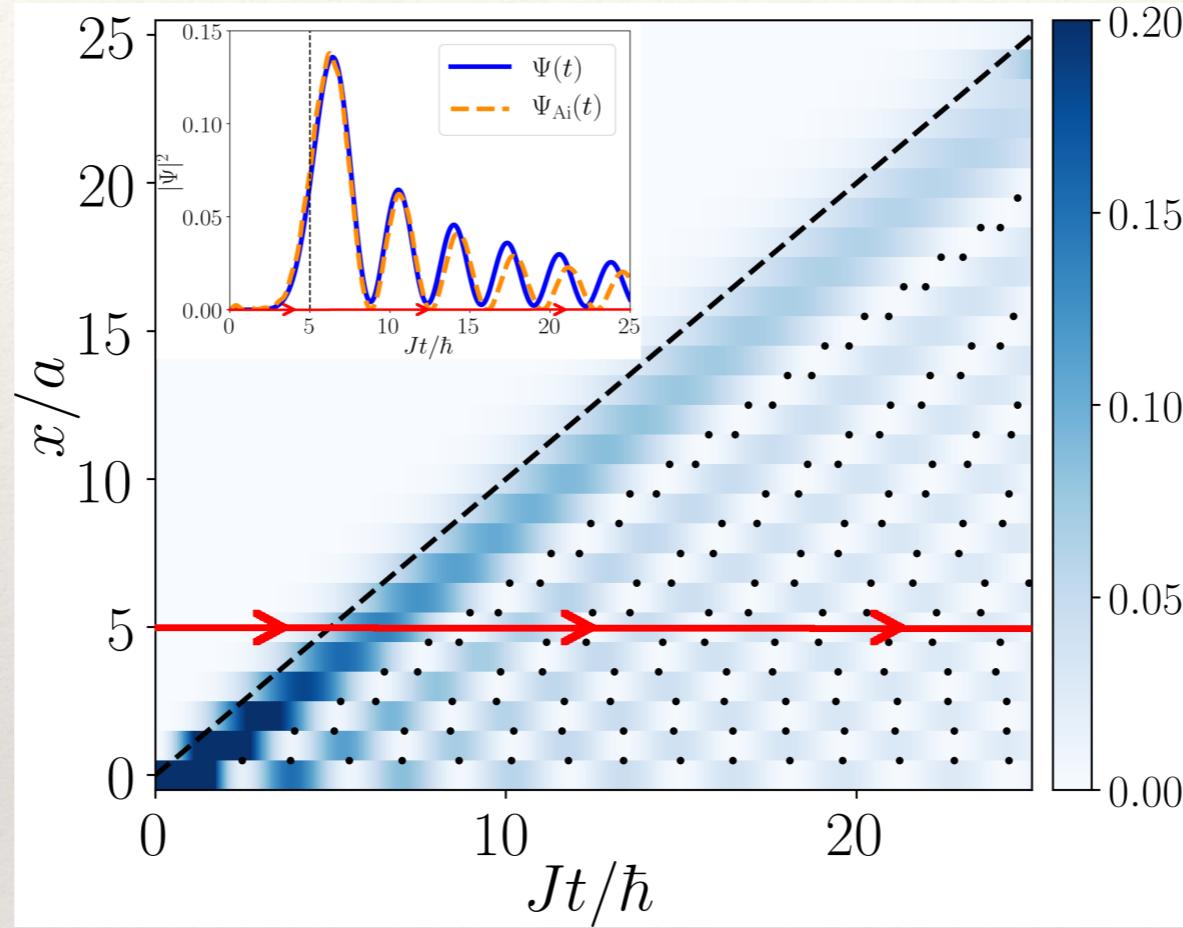


$$\Phi(k, x, t) = kx - 2Jt \sqrt{(\cos(ka) - g)^2 + \sin^2(ka)/\hbar}$$

generating
function



TFIM light cone



$$\Psi_{\text{Ai}}(C^j; J) \approx \frac{1}{2\pi t^{1/3}} \left(\frac{2Jg^{\frac{2-j}{3}}}{v_{\text{I}}\hbar} \right)^{\frac{1}{2}} \int_{\alpha_j}^{\beta_j} ds_j e^{\frac{iJ}{\hbar}\Phi_1(s_j; C^j)} \xrightarrow{t \rightarrow \infty} \left(\frac{J}{\hbar} \right)^{\frac{1}{6}} \text{Ai} \left[\left(\frac{J}{\hbar} \right)^{\frac{2}{3}} C^j \right]$$

$$C_j(x, t) = 2(x/v_{\text{I}} - t)(g^{2-j}/\sqrt{t})^{2/3}$$

$j = \{1, 2\}$ when $\{g > 1, g < 1\}$

$$v_{\text{LR}} \equiv \begin{cases} \frac{2Jag}{\hbar} & 0 < |g| < 1 \\ \frac{2Ja}{\hbar} & 1 < |g| \end{cases}$$

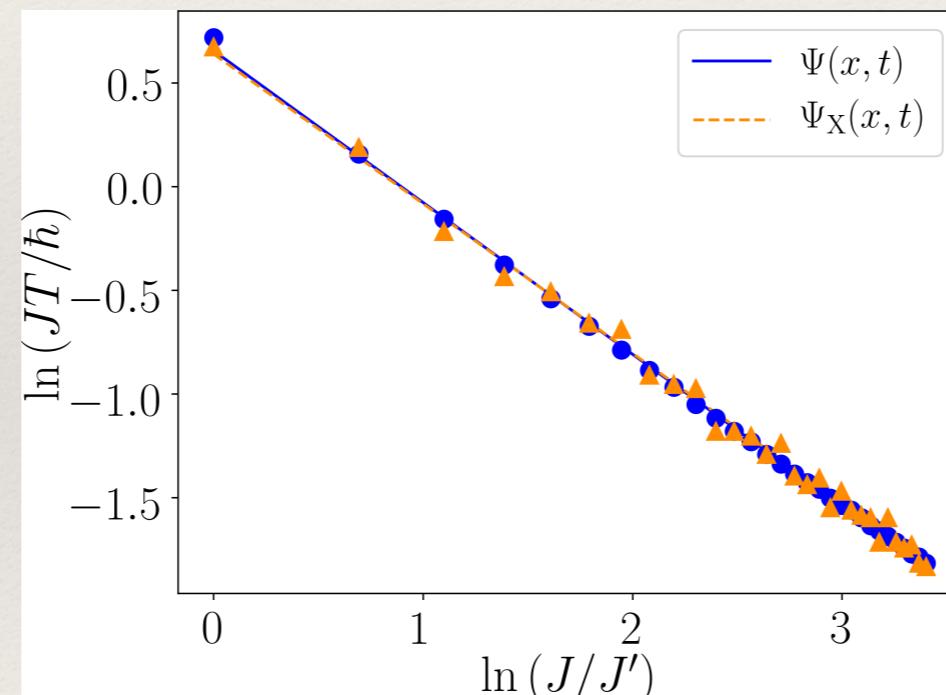
Self-similar scaling

Analytical prediction:

Arnol'd index Berry index

$$\Psi_{\text{Ai}}(C^j; J) \xrightarrow{t \rightarrow \infty} \left(\frac{J}{\hbar}\right)^{\frac{1}{6}} \text{Ai} \left[\left(\frac{J}{\hbar}\right)^{\frac{2}{3}} C^j \right]$$

Exact results (numerical):



Oscillation period, T , is found from the time difference between maxima in the wavefunction immediately following the light cone for the TFIM quench

XY spin chain

$$H = -J \sum_{\langle ij \rangle} \left(\frac{(1+\gamma)}{2} \sigma_i^x \sigma_j^x + \frac{(1-\gamma)}{2} \sigma_i^y \sigma_j^y \right) - h \sum_i \sigma_i^z$$

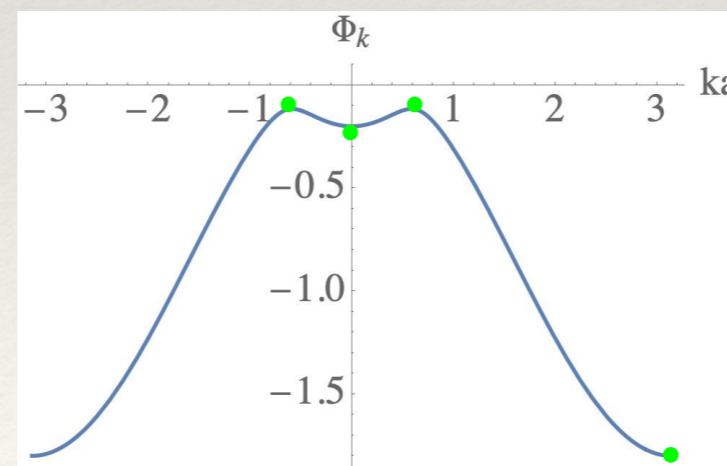
- Reduces to transverse field Ising model when $\gamma=1$
- Can be exactly diagonalized to give fermionic quasiparticles with **dispersion relation**:

$$\epsilon_k = 2J \sqrt{(\cos(ka) - g)^2 + \gamma^2 \sin^2(ka)} \quad g = h/J$$



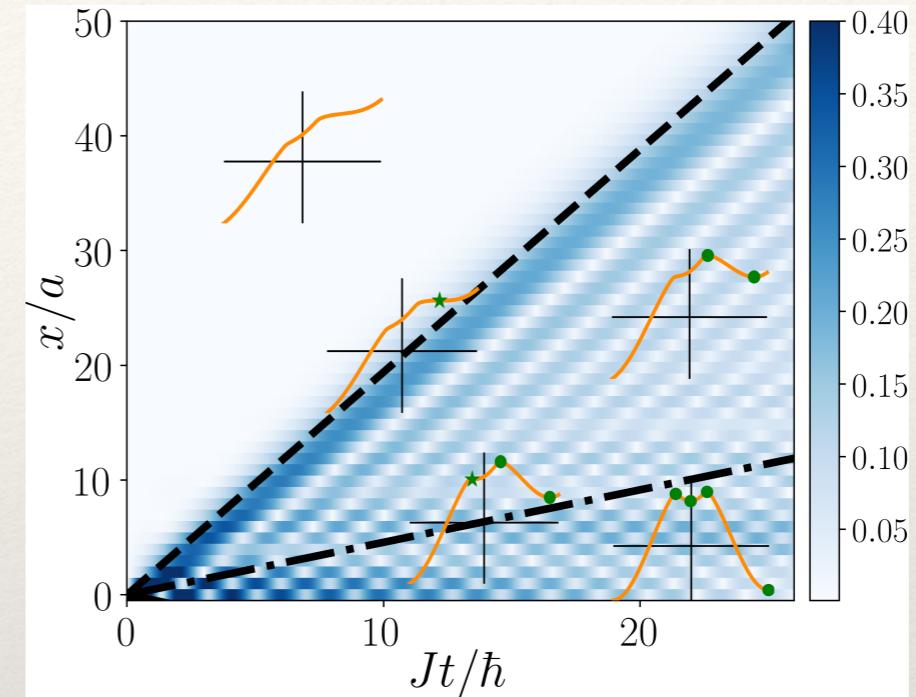
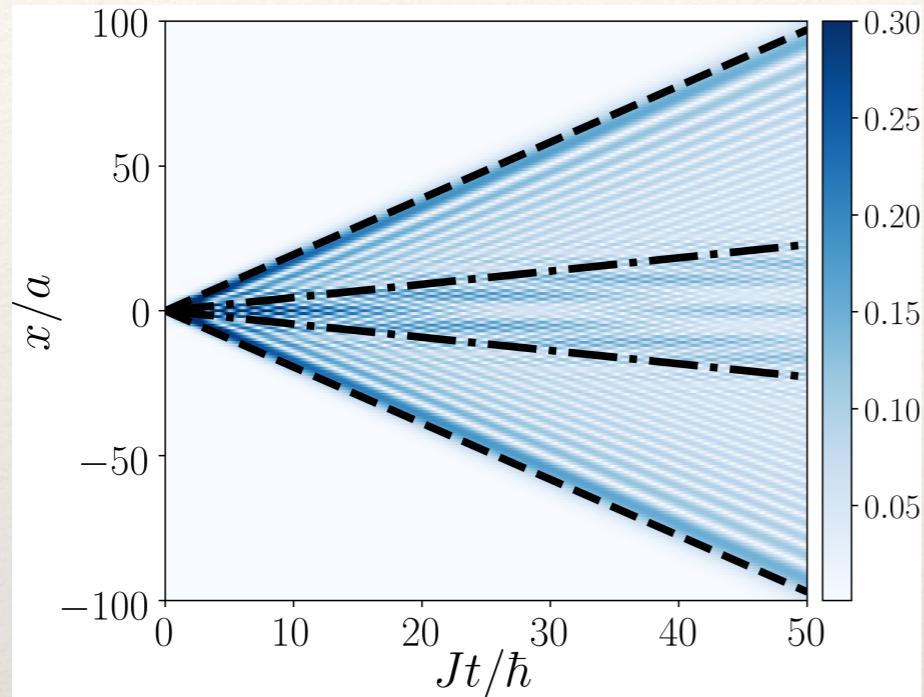
$$\Phi(k, x, t) = kx - 2Jt \sqrt{(\cos(ka) - g)^2 + \gamma^2 \sin^2(ka)/\hbar}$$

generating
function



4 stationary points

Double cones



Inner cone wave function:

$$\Psi_{\text{Pe}}(C_1, C_2; J) \approx \frac{1}{2\pi} \left(\frac{J(\gamma^2 + g - 1)}{\hbar v_{\text{I}}(g - 1)C_2} \right)^{\frac{1}{2}} \int_{-S}^S ds e^{-\frac{iJ}{\hbar}\Phi_2(s; C_1, C_2)} \xrightarrow{t \rightarrow \infty} \left(\frac{J}{\hbar} \right)^{\frac{1}{4}} \text{Pe} \left[\left(\frac{J}{\hbar} \right)^{\frac{3}{4}} C_1, \left(\frac{J}{\hbar} \right)^{\frac{1}{2}} C_2 \right]$$

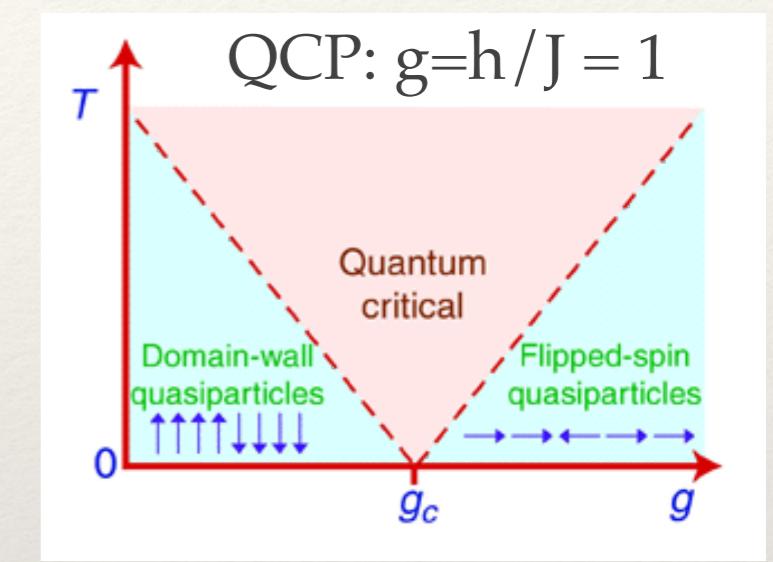
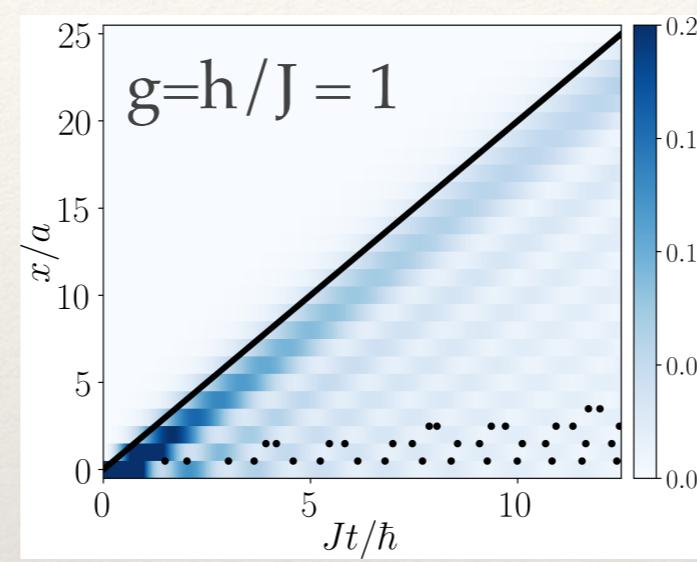
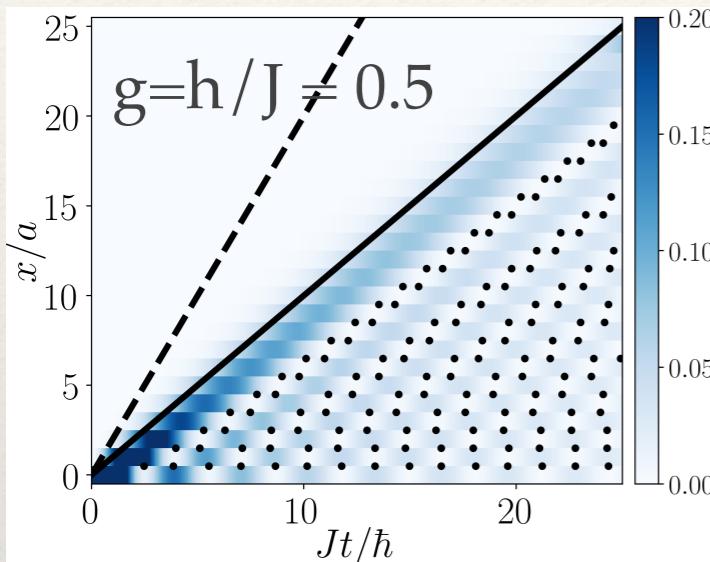
where:

$$C_1 = -\sqrt{2}x/[v_{\text{I}}(t\Gamma)^{\frac{1}{4}}]$$

$$C_2 = -\sqrt{t}(\gamma^2 + g - 1)/[\sqrt{\Gamma}(g - 1)]$$

$$\Gamma = \frac{(g^3 - 1 - 2\gamma^2 + 3\gamma^4 + g(3 - 2\gamma^2) + g^2(4\gamma^2 - 3))}{12(g - 1)^3}$$

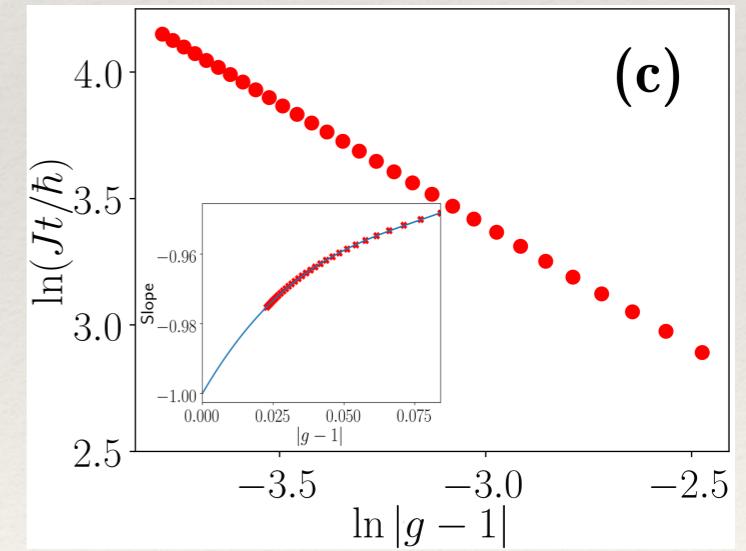
Vortices



- Less vortices as QCP at $g=g_c=1$ is approached.
- Critical slowing down
- Vortices much more sensitive to QCP than light cone
- Can extract **dynamical critical exponent z** from vortex-antivortex annihilation as $g \rightarrow g_c$

$$\tau \propto \xi^z$$

$$\xi = |g - g_c|^{-\nu}$$



We recover: $\nu z = 0.9999 \pm 0.0004$ consistent with known scaling $\nu z = 1$

Summary

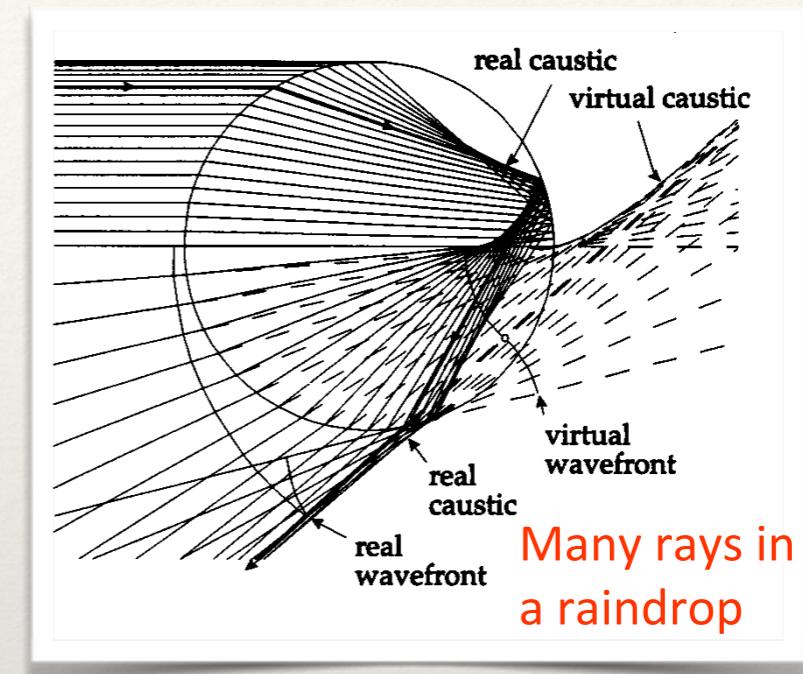
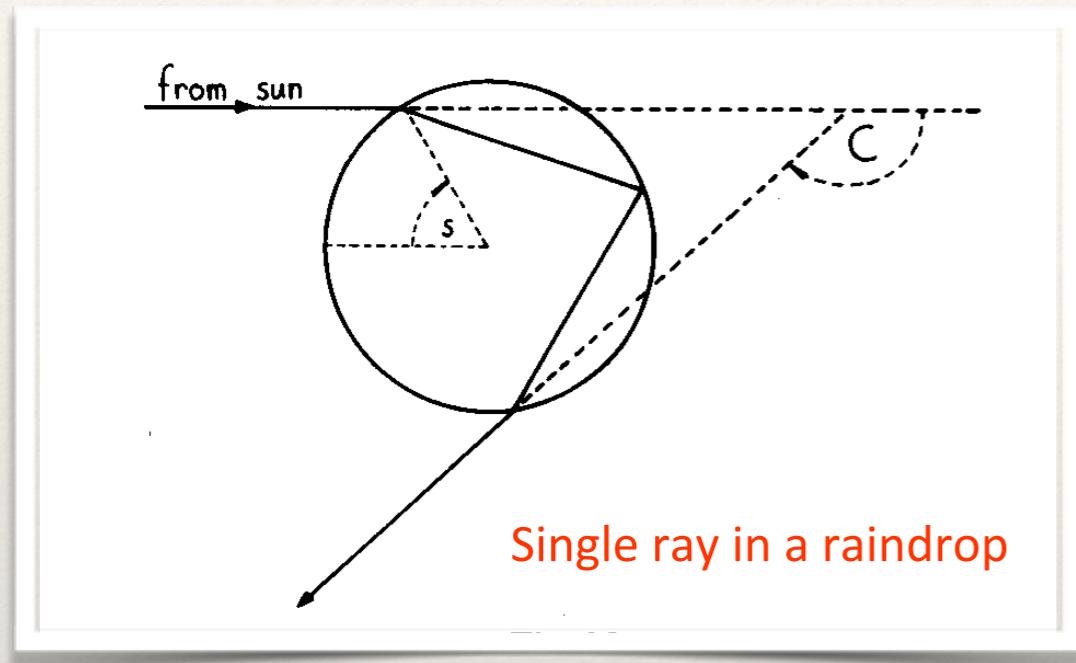
- Light cones in condensed matter systems can be understood as **caustics**
- Underlying mathematical description is catastrophe theory
- Structural stability
- Universality classes each with their own characteristic wavefunctions and scaling exponents
- Universality in dynamics!

For more information see:

J. Mumford, E. Turner, D.W.L. Sprung, and D.H.J. O'Dell, *Quantum Spin Dynamics in Fock Space Following Quenches: Caustics and Vortices*, Phys. Rev. Lett. 122, 170402 (2019).

W. Kirkby, J. Mumford and D.H.J. O'Dell, *Light-cones and quantum caustics in quenched spin chains* [arXiv: 1710.01289](#)

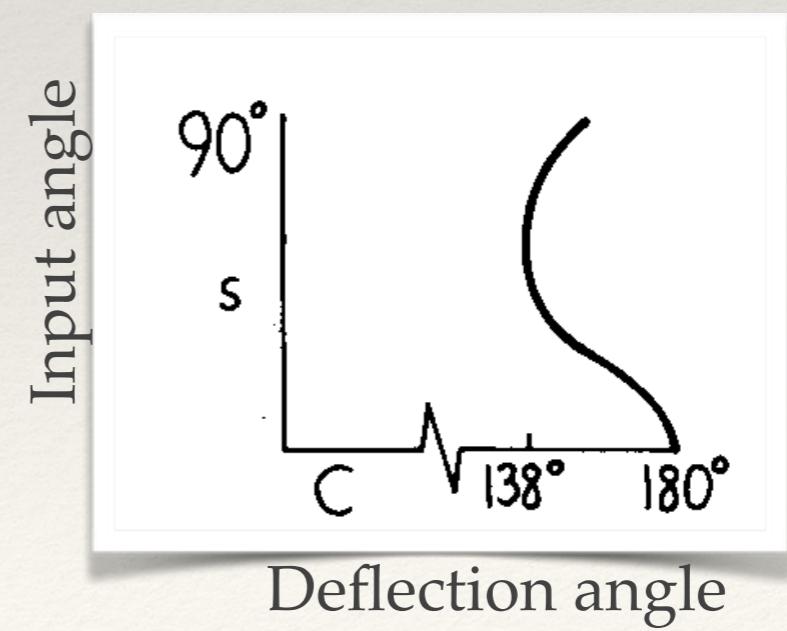
Fold caustic: the rainbow



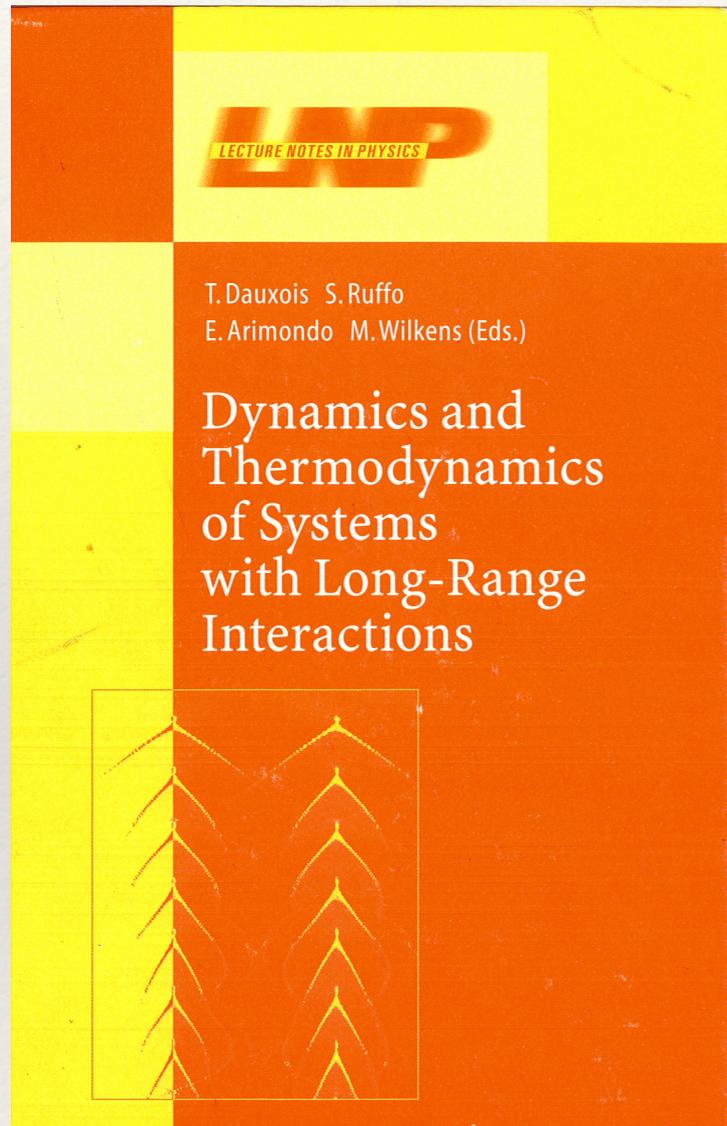
Caustic = envelope of a family of rays

In ray theory the light intensity **diverges** on a caustic: “a lot goes into a little”

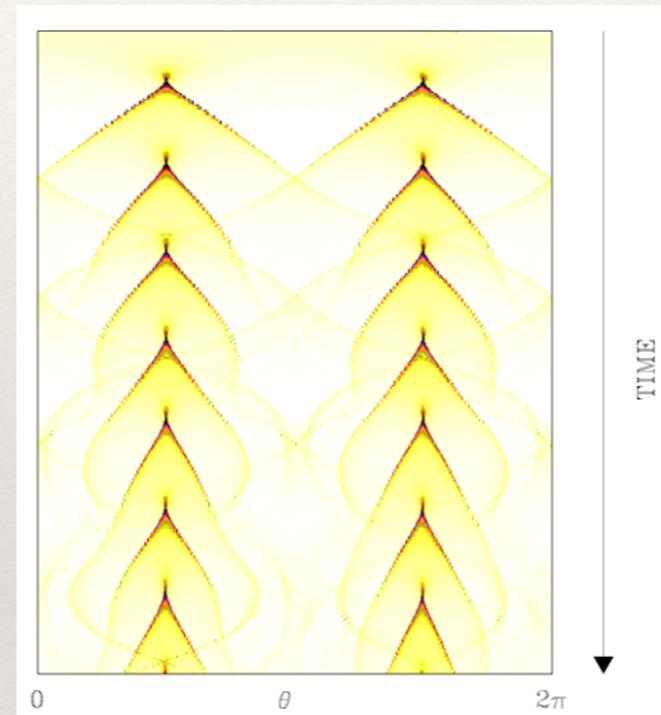
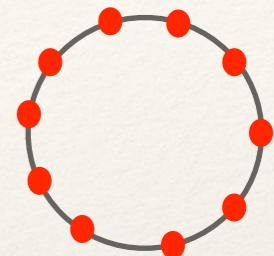
Caustics are the singularities of ray theory,
i.e. places where it fails



Dynamics of N particles on a ring



$$\sum_i \frac{p_i^2}{2} + \frac{\epsilon}{2N} \sum_{i,j} [1 - \cos(\theta_i - \theta_j)]$$



Particle density as a function of time. Initial density on ring at $t=0$ is uniform.
Interaction is *repulsive*.

Julien Barré, Freddy Bouchet, Thierry Dauxois, and Stefano Ruffo, Phys. Rev. Lett. **89**, 110 (2002). **T. Dauxois, V. Latora, A. Rapisarda, S. Ruffo, A. Torcini**, “The Hamiltonian Mean Field Model: from Dynamics to Statistical Mechanics and back” in: **T. Dauxois, S. Ruffo, E. Arimondo, M. Wilkens (Eds.)**, Dynamics and Thermodynamics of Systems with Long-Range Interactions, Lecture Notes in Physics 602, Springer (2002).

Molecular wave packets

Angular Focusing, Squeezing, and Rainbow Formation in a Strongly Driven Quantum Rotor

I. Sh. Averbukh and R. Arvieu

PRL 87, 163601 (2001)

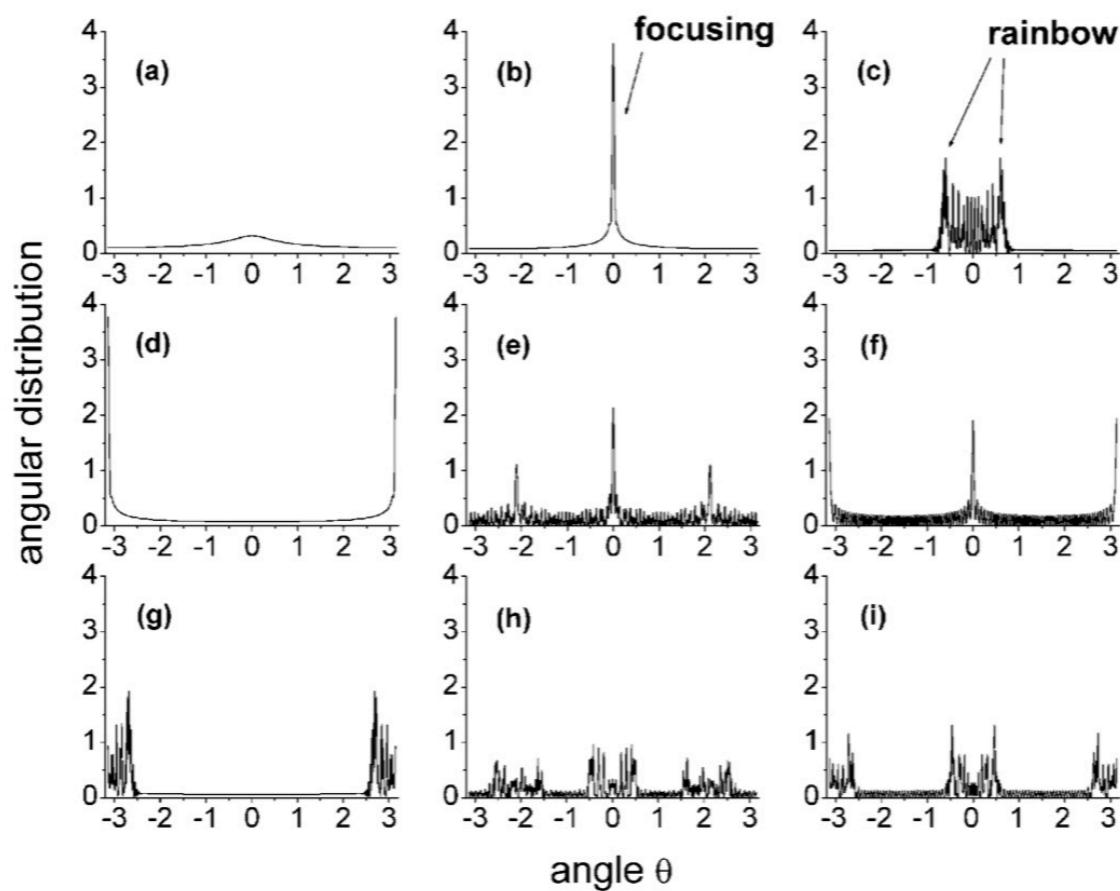


FIG. 1. Angular distribution of a quantum rotor excited by a strong δ kick ($P = 85$). The graphs correspond to (a) $\tau = 0.5\tau_f$, (b) $\tau = \tau_f$, (c) $\tau = 2\tau_f$, (d) $\tau = \tau_f + T_{\text{rev}}/2$, (e) $\tau = \tau_f + T_{\text{rev}}/3$, (f) $\tau = \tau_f + T_{\text{rev}}/4$, (g) $\tau = 1.8\tau_f + T_{\text{rev}}/2$, (h) $\tau = 1.8\tau_f + T_{\text{rev}}/3$, and (i) $\tau = 1.8\tau_f + T_{\text{rev}}/4$, respectively.

Caustics in atom optics

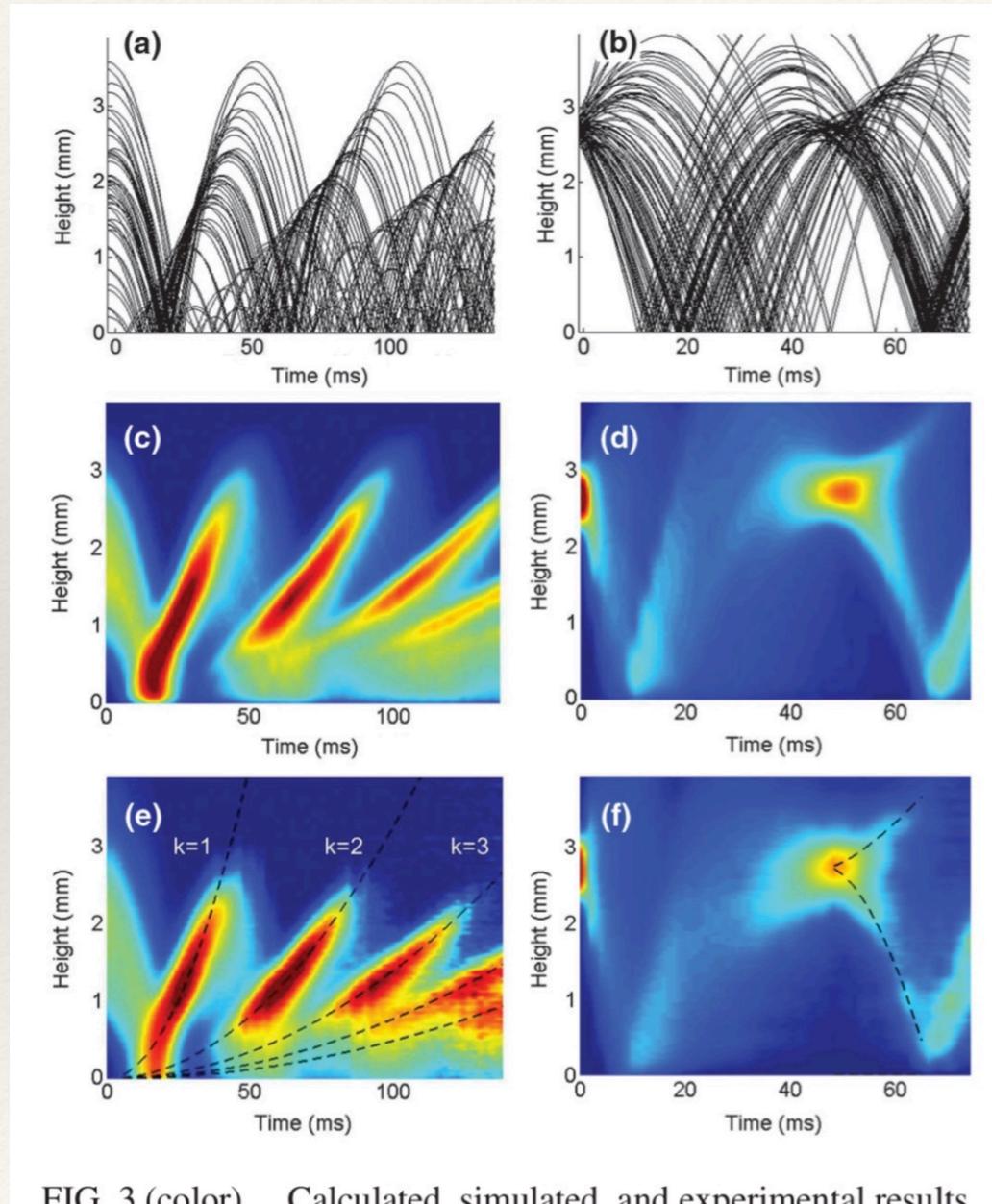


FIG. 3 (color). Calculated, simulated, and experimental results.

Demonstration of Fold and Cusp Catastrophes in an Atomic Cloud Reflected from an Optical Barrier in the Presence of Gravity

PRL 112, 120403 (2014)

Serge Rosenblum, Orel Bechler, Itay Shomroni, Roy Kaner, Talya Arusi-Parpar, Oren Raz, and Barak Dayan

Gravity-driven catastrophes

The Large Scale Structure of the Universe I. General Properties. One- and Two-Dimensional Models

V. I. ARNOLD

Moscow State University, U.S.S.R.

and

S. F. SHANDARIN and YA. B. ZELDOVICH

Institute of Applied Mathematics, Moscow, U.S.S.R.

(Received August 11, 1981)

Evolution of initially smooth perturbations in a cold self-gravitating medium in a Friedmann Universe gives rise to the formation of singularities in the distribution of density in a manner similar to that of catastrophe theory. Using the approximate nonlinear theory of gravitational instability the objects formed were found to possess a very oblate shape—a “pancake” (Zeldovich, 1970). This result is shown in this paper to be a general feature of the evolution of so-called Lagrangian systems (Arnold, 1980). Pancakes are one of the several kinds of generic singularity formed at the nonlinear stage of evolution of such a system. Some of the others are a cusp, a beak-to-beak, a swallow-tail. In this paper we present the full list of singularities for the one- and two-dimensional cases (Figures 1–9). The three dimensional singularities will be discussed in Part II of the paper. We discuss the geometrical and some dynamical properties of each kind of singularity. We give also asymptotic laws for the growth of the density near each kind of singularity. This list of singularities gives the elements from which the large scale structure of the Universe is constructed.

Our scheme naturally explains the existence of the flattened superclusters, as well as the rich clusters of galaxies by themselves and their chains. It is useful to note that the process of formation of smaller objects (galaxies) must be considered against the background of structures described in this paper.

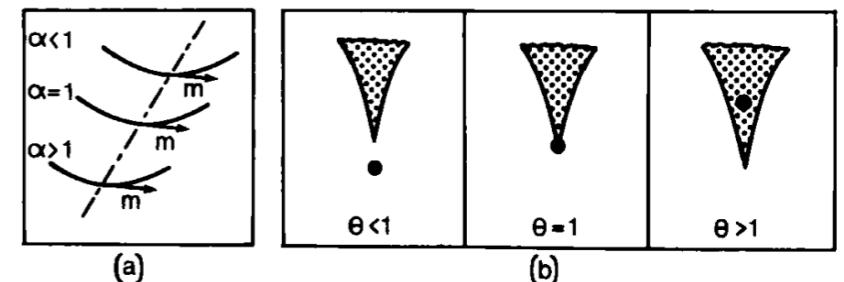


FIGURE 3 An edge of a pancake (“cusp” type A_3). The same designations as in Figure 2. The dash-dot line is formed by the edge points realized at different times.

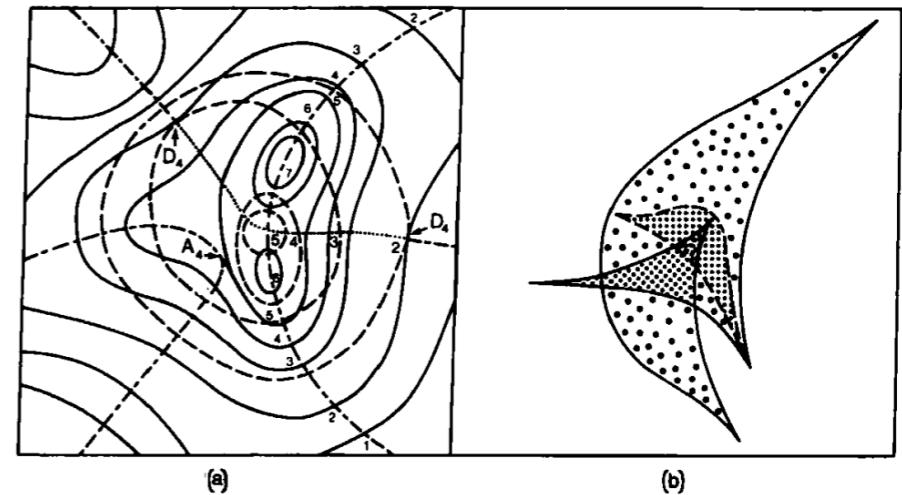


FIGURE 11 An example of the real structure arising in numerical simulations with statistical initial conditions. The designations are the same as in previous figures. (b) represents the Eulerian plane at the time of realization of one of two singularities of type D_4 (“purse”).

Self-similarity: scaling exponents

Catastrophe	Arnold Index β	Berry Indices σ_j	Berry Index γ
Fold	1/6	2/3	2/3
Cusp	1/4	3/4, 1/2	5/4
Swallowtail	3/10	4/5, 3/5, 2/5	9/5
Elliptic umbilic	1/3	2/3, 2/3, 1/3	5/3
Hyperbolic umbilic	1/3	2/3, 2/3, 1/3	5/3
Butterfly	1/3	5/6, 2/3, 1/2, 1/3	7/3
Parabolic umbilic	3/8	5/8, 3/4, 1/2, 1/4	17/8

e.g. Cusp (Pearcey function)

$$\Psi_{\text{cusp}}(C_1, C_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(s^4/4 + C_2 s^2/2 + C_1 s)} ds$$

$$\Psi_{\text{physical}} \{C_1, C_2; N_2\} = \left(\frac{N_2}{N_1}\right)^{\beta} \Psi \left\{ \left(\frac{N_2}{N_1}\right)^{\sigma_1} C_1, \left(\frac{N_2}{N_1}\right)^{\sigma_2} C_2; N_1 \right\}$$