# Predictions for the Dirac CP-Violating Phase from Sum Rules

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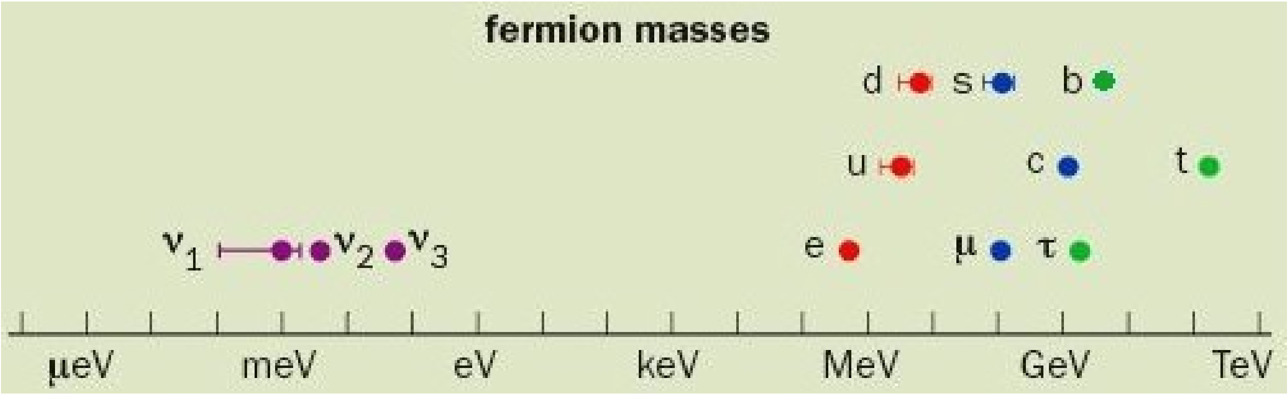
Based on: L.A. Delgadillo, L.L. Everett, R. Ramos and AS, Phys.Rev. D97 (2018) no.9, 095001 [arXiv:1801.06377]

# The Standard Model

Triumph of modern science, but incomplete-
Fails to predict the measured fermion masses and mixings.


http://www.particleadventure.org/standard_model.html

# What We Measure




## Quark Mixing

$$U_{CKM} = R_1(\theta_{23}^{CKM})R_2(\theta_{13}^{CKM}, \delta_{CKM})R_3(\theta_{12}^{CKM})$$

$$\theta_{13}^{CKM} = 0.2° \pm 0.1°$$

$$\theta_{23}^{CKM} = 2.4° \pm 0.1°$$

$$\theta_{12}^{CKM} = 13.0° \pm 0.1°$$

$$\delta_{CKM} = 60° \pm 14°$$

## Lepton Mixing

$$U_{PMNS} = R_1(\theta_{23})R_2(\theta_{13}, \delta_{CP})R_3(\theta_{12})P$$

NuFIT 4.0 (2018): 1811.05487

$$\theta_{13}^{MNSP} = (8.61°)^{+0.13}_{-0.13}$$

$$\theta_{23}^{MNSP} = (49.6°)^{+1.0}_{-1.2}$$

$$\theta_{12}^{MNSP} = (33.82°)^{+0.78}_{-0.76}$$

$$\delta_{CP} = (215°)^{+40}_{-29}$$


http://lbne.fnal.gov/how-work.shtml

Quarks look like deviations from unity. What about the leptons?

# Popular Starting Points

$$\tilde{U}_\nu = R_{23}\left(\theta_{23}^\nu\right) R_{12}\left(\theta_{12}^\nu\right) = \begin{pmatrix} \cos\theta_{12}^\nu & \sin\theta_{12}^\nu & 0 \\ -\frac{\sin\theta_{12}^\nu}{\sqrt{2}} & \frac{\cos\theta_{12}^\nu}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sin\theta_{12}^\nu}{\sqrt{2}} & \frac{\cos\theta_{12}^\nu}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Marzocca, et al. (2013)
Petcov (2014)
Girardi, Petcov, Titov (2015)

$$r = (1 + \sqrt{5})/2$$

TriBiMaximal (TBM) Mixing: $\sin^2\theta_{12}^\nu = 1/3$ (P. Harrison, D. Perkins, W. Scott (2002) ; Z. Xing (2002); X. He, A. Zee (2003))

BiMaximal (BM) Mixing: $\sin^2\theta_{12}^\nu = 1/2$ F. Vissani (1997); V. Barger, S. Pakvasa, T. Weiler, K. Whisnant, (1998); A. Baltz, A. Goldhaber, M. Goldhaber (1998)

Golden Ratio A (GRA) Mixing: $\sin^2\theta_{12}^\nu = (2+r)^{-1} \cong 0.276$ A. Datta, F. Ling, P. Ramond (2003); L. Everett, AS (2008)

Golden Ratio B (GRB) Mixing: $\sin^2\theta_{12}^\nu = (3-r)/4 \cong 0.345$ W. Rodejohann (2009);

HexaGonal (HG) Mixing: $\sin^2\theta_{12}^\nu = 1/4$ C. Albright, A. Dueck and W. Rodejohann (2010); J. E. Kimand M. Seo (2011)

These starting points can be shown to come from simple discrete flavor symmetry groups like $A_4, S_4, A_5$, etc…

However, they all have a vanishing reactor mixing angle $\theta_{13}$. How can we fix this?

# Charged Lepton Corrections

$$U_{\rm MNSP} = U_e^\dagger U_\nu$$

As on previous slide, assume the neutrino mixing matrix is given as $U_\nu = R_{23}(\theta_{23}^\nu) R_{12}(\theta_{12}^\nu)$.

$$R_{23}^\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^\nu & s_{23}^\nu \\ 0 & -s_{23}^\nu & c_{23}^\nu \end{pmatrix} \qquad R_{12}^\nu = \begin{pmatrix} c_{12}^\nu & s_{12}^\nu & 0 \\ -s_{12}^\nu & c_{12}^\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Postulate a theoretical form for $U_e$, *i.e.,* one rotation, two rotations, etc. For simplicity assume $U_e$ consists of only one rotation. Then it can be a 1-3 rotation or a 1-2 rotation (a 2-3 rotation only shifts the atmospheric angle).

$$U_{12}^e = \begin{pmatrix} c_{12}^e & s_{12}^e e^{-i\delta_{12}^e} & 0 \\ -s_{12}^e e^{i\delta_{12}^e} & c_{12}^e & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U_{13}^e = \begin{pmatrix} c_{13}^e & 0 & s_{13}^e e^{-i\delta_{13}^e} \\ 0 & 1 & 0 \\ -s_{13}^e e^{i\delta_{13}^e} & 0 & c_{13}^e \end{pmatrix}$$

$$s_{ij}^e = \sin\theta_{ij}^e \text{ and } c_{ij}^e = \cos\theta_{ij}^e$$

First focus on a 1-2 charged lepton rotation. How does this charged lepton rotation change the initial mixing predictions?

$$U_e = U_{12}^e(\theta_{12}^e, \delta_{12}^e)$$

# The Effect of a 1-2 Rotation

$$U_{\mathsf{MNSP}} \equiv U = U_e^\dagger U_\nu = U_{12}^{e\dagger} R_{23}^\nu R_{12}^\nu$$

$$U_{e1} = c_{12}^e c_{12}^\nu + c_{23}^\nu e^{-i\delta_{12}^e} s_{12}^e s_{12}^\nu, \quad U_{e2} = c_{12}^e s_{12}^\nu - c_{12}^\nu c_{23}^\nu e^{-i\delta_{12}^e} s_{12}^e;$$

$$U_{e3} = -e^{-i\delta_{12}^e} s_{12}^e s_{23}^\nu, \quad U_{\mu 1} = -c_{12}^e c_{23}^\nu s_{12}^\nu + c_{12}^\nu e^{i\delta_{12}^e} s_{12}^e$$

$$U_{\mu 2} = c_{12}^e c_{12}^\nu c_{23}^\nu + e^{i\delta_{12}^e} s_{12}^e s_{12}^\nu, \quad U_{\mu 3} = c_{12}^e s_{23}^\nu,$$

$$U_{\tau 1} = s_{12}^\nu s_{23}^\nu, \quad U_{\tau 2} = -c_{12}^\nu s_{23}^\nu,$$

$$U_{\tau 3} = c_{23}^\nu.$$

Compare this to the PDG parameterization of the MNSP Matrix:

$$U^{\mathsf{PDG}} = \begin{pmatrix} c_{12}c_{13} & c_{13}s_{12} & e^{-i\delta}s_{13} \\ -c_{23}s_{12} - c_{12}e^{i\delta}s_{13}s_{23} & c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13} & -c_{12}s_{23} - c_{23}e^{i\delta}s_{12}s_{13} & c_{13}c_{23} \end{pmatrix} P_{\mathsf{Maj}}$$

By equating both parameterizations, it should become clear that it is possible to express the PDG parameters in terms of the model parameters….

# Enter Sum Rules

Specifically, one can find a relationship between the Dirac CP-violating phase δ, the experimentally measured PDG, and the model parameters, i.e.,

$$\cos\delta = \frac{(1/t_{23} + s_{13}^2 t_{23})(s_{12}^{\nu})^2 - (s_{12}^2/t_{23} + c_{12}^2 s_{13}^2 t_{23})}{s'_{12} s_{13}}$$

(D. Marzocca, S.T. Petcov, et al. (2011); D. Marzocca, S.T. Petcov, et al. (2013); P. Ballet, S.F. King, et al. (2014) S.T. Petcov (2015); I. Girardi, S.T. Petcov, et al. (2015);)

Can be derived easily by taking the ratio (P. Ballet, S.F. King, et al. (2014)):

$$\frac{|U_{\tau 1}|}{|U_{\tau 2}|} = t_{12}^{\nu} \qquad \frac{|U_{\tau 1}^{PDG}|}{|U_{\tau 2}^{PDG}|} = \frac{|s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|}{|c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta}|}$$

In principle there exists 9!/2!7!-1=35 other additional ratios that must hold to guarantee the unitarity of the matrices. For example…...

# More Examples of Sum Rules

Continue by considering the rest of the ratios in the 3rd row of the MNSP matrix:

$$\frac{|U_{\tau 1}|}{|U_{\tau 3}|} = s^{\nu}_{12} t^{\nu}_{23}$$

$$s^{\nu}_{12} t^{\nu}_{23} = \frac{|s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta}|}{|c_{13}c_{23}|}$$

$$\cos\delta = \frac{1}{t_{23}s_{13}\sin(2\theta_{12})}(s^2_{12}t^2_{23} + c^2_{12}s^2_{13} - s^{\nu 2}_{12}t^{\nu 2}_{23}c^2_{13})$$

"Sum rule 2"

$$\frac{|U_{\tau 2}|}{|U_{\tau 3}|} = c^{\nu}_{12} t^{\nu}_{23}$$

$$c^{\nu}_{12} t^{\nu}_{23} = \frac{|c_{12}s_{23} + c_{23}s_{12}s_{13}e^{i\delta}|}{|c_{13}c_{23}|}$$

$$\cos\delta = \frac{1}{t_{23}s_{13}\sin(2\theta_{12})}(c^{\nu 2}_{12}t^{\nu 2}_{23}c^2_{13} - c^2_{12}t^2_{23} - s^2_{12}s^2_{13})$$

"Sum Rule 3"

We can continue taking ratios of elements to find out that these ratios can be classified.

# Classification of Sum Rules

**The sum rules fall into 4 cases:**

I. They can involve involve only $\cos\delta$

II. They can involve only $\cos(\delta^e_{12})$

III. They can involve both phases.

IV. They can involve neither phase.

Start by considering Case IV:

$$cs_{ij} = \csc(\theta_{ij})$$
$$sc_{ij} = \sec(\theta_{ij})$$
$$ct_{ij} = \cot(\theta_{ij})$$

$$|U_{e3}|/|U_{\tau 3}| : \quad (s^e_{12})^2 (t^\nu_{23})^2 = sc^2_{23} t^2_{13}$$
$$|U_{\mu 3}|/|U_{\tau 3}| : \quad (c^e_{12})^2 (t^\nu_{23})^2 = t^2_{23},$$
$$|U_{e3}|/|U_{\mu 3}| : \quad (t^e_{12})^2 = cs^2_{23} t^2_{13}$$

What about the other categories?

# Case I Sum Rules

There exists 7 sum rules of this type:

$$2\cos\delta = (c^{\nu}_{12})^2\left[cs_{13}ct_{12}t_{23}(t^2_{12}-(t^{\nu}_{12})^2)+ct_{23}s_{13}t_{12}(ct^2_{12}-(t^{\nu}_{12})^2)\right],$$

$$2\cos\delta = ct_{23}\left[ct_{12}s_{13}-c_{13}cs_{12}ct_{13}sc_{12}(s^{\nu}_{12})^2(t^{\nu}_{23})^2\right]+cs_{13}t_{12}t_{23},$$

$$2\cos\delta = ct_{23}\left[c_{13}cs_{12}ct_{13}sc_{12}(c^{\nu}_{12})^2(t^{\nu}_{23})^2-s_{13}t_{12}\right]-cs_{13}ct_{12}t_{23},$$

$$2\cos\delta = -cs_{12}cs_{23}s_{13}sc_{12}sc_{23}(cs^{e}_{12})^2(s^{\nu}_{12})^2+cs_{13}t_{12}t_{23}+ct_{12}ct_{23}s_{13},$$

$$2\cos\delta = cs_{12}cs_{23}s_{13}sc_{12}sc_{23}(c^{\nu}_{12})^2(cs^{e}_{12})^2-cs_{13}ct_{12}t_{23}-ct_{23}s_{13}t_{12},$$

$$2\cos\delta = t_{23}\left[cs_{13}t_{12}-c_{13}cs_{12}ct_{13}sc_{12}(sc^{e}_{12})^2(s^{\nu}_{12})^2\right]+ct_{12}ct_{23}s_{13},$$

$$2\cos\delta = c_{13}cs_{12}ct_{13}sc_{12}t_{23}(c^{\nu}_{12})^2(sc^{e}_{12})^2-cs_{13}ct_{12}t_{23}-ct_{23}s_{13}t_{12}.$$

These are all very complicated but all must be yield the same value for cos(δ).

# Sum Rules (Case II)

For this case there are another 7 sum rules:

$$|U_{e1}|/|U_{e3}| : 2\cos(\delta^e_{12}) = t^e_{12}\left(c^2_{12}ct^2_{13}cs^\nu_{12}s^\nu_{23}sc^\nu_{12}t^\nu_{23} - c^\nu_{23}t^\nu_{12}\right) - ct^e_{12}ct^\nu_{12}sc^\nu_{23},$$

$$|U_{e2}|/|U_{e3}| : 2\cos(\delta^e_{12}) = c^\nu_{23}ct^\nu_{12}t^e_{12} - ct^2_{13}s^2_{12}cs^\nu_{12}t^e_{12}s^\nu_{23}sc^\nu_{12}t^\nu_{23} + ct^e_{12}sc^\nu_{23}t^\nu_{12},$$

$$|U_{e2}|/|U_{e1}| : 2\cos(\delta^e_{12}) = c^2_{12}\left[c^\nu_{23}t^e_{12}t^\nu_{12}\left((ct^\nu_{12})^2 - t^2_{12}\right) + ct^e_{12}\left(sc^\nu_{23}t^\nu_{12} - t^2_{12}ct^\nu_{12}sc^\nu_{23}\right)\right]$$

$$|U_{e1}|/|U_{\mu 3}| : 2\cos(\delta^e_{12}) = ct^e_{12}\left(c^2_{12}cs^2_{23}cs^\nu_{12}s^\nu_{23}sc^\nu_{12}t^\nu_{23} - ct^\nu_{12}sc^\nu_{23}\right) - c^\nu_{23}t^e_{12}t^\nu_{12},$$

$$|U_{e1}|/|U_{\tau 3}| : 2\cos(\delta^e_{12}) = c^\nu_{23}\left(c^2_{12}sc^2_{23}cs^e_{12}cs^\nu_{12}sc^e_{12}sc^\nu_{12} - t^e_{12}t^\nu_{12}\right) - ct^e_{12}ct^\nu_{12}sc^\nu_{23},$$

$$|U_{e2}|/|U_{\mu 3}| : 2\cos(\delta^e_{12}) = c^\nu_{23}ct^\nu_{12}t^e_{12} + ct^e_{12}\left(sc^\nu_{23}t^\nu_{12} - cs^2_{23}s^2_{12}cs^\nu_{12}s^\nu_{23}sc^\nu_{12}t^\nu_{23}\right),$$

$$|U_{e2}|/|U_{\tau 3}| : 2\cos(\delta^e_{12}) = c^\nu_{23}\left(ct^\nu_{12}t^e_{12} - s^2_{12}sc^2_{23}cs^e_{12}cs^\nu_{12}sc^e_{12}sc^\nu_{12}\right) + ct^e_{12}sc^\nu_{23}t^\nu_{12}.$$

All of these sum rules must be equal because they result from the same unitary matrix…. Case III has the remaining 19 sum rules, but they are too complicated to write, but will be used in analyses… We can solve this set of 36 equation to reveal….

# A Simple set of 4 Sum Rules

The 36 sum rules, upon equating parameters, reduce to only 4 sum rules:

$$\cos\delta = \frac{1}{s'_{12}s_{13}|c^{\nu}_{23}|\sqrt{(s^{\nu}_{23})^2 - s^2_{13}}}\left(((s^{\nu}_{23})^2 - s^2_{13})s^2_{12} + s^2_{13}c^2_{12}(c^{\nu}_{23})^2 - (s^{\nu}_{12})^2(s^{\nu}_{23})^2c^2_{13}\right)$$

(Equivalent to original result when the remainder of sum rules are imposed:)

$$\cos(\delta^e_{12}) = \frac{\mathrm{sign}(sc_{23})cs_{23}sc_{23}cs^{\nu}_{12}sc^{\nu}_{12}\left[sc^{\nu}_{23}c'^{\nu}_{12}(c'_{23} - (c^{\nu}_{23})^2) + s^{\nu}_{23}t^{\nu}_{23}c'_{12}\right]}{4\sqrt{(t^{\nu}_{23})^2 - t^2_{23}}}$$

$$(t^e_{12})^2 = \frac{t^2_{13}}{s^2_{23}}$$

$$t^2_{13} = c^2_{23}\left[(t^{\nu}_{23})^2 - t^2_{23}\right]$$

What do these sum rules look like for a 1-3 charged lepton rotation?

# Sum Rules for a 1-3 Charged Lepton Rotation

$$U_e = U_{13}^e(\theta_{13}^e, \delta_{13}^e)$$

Implementing the same methodology as in the 1-2 rotation case, i.e., taking all 36 ratios then solving that system and equating all of their parameters, the 1-3 sum rules are revealed to be (see 1801.06377 for complete listing of original 36):

$$\sin^2(\theta_{13}) = (c_{23}^\nu)^2(s_{13}^e)^2$$

$$\sin^2(\theta_{23}) = \frac{(s_{23}^\nu)^2}{1-(c_{23}^\nu)^2(s_{13}^e)^2} = \frac{(s_{23}^\nu)^2}{1-\sin^2\theta_{13}}$$

$$\sin^2(\theta_{12}) = \frac{(c_{13}^e)^2(s_{12}^\nu)^2 + 2\cos\delta_{13}^e c_{12}^\nu c_{13}^e s_{12}^\nu s_{13}^e s_{23}^\nu + (c_{12}^\nu)^2(s_{13}^e)^2(s_{23}^\nu)^2}{1-\sin^2\theta_{13}}$$

$$\cos\delta = \frac{1}{s_{12}' s_{13}|s_{23}^\nu|\sqrt{c_{13}^2-(s_{23}^\nu)^2}}\left((s_{12}^\nu)^2 c_{13}^2 (c_{23}^\nu)^2 - s_{12}^2 c_{13}^2 + (s_{12}^2 - s_{13}^2 c_{12}^2)(s_{23}^\nu)^2\right)$$

Now let us analyze the allowed parameter space of cos(δ) using these sum rules and a particular lepton mixing starting point.

# BM Mixing and a 1-2 Rotation

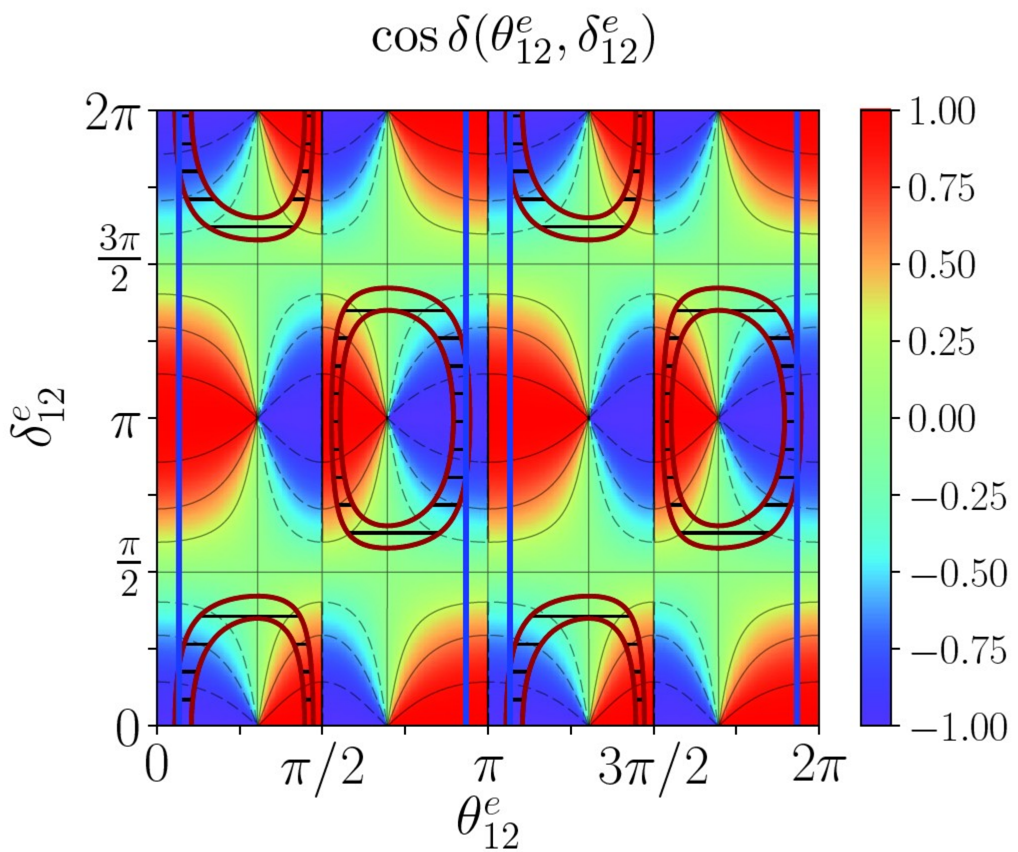
It is possible to showcase the allowed regions of cos(δ) by using a contour plot:

$$\sin^2(\theta_{13}) \equiv s_{13}^2 = (s_{12}^e)^2/2$$

$$s_{23}^2 = \frac{(c_{12}^e)^2/2}{(1-(s_{12}^e)^2/2)} = \frac{1-2s_{13}^2}{2(1-s_{13}^2)}.$$

$$0.4878 \leq s_{23}^2 \leq 0.4904$$

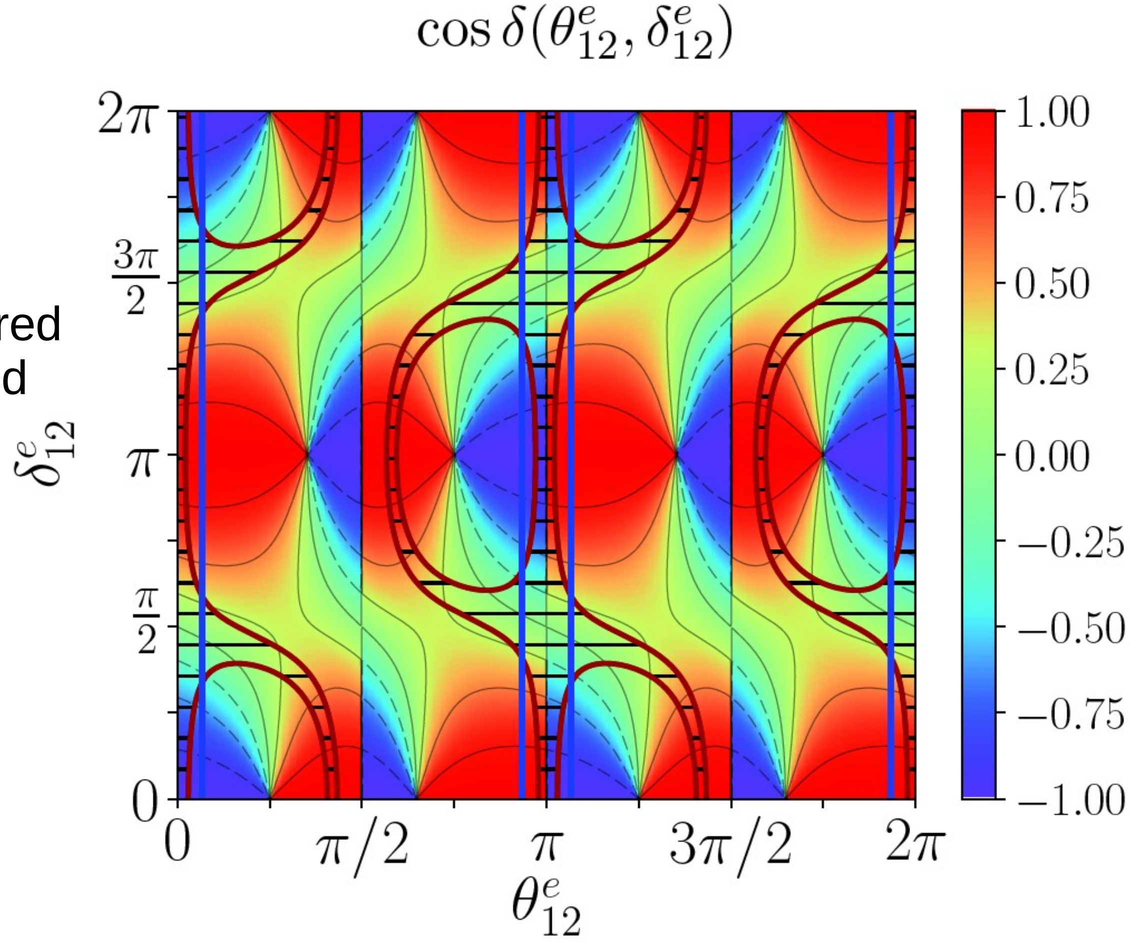
**(First octant!)**




Blue bands and regions between red contours represent regions allowed by $\sin^2(\theta_{13})$
And $\sin^2(\theta_{12})$ at 3σ, respectively. Notice cos(δ)=~-1 is preferred value.

How about for another popular mixing scenario?

# TBM Mixing and a 1-2 Rotation




Blue bands and regions between red contours represent regions allowed by $\sin^2(\theta_{13})$ and $\sin^2(\theta_{12})$ at 3σ, respectively.

$$\sin^2(\theta_{13}) \equiv s_{13}^2 = (s_{12}^e)^2/2$$

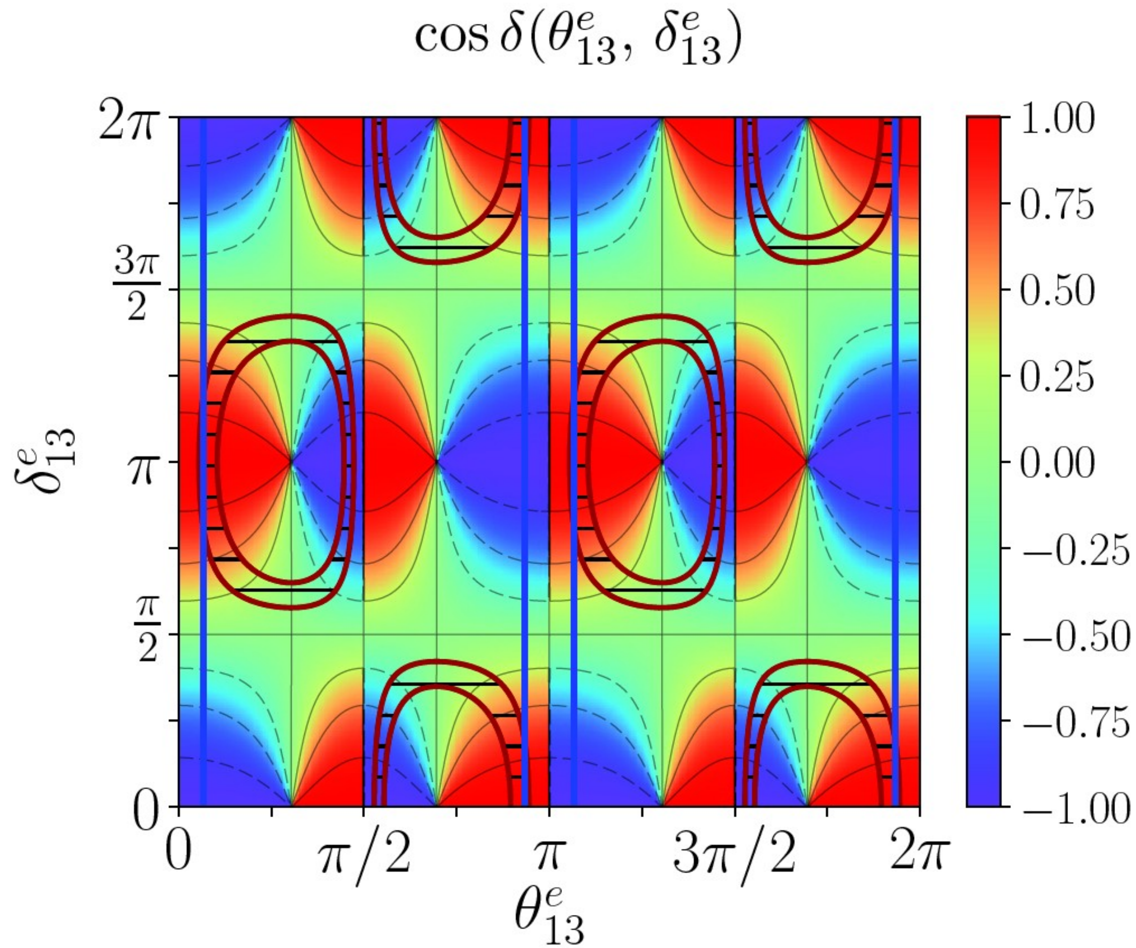
$$s_{23}^2 = \frac{(c_{12}^e)^2/2}{(1-(s_{12}^e)^2/2)} = \frac{1-2s_{13}^2}{2(1-s_{13}^2)}.$$

$$0.4878 \leq s_{23}^2 \leq 0.4904$$

**(First octant!)**

Because the solar mixing angle does not start as maximal (like in BM mixing) there is a larger region of parameter space in which the reactor and solar mixing angle constraints can be satisfied. This also happens for GR1, GR2, and Hexagonal Mixing. Thus, let this case serve as a representative for them. **How do these plots change for a 1-3 rotation?**

# BM Mixing and a 1-3 Rotation




Blue bands and regions between red contours represent regions allowed by $\sin^2(\theta_{13})$ and $\sin^2(\theta_{12})$ at 3σ, respectively.

$$s_{13}^2 = (s_{13}^e)^2/2,$$

$$s_{23}^2 = \frac{1}{2(1-s_{13}^2)},$$

$$0.5096 < s_{23}^2 < 0.5122$$

**(Second octant!)**

Notice again that due to the unperturbed maximal starting point, BM has a preferred value of cos(δ)=1.

# TBM Mixing and a 1-3 Rotation

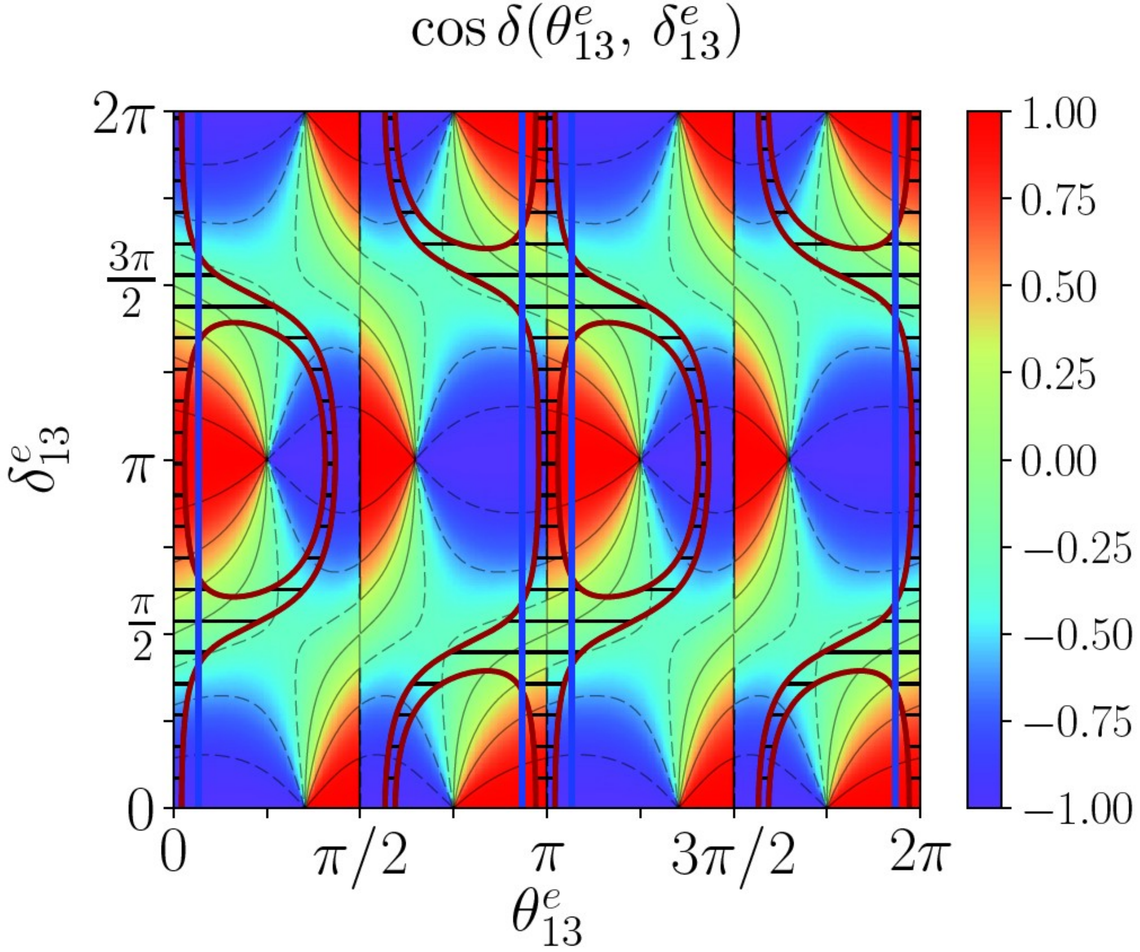
Blue bands and regions between red contours represent regions allowed by $\sin^2(\theta_{13})$ and $\sin^2(\theta_{12})$ at 3σ, respectively.

$$s_{13}^2 = (s_{13}^e)^2/2,$$

$$s_{23}^2 = \frac{1}{2(1 - s_{13}^2)},$$

$$0.5096 < s_{23}^2 < 0.5122$$

**(Second octant!)**




We see overlap of allowed regions in case of TBM and a 1-2 rotation and a small shift in the allowed values for the phase $\delta^e_{13}$.

# Conclusion

- The question of why particles have the masses and mixings that they do still remains unsolved, i.e., the flavor problem. Moreover, the exact forms of the mixing matrices $U_e$ and $U_\nu$ remain a mystery.
- By assuming a well-known starting point for $U_\nu$ it is possible to analyze the phenomenological predictions of this starting point by applying single rotation matrix for $U_e$ (see **1801.06377** for the double-rotation cases and more details of the single rotation cases).
- This additional charged lepton rotation gives rise to sum rules which allow for correlations between parameters. Perhaps the most important of these correlations are the correlations between the atmospheric and reactor angles, i.e., the sum rules predict a smaller region of parameter space for the atmospheric angle once the reactor angle measurement is specified.
- Further experimental bounding on the atmospheric angle will have the power to make definitive statements on whether or not such a simple model is ruled out.
- Furthermore, even if atmospheric constraints are satisfied in such a simple model, the results for the solar mixing and CP-violating parameter δ will further separate these scenarios.
- Studies such as this highlight that with the anticipated improvements and measurements of lepton mixing, we may be on the verge of making great progress in understanding the flavor problem.

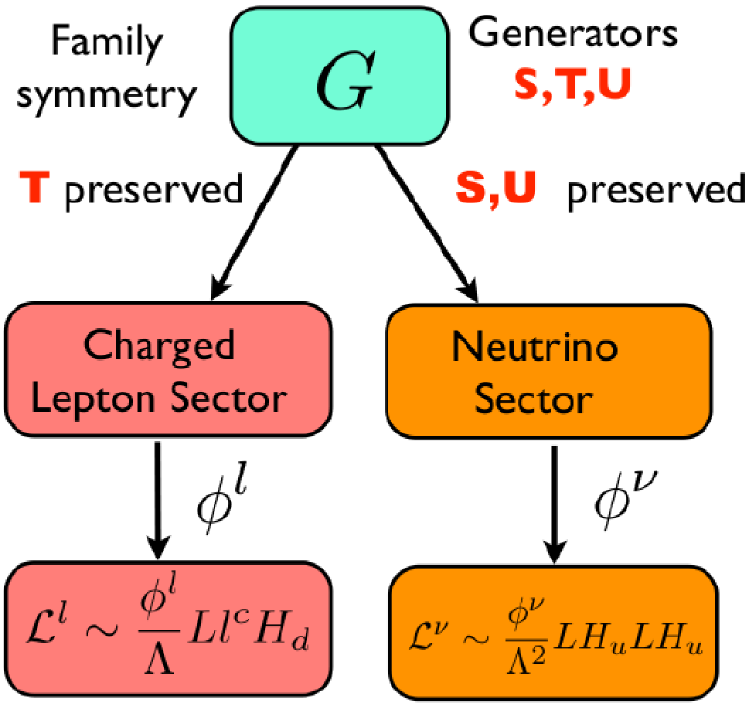
# Backup Slides

# Motivated by Symmetry

Introduce set of flavon fields (e.g. $\phi^\nu$ and $\phi^\ell$ ) whose vevs break $G_f$ to $G_\nu$ in the neutrino sector and $G_e$ in the charged lepton sector.

$$T\langle\phi^\ell\rangle \approx \langle\phi^\ell\rangle \qquad S\langle\phi^\nu\rangle = U\langle\phi^\nu\rangle = \langle\phi^\nu\rangle$$

Non-renormalizable couplings of flavons to mass terms can be used to explain the smallness of Yukawa Couplings.




S.F. King, C. Luhn (2013)

What are some popular examples of flavor symmetries?