Series Expansion of the Quark Mass Renormalization Group Equation: An Application of Rubi

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Regularization and Renormalization

Basic Structure

- Ultraviolet (UV) singularities + Infrared (IR) singularities

Divergences in the high momentum limit

Divergences in the low momentum limit

- Regularization and Renormalization obtains finite and meaningful results from expressions containing divergences

- **Regularization** - Regulator parameter to deal with divergent integrals
  - For UV divergences: Introduce $\epsilon$, the minimal space distance
  - Results will contain terms proportional $1/\epsilon$ which are not well defined in the limit $\epsilon \to 0$
Regularization and Renormalization

• **Renormalization**
  - There must exist observed values equal to some physical quantities that are expressed by seemingly divergent expressions.
  - Removes UV divergences by absorbing divergences into parameters of Lagrangian.
  - Calculating subtraction terms and defining Z coefficients.

• Regularization and renormalization enable us to calculate finite values for many quantities that appear divergent.
Renormalization Group Equations

- Non-physical scale parameter $\mu$ introduced to represent the point where divergences are subtracted to render amplitudes finite

- 2 renormalized quantities: the quark mass $m_q(\mu^2)$ and strong coupling $\alpha_s(\mu^2)$

- Scale dependence governed by corresponding RGE’s which rely on QCD’s anomalous dimensions as input

\[
\frac{da_s}{d \ln s} = \beta(a_s) = -a_s^2 (\beta_0 + a_s \beta_1 + a_s^2 \beta_2 + a_s^3 \beta_3 + a_s^4 \beta_4)
\]

\[
\frac{1}{m_q} \frac{d m_q}{d \ln s} = \gamma(a_s) = -a_s (\gamma_0 + a_s \gamma_1 + a_s^2 \gamma_2 + a_s^3 \gamma_3 + a_s^4 \gamma_4)
\]

where the energy parameter is set such that $s = \mu^2$
Renormalization Group Equations

\begin{align*}
\beta_0 &= \frac{1}{4} \left( 11 - \frac{2}{3} n_f \right) \\
\beta_1 &= \frac{1}{16} \left( 102 - \frac{38}{3} n_f \right) \\
\beta_2 &= \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2 \right) \\
\beta_3 &= \frac{1}{4^4} \left\{ \frac{149753}{6} + 3564 \zeta_3 - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f + \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 \\ &\quad + \frac{1093}{729} n_f^3 \right\} \\
\beta_4 &= \frac{1}{4^5} \left\{ \frac{8157455}{16} + \frac{621885}{2} \zeta_3 - \frac{88209}{2} \zeta_4 - 288090 \zeta_5 \\ &\quad + n_f \left( -\frac{336460813}{1944} - \frac{4811164}{81} \zeta_3 + \frac{33935}{6} \zeta_4 + \frac{1358995}{27} \zeta_5 \right) \\ &\quad + n_f^2 \left( \frac{25960913}{1944} + \frac{698531}{81} \zeta_3 - \frac{10526}{9} \zeta_4 - \frac{381760}{81} \zeta_5 \right) \\ &\quad + n_f^3 \left( -\frac{630559}{5832} - \frac{48722}{243} \zeta_3 + \frac{1618}{27} \zeta_4 + \frac{460}{9} \zeta_5 \right) + n_f^4 \left( \frac{1205}{2916} - \frac{152}{81} \zeta_3 \right) \right\}
\end{align*}
Quark Mass RGE

\[
\frac{da_s}{d \ln s} = \beta(a_s) = -a_s^2 (\beta_0 + a_s \beta_1 + a_s^2 \beta_2 + a_s^3 \beta_3 + a_s^4 \beta_4)
\]

\[
\frac{1}{m_q} \frac{d\bar{m}_q}{d \ln s} = \gamma(a_s) = -a_s (\gamma_0 + a_s \gamma_1 + a_s^2 \gamma_2 + a_s^3 \gamma_3 + a_s^4 \gamma_4)
\]

- Linearly separable differential equation
- More lucid to solve in terms of a power expansion – provides insight into the renormalization scheme dependence of the running quark mass on the energy scale parameter at higher powers
- This is important in accurately determining the light quark mass at a chosen scale.
\[
\frac{d\overline{m}_q}{\overline{m}_q} = \frac{\gamma(a_s)}{\beta(a_s)} \, da_s \\
\ln \left( \frac{\overline{m}_q(s)}{\overline{m}_q(s^*)} \right) = \int_{a_s(s^*)}^{a_s(s)} \, da'_s \frac{\gamma(a'_s)}{\beta(a'_s)} \\
\overline{m}_q(s) = \overline{m}_q(s^*) \exp \left( \int_{a_s(s_0)}^{a_s(s)} \, da'_s \frac{\gamma(a'_s)}{\beta(a'_s)} \right)
\]
Quark Mass RGE

\[ \overline{m}_q(s) = \overline{m}_q(s^*) \exp \left( \int_{a_s(s_0)}^{a_s(s)} \frac{\gamma(a'_s)}{\beta(a'_s)} \right) \]

\[
\begin{align*}
\alpha_s(s) &= \alpha_s(s^*) + \alpha_s^2(s^*) \left( -\beta_0 \eta \right) + \alpha_s^3(s^*) \left( -\beta_1 \eta + \beta_0^2 \eta^2 \right) \\
&\quad + \alpha_s^4(s^*) \left( -\beta_2 \eta + \frac{5}{2} \beta_0 \beta_1 \eta^2 - \beta_0^3 \eta^3 \right) \\
&\quad + \alpha_s^5(s^*) \left( -\beta_3 \eta + \frac{3}{2} \beta_1^2 \eta^2 + 3 \beta_0 \beta_2 \eta^2 - \frac{13}{3} \beta_0^2 \beta_1 \eta^3 + \beta_0^4 \eta^4 \right) \\
&\quad + \alpha_s^6(s^*) \left( -\beta_4 \eta + \frac{7}{2} \beta_0 \beta_1 \eta^2 + \frac{7}{2} \beta_0 \beta_3 \eta^2 - \frac{35}{6} \beta_0 \beta_1^2 \eta^3 - 6 \beta_0^2 \beta_2 \eta^3 \\
&\quad + \frac{77}{12} \beta_0^3 \beta_1 \eta^4 - \beta_0^5 \eta^5 \right),
\end{align*}
\]

where \( \eta \equiv \ln(s/s^*) \) and \( \alpha_s(s) \equiv \frac{\alpha_s(s)}{\pi} \).
Computer Algebra Systems

- Computer Algebra Systems have built-in symbolic integral routines.
- Most based on the Risch Algorithm, which can achieve the integration of any rational fraction.
- But the integral can get so messy once it’s integrated that it is beyond a recoverable form for symbolic software to simplify it.
- Quark mass renormalization integral is just such an example ...

- Use Rubi!

- Instead of applying a recursive algorithm like the Risch, Rubi looks at the integrand and identifies if it matches any of its known integration rules. These rules are in a human readable form like:

\[ \int x^a \, dx = a \cdot x^{a-1} \]
Rule Based Integration (RUBI)

- Implemented as a Mathematica package that gives the user an option to inspect integration steps and application conditions.

- Applies extensive system of symbolic integration rules: Currently has +6600 rules implemented in Mathematica’s pattern matching language.

- Some of these rules are based on well integration formulas (Abramowitz, 2012; Burington, 1973; Gradshteyn, 2014; Zwillinger, 2011).

- Other rules derived during Rubi’s development.

Rule-based Integration (Rubi): An Extensive System of Symbolic Integration Rules
https://rulebasedintegration.org/
Integration tests for Computer Algebra Systems

Summary of Integration Test Results

- Rubi 4.16.1: 99.81%
- Mathematica 11.3: 72.90%
- Maple 2018.2: 54.09%
Rubi’s attempt to solve the Quark Mass RGE

\[ F(a'_s) = \int da'_s \frac{\gamma(a'_s)}{\beta(a'_s)} \]

\[ = \frac{\gamma_0 \ln(a'_s)}{\beta_0} - \frac{1}{4 \beta_0 \beta_4} \left\{ \left( \beta_4 \gamma_0 - \beta_0 \gamma_4 \right) \ln \left( \beta_0 + \beta_1 a'_s + \beta_2 a'_s^2 + \beta_3 a'_s^3 + \beta_4 a'_s^4 \right) \right. \]

\[ + I_0 \left( 3 \beta_1 \beta_4 \gamma_0 - 4 \beta_0 \beta_4 \gamma_1 + \beta_0 \beta_1 \gamma_4 \right) + 2 I_1 \left( \beta_2 \beta_4 \gamma_0 - 2 \beta_0 \beta_4 \gamma_2 + \beta_0 \beta_2 \gamma_4 \right) \]

\[ + I_2 \left( \beta_3 \beta_4 \gamma_0 - 4 \beta_0 \beta_4 \gamma_3 + 3 \beta_0 \beta_3 \gamma_4 \right) \left\} \right. \]

where

\[ I_n = \int da'_s \frac{da'_n}{\beta_0 + \beta_1 a'_s + \beta_2 a'_s^2 + \beta_3 a'_s^3 + \beta_4 a'_s^4} \]

• At this stage, Mathematica is able to re-write these integrals in terms of RootSum objects (without logarithmic divergences)
QCD Renormalization

Results

\[ \bar{m}_q(s) = \bar{m}_q(s^*) \left\{ 1 - a(s^*) \gamma_0 \eta + \frac{1}{2} a^2(s^*) \eta \left[ -2 \gamma_1 + \gamma_0 (\beta_0 + \gamma_0) \eta \right] \ight. \\
- \frac{1}{6} a^3(s^*) \eta \left[ 6 \gamma_2 - 3 (\beta_1 \gamma_0 + 2 (\beta_0 + \gamma_0) \gamma_1) \eta + \gamma_0 (2 \beta_0^2 + 3 \beta_0 \gamma_0 + \gamma_0^2) \eta^2 \right] \\
+ \frac{1}{24} a^4(s^*) \eta \left[ -24 \gamma_3 + 12 (\beta_2 \gamma_0 + 2 \beta_1 \gamma_1 + \gamma_1^2 + 3 \beta_0 \gamma_2 + 2 \gamma_0 \gamma_2) \eta \\
- 4 (6 \beta_0^2 \gamma_1 + 3 \gamma_0^2 (\beta_1 + \gamma_1) + \beta_0 \gamma_0 (5 \beta_1 + 9 \gamma_1) \eta^2 + \gamma_0 (6 \beta_0^3 + 11 \beta_0^2 \gamma_0 \\
+ 6 \beta_0 \gamma_0^2 + \gamma_0^3) \eta^3 \right] \\
+ \frac{1}{120} a^5(s^*) \eta \left[ -120 \gamma_4 + \frac{1}{\beta_0} 60 \left( -7 \beta_1 \beta_2 \gamma_0 + 4 \beta_0^2 \gamma_3 + \beta_0 (7 \beta_1 \gamma_0 + \beta_3 \gamma_0 \\
+ 2 \beta_2 \gamma_1 + 3 \beta_1 \gamma_2 + 2 \beta_1 \gamma_2 + 2 \gamma_0 \gamma_3) \right) \eta - 20 \left( 3 \beta_1^2 \gamma_0 + \beta_1 (14 \beta_0 + 9 \gamma_0) \gamma_1 \\
+ 3 (2 \beta_0 + \gamma_0) (\beta_2 \gamma_0 + \gamma_1^2 + 2 \beta_0 \gamma_2 + \gamma_0 \gamma_2) \eta^2 + 10 \left( 12 \beta_0^3 \gamma_1 + \gamma_0^3 (3 \beta_1 + 2 \gamma_1) \\
+ \beta_0 \gamma_0^2 (13 \beta_1 + 12 \gamma_1) + \beta_0^2 \gamma_0 (13 \beta_1 + 22 \gamma_1) \right) \eta^3 - \gamma_0 \left( 24 \beta_0^4 + 50 \beta_0^3 \gamma_0 \\
+ 35 \beta_0^2 \gamma_0^2 + 10 \beta_0 \gamma_0^3 + \gamma_0^4 \right) \eta^4 \right] + \mathcal{O}(a^6(s^*)) \right\} \]
The local error function, \( f(s_j) = r(s_j) - k(s_j) \), where \( r(s_j) \) is the reference value of \( m_{ud}(s_j) \) with a scale dependence calculated by direct numerical integration of the quark mass RG equation, and \( k(s_j) \) is the value of \( m_{ud}(s_j) \) with a scale dependence as either the five-loop series expansion (orange) or the four-loop series expansion (blue).
Conclusion

- Effect of fifth-loop correction term in the quark mass renormalization perturbative expansion is small, but does serve to increase it’s accuracy by about 0.5%

- **Result is not as important as the method to obtain the result**

- Can achieve same result using Mathematica only

- But requires unintuitive method and only works if series expansion converges uniformly

- The case for using **Rubi** as a tool in this situation, and in other Science, Technology, Engineering and Mathematics (STEM) research areas, is thus: it provides a **lucid and intuitive approach to solving integrals**, which **other CAS systems are often unable** to solve directly
An Application of Rubi: Series Expansion of the Quark Mass Renormalization Group Equation


_GitHub Repository https://github.com/AlexesMes/light-quark-masses._

QUESTIONS ?