Galician Gravitational Waves' Week

Introductory Lectures

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Chapter 1: Linearized Gravitational Waves

- Expansion around flat space

- The TT gauge

- Interaction with test masses

- Energy and Momentum of GW radiation

- Propagation on curved backgrounds

Expansion around flat space

- Expand around the Minkowski metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad , \qquad |h_{\mu\nu}| \ll 1$$

- General coordinate transformations $x^\mu \to x'^\mu(x) \;\Rightarrow\; g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$

$\Rightarrow$ Global *Lorentz covariance* $x'^\mu = \Lambda^\mu{}_\nu x^\nu$

$$g'_{\mu\nu}(x') \;=\; \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu (\eta_{\rho\sigma} + h_{\rho\sigma}(x)) \;=\; \eta_{\mu\nu} + \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu h_{\rho\sigma}(x)$$

hence $h_{\mu\nu}$ is a Lorentz tensor (only small boosts) $h'_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu h_{\rho\sigma} \ll 1$

$\Rightarrow$ Local infinitesimal *gauge symmetry* $x'^\mu = x^\mu + \xi^\mu(x)$

$$h'_{\mu\nu}(x') = h_{\mu\nu}(x) - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu)$$

Consistency $\;\Rightarrow\; |\partial_\mu \xi_\nu| \sim h \ll 1$ (small diffeomorphisms)

$\Rightarrow$ $h_{\mu\nu}$ is a symmetric rank two Lorentzian tensor that transforms as an abelian conexion

Linearization of Einstein equations

- The Christoffel symbols are already first order

$$\Gamma^{\mu}{}_{\nu\rho} = g^{\mu\lambda}\left(g_{\lambda\rho,\mu} + g_{\nu\lambda,\rho} - g_{\nu\rho,\lambda}\right)$$

$$= \eta^{\mu\lambda}\left(h_{\lambda\rho,\mu} + h_{\nu\lambda,\rho} - h_{\nu\rho,\lambda}\right) + \dots = \Gamma^{(1)\mu}{}_{\nu\rho} + \mathcal{O}(h^2)$$

- Expand the Riemann tensor

$$R^{\mu}{}_{\nu\rho\sigma} = \Gamma^{\mu}{}_{\nu\sigma,\rho} - \Gamma^{\mu}{}_{\nu\rho,\sigma} + \mathcal{O}(\Gamma^2)$$

$$= \frac{1}{2}\eta^{\mu\lambda}[h_{\lambda\sigma,\nu\rho} + h_{\nu\lambda,\sigma\rho} - h_{\nu\sigma,\lambda\rho}] - \frac{1}{2}\eta^{\mu\lambda}[h_{\lambda\rho,\nu\sigma} + h_{\nu\lambda,\rho\sigma} - h_{\nu\rho,\lambda\sigma}] + \mathcal{O}(h^2)$$

$$= R^{(1)\mu}{}_{\nu\rho\sigma} + \mathcal{O}(h^2)$$

$R^{(1)\mu}{}_{\nu\rho\sigma}$ is gauge invariant $h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu) \;\Rightarrow\; R'^{(1)\mu}{}_{\nu\rho\sigma} = R^{(1)\mu}{}_{\nu\rho\sigma}$.

Linearization of Einstein equations

- Expand the Einstein tensor

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\left(h^{\lambda}{}_{\nu,\mu\lambda} + h^{\lambda}{}_{\mu,\nu\lambda} - h_{\mu\nu,}{}^{\lambda}{}_{\lambda} - h_{,\mu\nu}\right) - \frac{1}{2}\eta_{\mu\nu}\left(h^{\lambda\rho}{}_{,\lambda\rho} - h_{,}{}^{\lambda}{}_{\lambda}\right)$$

- rewrite Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ as follows

$$G^{(1)}_{\mu\nu} = 8\pi\left(T_{\mu\nu} + T^{(h)}_{\mu\nu}\right)$$

with $T^{(h)}_{\mu\nu} = G^{(1)}_{\mu\nu} - G_{\mu\nu}$.

- no we have a stantard Minkowskian conservation law

$$\partial^{\mu} G^{(1)}_{\mu\nu} = 0 \quad \Longleftrightarrow \quad \partial^{\mu}\left(T_{\mu\nu} + T^{(h)}_{\mu\nu}\right) = 0.$$

- linearized approximation $T^{(h)}_{\mu\nu} \hookrightarrow 0$

$$\left(h^{\lambda}{}_{\nu,\mu\lambda} + h^{\lambda}{}_{\mu,\nu\lambda} - h_{\mu\nu,}{}^{\lambda}{}_{\lambda} - h_{,\mu\nu}\right) - \frac{1}{2}\left(h^{\lambda\rho}{}_{,\lambda\rho} - h_{,}{}^{\lambda}{}_{\lambda}\right) = 16\pi T_{\mu\nu}$$

Linearization of Einstein equations

- define the trace reversed metric

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad \Longleftrightarrow \quad \bar{h} = -h$$

$$\text{gauge transf.} \quad \rightarrow \quad \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial_\rho\xi^\rho)$$

- the linearized Einstein equations boil down to

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\bar{h}_{\rho\sigma}{}^{,\rho\sigma} - \bar{h}_{\mu\rho}{}^{,\rho}{}_{\nu} - \bar{h}_{\nu\rho}{}^{,\rho}{}_{\mu} = -16\pi T_{\mu\nu}$$

- using a small gauge transformation $x^\mu \rightarrow x'^\mu$ the condition $\bar{h}'_{\mu\nu}{}^{,\nu} = 0$ is always reachable

we arrive at the linearized Einstein equations in the harmonic gauge

$$\Box\bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad ; \quad \bar{h}_{\mu\nu}{}^{,\nu} = 0$$

GWs in vacuum

- Massless wave equation in harmonic coordinates $\Box x^\mu = 0$

$$\Box \bar{h}_{\mu\nu} = 0 \quad ; \qquad \bar{h}_{\mu\nu}{}^{,\nu} = 0$$

try with plane waves

$$\bar{h}_{\mu\nu}(x) = \bar{e}_{\mu\nu} \frac{1}{2} \exp(i k_\lambda x^\lambda) + c.c\,.$$

- solution involves a null propagation vector and a transverse polarization tensor $\bar{e}_{\mu\nu}$.

$$k_\lambda k^\lambda = 0 \qquad , \qquad \bar{e}_{\mu\nu} k^\nu = 0$$

with general solution

$$k^\mu = k(1, \hat{\mathbf{n}})\,, \qquad (|\hat{\mathbf{n}}| = 1)$$

$\Rightarrow$ linearized plane GWs over Minkowski space propagate at the speed of light $c = 1$.

$\Rightarrow$ not all degrees of freedom are physical

TT gauge

- there exist residual gauge transformations $x^\mu \to x^\mu + \xi^\mu$, $\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu}$ with

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho)$$

now, if $\bar{h}_{\mu\nu}$ is in the harmonic gauge, $\bar{h}'_{\mu\nu}$ will also be $\iff$ $\Box \xi_\mu = 0$.

- We can chose ξ^μ to impose 4 conditions on $\bar{h}'_{\mu\nu}$
  - choose ξ^0 to fix $\bar{h}' = \bar{h}'^\mu{}_\mu = 0$
  - choose ξ^i to fix $\bar{h}'_{0i} = 0$.

From the harmonic gauge condition we get

$$\partial^\nu h'_{0\nu} = \partial^0 h'_{00} = 0 \quad \Rightarrow \quad h'_{00} = \text{const.} = 0$$

- In summary, (skip primes) the TT (transverse traceless) gauge defined by

$$h^{TT}_{0\mu} \; = \; h^{TT} \; = \; \partial^j h^{TT}_{ij} = 0$$

is always reachable (notice that $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$).

TT gauge - the Λ tensor

Consider a plane wave solution $\bar{h}_{\mu\nu}(x) = \frac{1}{2}\bar{e}_{\mu\nu}e^{ikx} + c.c.$ in the harmonic gauge

$$k^\mu = \omega(1, \hat{\mathbf{n}}) \quad , \quad \bar{e}_{\mu\nu}k^\nu = 0$$

To bring it to the TT gauge $\bar{h}_{\mu\nu}(x) \to h^{TT}_{\mu\nu}(x)$ perform the following the steps

1. $\bar{h}_{\mu\nu} \implies (\bar{h}_{0\mu} = 0, \bar{h}_{ij})$
2. construct the *transverse* projector $P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_i n_j \Rightarrow n^i P_{ij} = P_{ij} n^j = 0$ with $\mathrm{tr}\, P = 2$.
3. define the Λ tensor

$$\boxed{\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}}$$

   is transverse in any index, and traceless $\Lambda_{ii,kl} = \Lambda_{ij,kk} = 0$.

4. finally, project onto the TT gauge

$$\bar{h}_{\mu\nu}(x) \quad \implies \quad h^{TT}_{ij}(x) = \Lambda_{ij,kl}(\hat{\mathbf{n}})\bar{h}_{kl}(x)$$

TT gauge example

- Propagation along the x^3 axis: let $k^\mu = \omega(1, \mathbf{n})$ with $\mathbf{n} = (0,0,1) \implies P_{11} = P_{22} = 1$ and $P_{ij} = 0$ otherwise. Then the only nonvanishing $\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \neq 0$ have $i,j =, 1, 2$

$$\Lambda_{11,11} = \Lambda_{22,22} = \frac{1}{2} \quad ; \quad \Lambda_{11,22} = \Lambda_{22,11} = -\frac{1}{2} \quad ; \quad \Lambda_{12,12} = \Lambda_{21,21} = 1$$

- now $h^{TT}_{ij}(x) = \Lambda_{ij,kl}(\hat{\mathbf{n}})\bar{h}_{kl}(x)$ reveals the two physical polarizations: $h_+, h_\times$

$$h^{TT}_{\mu\nu}(t,0,0,z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\bar{h}_{xx} - \bar{h}_{yy}) & \bar{h}_{xy} & 0 \\ 0 & \bar{h}_{xy} & \frac{1}{2}(-\bar{h}_{xx} + \bar{h}_{yy}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- under the little group $h' = R^T h R \implies (h_+ \pm i h_\times)$ transform with helicity 2

$$R^\mu{}_\nu(\theta\hat{\mathbf{n}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \implies (h'_+ \pm i h'_\times) = e^{\pm i 2\theta}(h_+ \pm i h_\times)$$

Test particles at rest

Consider a particle $x^\mu = (0, L_*, 0, 0)$ at rest $u_A^\mu = (1, 0, 0, 0)$

- Coordinate distances do not change

$$\frac{du_A^\mu}{d\tau} = -\Gamma^\mu_{\beta\alpha} u_A^\beta u_A^\alpha = -\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\beta\alpha}(h^{TT}_{\alpha 0,0} + h^{TT}_{0\alpha,0} - h^{TT}_{00,\alpha}) = \quad 0\,.$$

- Proper distances do change

$$L = \int ds = \int_0^{L_*} dx(1 + h_{xx}(t,0))^{1/2} \sim L_* \left(1 + \frac{1}{2} h_{xx}(t,0)\right) + \ldots$$

Define $\delta L = L - L_*$ we obtain

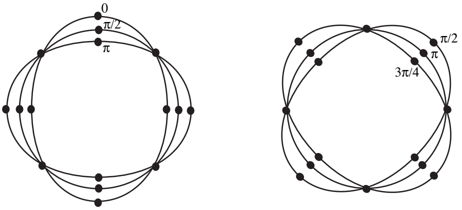
$$\frac{\delta L}{L_*} = \frac{1}{2} h_{xx}(t,0) \sim \frac{1}{2} e_{xx} \cos(\omega t + \delta)\,.$$

Test particles at rest

- Set particles at positions $x^\mu = (\tau, L_* n_x, L_* n_y, 0)$ on a unit circle $\vec{n}^2 = 1$

$$\frac{\delta L}{L_*}(\mathbf{n}) = \frac{1}{2} h_{ij} n^i n^j = \frac{1}{2} h_+^{TT}(n_x^2 - n_y^2) + h_\times^{TT} n_x n_y$$

- Plane polarizations



$h_{xx}^{TT} = \cos(wt-kz)$ $\quad$ $h_{xy}^{TT} = \cos(wt-kz-\pi/2)$

- Circular polarizations

$$g_\pm = e_+ \cos(\omega t - kz) \pm e_\times \cos(\omega t - kz \mp \frac{\pi}{2})$$

Energy of GWs

• Decompose $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\lim_{|x|\to\infty} h_{\mu\nu}(x) = 0$. Split the vacuum Einstein's equations as follows

$$G_{\mu\nu} = 0 \quad \longrightarrow \quad G^{(1)}_{\mu\nu} = 8\pi T^h_{\mu\nu}$$

with

$$\boxed{T^h_{\mu\nu} = \frac{1}{8\pi}(G^{(1)}_{\mu\nu} - G_{\mu\nu})} \tag{1}$$

which is locally conserved and contains h

$$\partial^\mu T^h_{\mu\nu}(h) = 0$$

so can be considered the Energy-Momentum tensor of gravitation.

• in a series expansion in h, since $G^{(1)}_{\mu\nu} = 0$, we get that the energy-momentum tensor of the linearized solution is

$$T^h_{\mu\nu}(h) = \frac{1}{8\pi} G^{(2)}_{\mu\nu}(h) + \mathcal{O}(h^3) = \frac{1}{8\pi}\left(R^{(2)}_{\mu\nu}(h) - \frac{1}{2}\eta_{\mu\nu} R^{(2)}(h)\right) + \mathcal{O}(h^3)$$

Energy of GWs

- an explicit tedious calculation gives

$$\begin{aligned} R^{(2)}_{\mu\nu}(h) &= \frac{1}{2} h^{\alpha\beta}\left(h_{\alpha\beta,\mu\nu} - h_{\mu\beta,\alpha\nu} - h_{\nu\beta,\mu\alpha} + h_{\mu\nu,\alpha\beta}\right) \\ & -\frac{1}{4}(2h^{\beta}{}_{\alpha,\beta} - h^{\beta}{}_{\beta,\alpha})(h^{\alpha}{}_{\mu,\nu} + h^{\alpha}{}_{\nu,\mu} - h_{\mu\nu,}{}^{\alpha}) \\ & +\frac{1}{4}(h_{\alpha\nu,\beta} + h_{\alpha\beta,\nu} - h_{\beta\nu,\alpha})(h^{\alpha}{}_{\mu,}{}^{\beta} + h^{\alpha\beta}{}_{,\mu} - h^{\beta}{}_{\mu,}{}^{\alpha}) \,. \end{aligned}$$

- take a space or time average $T \gg 1/\omega$

$$t_{\mu\nu} \equiv \langle T^{h}_{\mu\nu}\rangle \equiv \frac{1}{T}\int_{-T/2}^{+T/2} T^{h}_{\mu\nu}(t,\mathbf{x})dt = \frac{1}{8\pi}\left\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\right\rangle$$

Notice that under the average, derivatives of solutions $h = h(t-z)$ can be partially integrated

$$\left\langle \frac{1}{2}h^{\alpha\beta}h_{\alpha\beta,\mu\nu} - \ldots\right\rangle = \left\langle \frac{1}{2}h^{\alpha\beta}{}_{,\nu}h_{\alpha\beta,\mu} - \ldots\right\rangle$$

Energy and Momentum of GWs

- Exercise: for solutions $\Box h_{\mu\nu} = 0$ in the traceless, $h = 0$, harmonic, $h_{\mu\nu}{}^{,\nu} = 0$, gauge, and integrating by parts show that this expression collapses to

$$t_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu h_{\alpha\beta} \partial_\nu h^{\alpha\beta} \rangle$$

- It is residual-gauge invariant $x^\mu \to x^\mu + \xi^\mu$, $\Box \xi^\mu = 0$, hence we can switch $h_{\mu\nu} \to h_{ij}^{TT}$.

$$t_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu h_{ij}^{TT} \partial_\nu h_{ij}^{TT} \rangle = \frac{1}{16\pi} \langle \partial_\mu h_{+}^{TT} \partial_\nu h_{+}^{TT} + \partial_\mu h_{-}^{TT} \partial_\nu h_{-}^{TT} \rangle$$

$\Rightarrow$ Energy

$$E = \int_V d^3x\, t^{00} = \frac{1}{32\pi} \int_V d^3x \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle = \frac{1}{16\pi} \int_V d^3x \langle \dot{h}_{+}^2 + \dot{h}_{\times}^2 \rangle$$

$\Rightarrow$ Momentum

$$P^k = \int_V d^3x\, t^{0k} = \frac{1}{32\pi} \int_V d^3x \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

Energy and Momentum Flux

- Energy flux

For a wave $h_{ij}^{TT}(t,z) = h_{ij}^{TT}(t-z) \implies \partial_z h_{ij}^{TT} = -\partial_t h_{ij}^{TT} \implies t_{0z} = -t_{00}$, and the energy flux t_{0z} traversing a z-perpendicular surface element dA, drains E at a rythm

$$\frac{dE}{dt} = -dA\, t_{0z} = dA\, t_{00}$$

Then through a sphere $\int dA = r^2 \int d\Omega$ that contains the volume V the energy loss

$$\frac{dE}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle$$

- Linear momentum flux. Inside a volume V at large distance from the source $P^k = \int_V d^3x\, t_{0k}$.

$$\frac{dP^k}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle \dot{h}_{ij}^{TT} \partial^k h_{ij}^{TT} \rangle$$

- Angular momentum flux. Inside a volume V at large distance from the source

$$\frac{dJ^i}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle -\epsilon^{ijk} \dot{h}_{ab}^{TT} x^j \partial^k h_{ab}^{TT} + 2\epsilon^{ijk} h_{ak}^{TT} \dot{h}_{aj}^{TT} \rangle$$

Propagation on curved background

• Decompose $g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$ with $\lim_{|x|\to\infty} h_{\mu\nu}(x) = 0$. Expand $g^{\mu\nu} = \tilde{g}^{\mu\nu} - h_{\mu\nu} + \ldots$. Imposing the generalized harmonic gauge condition

$$\tilde{D}^{\nu}\bar{h}_{\mu\nu} = 0$$

the linearized curved equations of motion

$$R^{(1)}_{\mu\nu} = \underbrace{\tilde{\Box}\bar{h}_{\mu\nu}}_{\mathcal{O}(h/\lambda)} + \underbrace{2\tilde{R}_{\mu\rho\nu\sigma}\bar{h}^{\rho\sigma} - \tilde{R}_{\mu\rho}\bar{h}_{\nu}{}^{\rho} - \tilde{R}_{\nu\rho}\bar{h}_{\mu}{}^{\rho}}_{\mathcal{O}(h/L_B)}$$

$\lambda \ll L_B$

$$\tilde{\Box}\bar{h}_{\mu\nu} = 0 \qquad ; \qquad \tilde{D}^{\nu}\bar{h}_{\mu\nu} = 0$$

• Eikonal Approximation

$$\bar{h}_{\mu\nu}(x) = (A_{\mu\nu}(x) + \epsilon B_{\mu\nu}(x) + \ldots)e^{i\theta(x)/\epsilon}$$

to lowest order in ϵ and h

$$(3) \;\Rightarrow\; k^{\mu}A_{\mu\nu}(x) = 0$$

$$(4) \;\Rightarrow\; k_{\nu}k^{\nu} = 0 \;\Rightarrow\; (k^{\nu}\tilde{D}_{\nu})k_{\mu} = 0$$

given that $k_{\nu} = \partial_{\nu}\theta(x) \Rightarrow$ geometric optics approximation: rays (curves orthogonal to constant phase surfaces) follow null geodesic equation.

Summary of Lesson 1

• Linearized theory is that of a rank 2 symmetric Lorentzian field with a local gauge symmetry $h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

• in the harmonic gauge $\bar{h}^{\mu\nu}{}_{;\nu} = 0$ eqs. of motion reduce to linear wave equation $\Box \bar{h}_{\mu\nu} = 8\pi T_{\mu\nu}$.

• a residual gauge transformation allows to write the plane waves in the TT gauge in terms of two helicity 2 transverse polarizations.

$$\bar{h}_{\mu\nu} \hookrightarrow \left(h^{TT}_{0\mu} = 0 \, , \; h^{TT}_{ij} = \Lambda(\mathbf{n})_{ij;kl} \, \bar{h}_{kl} \right)$$

.

• Waves carry energy and momentum given by $t_{\mu\nu} = \frac{1}{32\pi} \langle \partial_\mu h_{\beta\alpha} \partial_\nu h^{\beta\alpha} \rangle$

• Radiation power by gravity waves is given by

$$P = \frac{dE}{dt} = \frac{r^2}{32\pi} \int d\Omega \, \langle \dot{h}^{TT}_{ij} \, \dot{h}^{TT}_{ij} \rangle$$

• In the geometrical optics approximation, gravity waves propagate on curved backgrounds following null geodesics, as electromagnetic waves do.

Chapter 2: Generation of Linearized Gravitational Waves

- Low velocity expansion

- Tensor spherical harmonics

- Mass Quadrupole Approximation

- Examples: oscillating and rotating 2-body systems

Weak-field sources

- Weakly sourced equations of motion in the harmonic gauge

$$\Box \bar{h}_{\mu\nu}(x) = -16\pi T_{\mu\nu}(x) \qquad ; \qquad \partial_\mu \bar{h}^\mu{}_\nu = 0$$

- Use the Green's function method

$$\bar{h}_{\mu\nu}(x) = 16\pi \int d^4x' \, G(x,x') \, T_{\mu\nu}(x')$$

where

$$\Box G(x,x') = \delta^4(x-x') \quad \Rightarrow \quad G^{ret}(x-x') = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t_{ret}-t')$$

with $t_{ret} = t - |\mathbf{x}-\mathbf{x}'|$

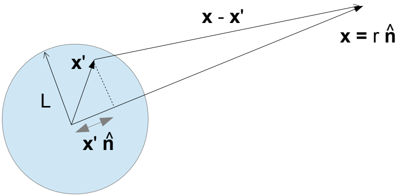
Weak-field sources

General solution

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\int d^3x' \, \frac{T_{\mu\nu}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

with $\mathbf{x} = r\,\hat{\mathbf{n}}$ we have

$$|\mathbf{x}'-\mathbf{x}| = r\left(1 - \frac{\mathbf{x}'\cdot\hat{\mathbf{n}}}{r} + \mathcal{O}(L^2/r^2)\right)$$



For $r \gg L$ go to the *wave zone approximation*

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4}{r}\int d^3x' \, T_{\mu\nu}(t - r + \mathbf{x}'\cdot\hat{\mathbf{n}}, \mathbf{x}')$$

where $h_{\mu\nu}$ becomes a spherical wave. At each point $\mathbf{x} = r\hat{\mathbf{n}}$ on the wavefront, we may express in the TT gauge

Low velocity expansion

$$
\begin{aligned}
h_{ij}^{TT}(t,\mathbf{x}) &= \Lambda_{ij,kl}(\hat{\mathbf{n}})\bar{h}_{kl}(t,\mathbf{x}) \qquad\qquad (\mathbf{x} = r\,\hat{\mathbf{n}}) \\
&= \frac{4}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\int d^3x'\,T_{kl}(t-r+\hat{\mathbf{n}}\cdot\mathbf{x}',\mathbf{x}') \\
&= \frac{4}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\int d^3x'\,\Big(T_{kl}(t-r,\mathbf{x}') + (\hat{\mathbf{n}}\cdot\mathbf{x}')\frac{d}{dt}T_{kl}(t-r,\mathbf{x}') + \\
&\qquad + \frac{1}{2}(\hat{\mathbf{n}}\cdot\mathbf{x}')^2\frac{d^2}{dt^2}T_{kl}(t-r,\mathbf{x}') + ...\Big) \\
&= \frac{4}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\Big(\int d^3x'\,T_{kl}(t-r,\mathbf{x}') + n_{i_1}\int d^3x'\,x'_{i_1}\frac{d}{dt}T_{kl}(t-r,\mathbf{x}') + \\
&\qquad + \frac{1}{2}n_{i_1}n_{i_2}\int d^3x'\,x'_{i_1}x'_{i_2}\frac{d^2}{dt^2}T_{kl}(t-r,\mathbf{x}') + ...\Big)
\end{aligned}
$$

- This is a low velocity expansion in powers of $v \ll 1$. Indeed, for $T_{kl}(t,x') \sim \tilde{T}_{kl}(\omega,\mathbf{x}')\cos(\omega t)$

$$
\begin{aligned}
(\hat{\mathbf{n}}\cdot\mathbf{x})^r\frac{d^r}{dt^r}T_{kl}(t,\mathbf{x}') &\sim (\hat{\mathbf{n}}\cdot\mathbf{x})^r(\omega)^r\,T_{kl}(t,\mathbf{x}') \le (L\omega)^r\,T_{kl}(t,\mathbf{x}') \\
&\sim v^r\,T_{kl}(t,x')
\end{aligned}
$$

- Defining the stress-tensor moments

$$
\boxed{S_{kl,i_1\ldots i_p}(t) = \int d^3x'\;T_{kl}(t,\mathbf{x}')\,x'_{i_1}\ldots x'_{i_p}}
$$

Low velocity expansion

- we arrive at the *multipole expansion* in the radiation zone $\mathbf{x} = r\hat{\mathbf{n}} \gg \mathbf{x}'$

$$h_{ij}^{TT}(t,\mathbf{x}) \;=\; \Lambda_{ij,kl}(\hat{\mathbf{n}})\,\frac{4}{r}\left[S^{kl}(t-r) + n_{i_1}\dot{S}^{kl,i_1}(t-r) + n_{i_1}n_{i_2}\frac{1}{2}\ddot{S}^{kl,i_1 i_2}(t-r) + ...\right]$$

$$= \Lambda_{ij,kl}(\hat{\mathbf{n}})\,\frac{4}{r}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\partial_t^n S^{kl,i_1\ldots i_n}(t-r)\,n_{i_1}(\theta,\phi)\ldots n_{i_n}(\theta,\phi)\right]$$

- Setting $\hat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ it is clear that we should be able to represent $h_{ij}^{TT}(t,\mathbf{x})$ as an expansion in tensor spherical harmonics

- Scalar field

$$\phi(x) = \frac{1}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}(t-r)Y_{lm}(\theta,\phi) \qquad\qquad \mathbf{L}^2 Y_{lm} = l(l+1)Y_{lm}$$

where

$$Y_{lm}(\theta,\phi) = C^{lm}\left(e^{i\phi}\sin\theta\right)^m \frac{d^m}{d(\cos\theta)^m}P_l(\cos\theta)$$

Tensor Spherical Harmonics

- Vector field

$$(\mathbf{V}(x))_i = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} R_{lm}(t,r)(\mathbf{Y}^R_{lm})_i + \sum_{l=1}^{\infty}\sum_{m=-l}\left[E_{lm}(t,r)(\mathbf{Y}^E_{lm})_i + B_{lm}(t,r)(\mathbf{Y}^B_{lm})_i\right]$$

with

$$(\mathbf{Y}^E_{lm})_i = \frac{r}{\sqrt{l(l+1)}}\,\partial_i Y_{lm}(\theta,\phi)$$

$$(\mathbf{Y}^B_{lm})_i = \frac{i}{\sqrt{l(l+1)}}\,L_i Y_{lm}(\theta,\phi)$$

$$(\mathbf{Y}^R_{lm})_i = n_i Y_{lm}(\theta,\phi)$$

- Symmetric tensor field

$$\begin{aligned}\mathbf{T}_{ij}(x) = &\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(L0_{lm}(t,r)(\mathbf{T}^{L0}_{lm})_{ij} + T0_{lm}(t,r)(\mathbf{T}^{T0}_{lm})_{ij}\right)\\ &+\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(E1_{lm}(t,r)(\mathbf{T}^{E1}_{lm})_{ij} + B1_{lm}(t,r)(\mathbf{T}^{B1}_{lm})_{ij}\right)\\ &\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left(E2_{lm}(t,r)(\mathbf{T}^{E2}_{lm})_{ij} + B2_{lm}(t,r)(\mathbf{T}^{B2}_{lm})_{ij}\right)\end{aligned}$$

Tensor Spherical Harmonics

$$
\begin{aligned}
(\mathbf{T}^{L0}_{lm})_{ij} &= n_i n_j Y_{lm}(\theta,\phi) \\
(\mathbf{T}^{T0}_{lm})_{ij} &= (n_i n_j - \delta_{ij}) Y_{lm}(\theta,\phi) \\
(\mathbf{T}^{E1}_{lm})_{ij} &= c^{(1)}_l (r/2)(n_i\partial_j + n_j\partial_i) Y_{lm}(\theta,\phi) \\
(\mathbf{T}^{B1}_{lm})_{ij} &= c^{(1)}_l (i/2)(n_i L_j + n_j L_i) Y_{lm}(\theta,\phi) \\
(\mathbf{T}^{E2}_{lm})_{ij} &= c^{(2)}_l r^2 \Lambda_{ij,i'j'}(\hat{\mathbf{n}})\partial_{i'}\partial_{j'} Y_{lm}(\theta,\phi) \\
(\mathbf{T}^{B2}_{lm})_{ij} &= c^{(2)}_l r(i/2)\Lambda_{ij,i'j'}(\hat{\mathbf{n}})(\partial_{i'} L_{j'} + \partial_{ij} L_{j'}) Y_{lm}(\theta,\phi)
\end{aligned}
$$

- With this, the general solution in the TT gauge only contains $\mathbf{T}^{E2}_{lm}$ and $\mathbf{T}^{B2}_{lm}$

$$
h^{TT}_{ij}(t,r,\theta,\phi) = \frac{1}{r}\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\left[u_{lm}(t-r)(\mathbf{T}^{E2}_{lm})_{ij}(\theta,\phi) + v_{lm}(t-r)(\mathbf{T}^{B2}_{lm})_{ij}(\theta,\phi)\right]
$$

(with $c_l = \left(\frac{2}{l(l+1)}\right)^{1/2}$) which are transverse $(\mathbf{T}^{E2}_{lm})_{ij} n_j = (\mathbf{T}^{B2}_{lm})_{ij} n_j = 0$,

Multipole Expansion

- remember the multipole expansion

$$h_{ij}^{TT}(t,r,\theta,\phi) = \Lambda_{ij,kl}(\hat{\mathbf{n}})\,\frac{4}{r}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\partial_t^n S^{kl,i_1\dots i_n}(t-r)\,n_{i_1}(\theta,\phi)\dots n_{i_n}(\theta,\phi)\right]$$

- in order to relate u_{lm} and v_{lm} to $S_{kl,i_1,\dots i_p}$ use the orthogonality relation

$$\int d\Omega(\mathbf{T}^{J}_{lm})_{ij}(\mathbf{T}^{J'}_{l'm'})_{ij} = \delta^{JJ'}\delta_{ll'}\delta_{mm'}$$

and simply project to get

$$\begin{aligned} u_{lm}(t) &= \sum_{\alpha=0}^{\infty}\frac{4}{\alpha!}\left(\partial_t^{\alpha}S^{kl,i_1,\dots,i_\alpha}(t)\right)\int d\Omega(\mathbf{T}^{E2}_{lm})^*_{ij}\Lambda_{ij,kl}n_{i_1}\dots n_{i_\alpha} \\ v_{lm}(t) &= \sum_{\alpha=0}^{\infty}\frac{4}{\alpha!}\left(\partial_t^{\alpha}S^{kl,i_1,\dots,i_\alpha}(t)\right)\int d\Omega(\mathbf{T}^{B2}_{lm})^*_{ij}\Lambda_{ij,kl}n_{i_1}\dots n_{i_\alpha} \end{aligned} \qquad (2)$$

Quadrupolar approximation

- The lowest order approximation is

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{4}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})S^{kl}(t-r)$$

- Define the moments of the *energy* , *momentum* and *stress* density

$$\begin{aligned}
M^{j_1\cdots j_n}(t) &= \int dx'^3\, T^{00}(t,\mathbf{x}')x'^{j_1}\ldots x'^{j_n} \\
P^{i,j_1\cdots j_n}(t) &= \int dx'^3\, T^{0i}(t,\mathbf{x}')x'^{j_1}\ldots x'^{j_n} \\
S^{ij,j_1\cdots j_n}(t) &= \int d^3x'\; T^{ij}(t,\mathbf{x}')\, x'^{j_1}\ldots x'^{j_n}
\end{aligned} \tag{3}$$

Lemma

$$S^{kl}(t) = \frac{1}{2}\ddot{M}^{kl}(t)$$

Exercise: Prove this lemma using only $\partial^\mu T_{\mu\nu} = 0$.

Theorem

$$\partial_t^n S^{ij,k_1\dots k_n}(t) \;= \mathcal{F}(\partial_t^{n+2} M^{ijk_1\dots k_n}(t), \partial_t^{n+1} P^{i,jk_1\dots k_n}(t), \dots)$$

For example

$$\begin{aligned} \dot{S}^{ij,k} &= \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{i,jk} + \ddot{P}^{j,ik} - 2\ddot{P}^{k,ij}\right) \\ &\vdots \end{aligned} \quad (4)$$

Mass quadrupole radiation

- far field in the quadrupolar approximation

$$h_{ij}^{TT}(t,\mathbf{x})_{quad} = \frac{2}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}_{kl}(t-r)$$

For example if $k^\mu = (\omega, 0, 0, \omega)$, then, along the z axis

$$\begin{aligned} h_+(t,0,0,z) &= \frac{1}{z}(\ddot{M}_{11} - \ddot{M}_{22})(t-z) \\ h_\times(t,0,0,z) &= \frac{2}{z}\ddot{M}_{12}(t-z) \end{aligned}$$

- If $x^\mu = \omega(1, \hat{\mathbf{n}})$ with $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ use rotated $M \to M' = \mathcal{R}^T M \mathcal{R}$

$$\begin{aligned} h_+(t;r\hat{\mathbf{n}}) &= \frac{1}{r}\Big[\ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) - \ddot{M}_{33}\sin^2\theta \\ &\quad -\ddot{M}_{12}\sin 2\phi(1+\cos^2\theta) + \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta\Big] \\ h_\times(t;r\hat{\mathbf{n}}) &= \frac{1}{r}\Big[(\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ &\quad -2\ddot{M}_{13}\cos\phi\sin\theta + 2\ddot{M}_{23}\sin\phi\sin\theta\Big] \end{aligned}$$

Radiated Energy

- Introduce the quadrupole moment $Q_{ij} = M_{ij} - \frac{1}{3}\delta_{ij} M$,

$$h_{ij}^{TT}(t, \mathbf{x})_{quad} = \frac{2}{r} \Lambda_{ij,kl}(\hat{\mathbf{n}}) \ddot{Q}_{kl}(t - r)$$

- Radiated energy

$$\frac{dE}{dt} = \frac{r^2}{32\pi} \int d\Omega \langle \dot{h}_{ij}^{TT} \dot{h}_{ij}^{TT} \rangle = \frac{1}{8\pi} \langle \dddot{Q}_{ij} \dddot{Q}_{kl} \rangle \int d\Omega \, \Lambda_{ij,kl}(\hat{\mathbf{n}})$$

Integrate $\int d\Omega \, \Lambda_{ij,kl}(\hat{\mathbf{n}}) = \frac{2\pi}{15}(11\delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$ to find the total radiated power

$$P = \frac{dE}{dt} = \frac{1}{5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle$$

- Spectrum: FT the quadruple moment $Q_{ij}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{Q}(\omega)_{ij} e^{-i\omega t}$ and integrating $\int_{-\infty}^{+\infty} dt$

$$E = \frac{1}{5\pi} \int_0^{\infty} d\omega \, \omega^6 \tilde{Q}_{ij}(\omega) \tilde{Q}^*_{kl}(\omega)$$

whence the radiation spectrum follows

$$\frac{dE}{d\omega} = \frac{1}{5\pi} \omega^6 |\tilde{Q}_{ij}(\omega)|^2$$

Non relativistic N-body system

- To lowest order in $v \ll 1$

$$T^{\mu\nu}(x) = \sum_{A=1}^{N} \int d\tau \, P_A^{\mu} \frac{dx_A^{\nu}}{d\tau} \delta^4(x - x_A)$$
$$= \sum_{A=1}^{N} m_A \frac{dx_A^{\mu}}{dt} \frac{dx_A^{\nu}}{dt} \delta^3(\mathbf{x} - \mathbf{x}_A(t)) + \mathcal{O}(v^2)$$

In particular $T^{00}(t, \mathbf{x}) = \sum_{A=1}^{N} m_A \delta^3(\mathbf{x} - \mathbf{x}_A(t))$ so the quadrupole mass reduces to

$$M_{ij} = \int d^3x \, T^{00}(t, \mathbf{x}) x^i x^j = \sum_{A=1}^{N} m_A x_A^i x_A^j \, .$$

- For a 2 body problem, $N = 2$. In the center of mass and relative coordinates

$$M = (m_1 + m_2) \; \rightarrow \; \mathbf{x}_M = \frac{m_1 \mathbf{x}_1 + m_2 \mathbf{x}_2}{m_1 + m_2} \quad ; \quad \mu = \left(\frac{m_2 m_1}{m_1 + m_2} \right) \; \rightarrow \; \mathbf{x}_r = \mathbf{x}_1 - \mathbf{x}_2$$

the quadrupole mass $M_{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = M \, x_{CM}^i x_{CM}^j + \mu \, x_r^i x_r^j$

Non relativistic 2-body system: oscillating linear system

- let

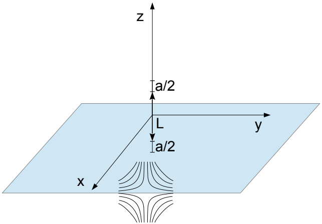
$$\mathbf{x}_{CM} = 0 \quad ; \quad \mathbf{x}_r = (0, 0, L + A\cos\omega_s t)$$

with $A\,\omega_s \ll 1$. Since $x_r^1 = x_r^2 = 0$ we only find one non-vanishing quadrupolar moment

$$M_{33}(t) = \mu(L + A\cos\omega_s t)^2 = \mu L^2 + \mu A\left(2L\cos\omega_s t + \frac{A}{2}\cos 2\omega_s t\right)$$

and hence

$$\begin{aligned} h_+^{TT}(t,\theta,\phi) &= -\frac{1}{r}\ddot{M}_{33}(t-r)\sin^2\theta \\ &= -\frac{2\mu\omega_s^2 A}{r}\left(L\cos\omega_s(t-r) + A\cos 2\omega_s(t-r)\right)\sin^2\theta \\ h_\times^{TT}(t,\theta,\phi) &= 0. \end{aligned}$$



Oscillating linear system

- putting some numbers for a lab sized setup $\mu = 1$ kg, $L = A = r = 1$m and $\omega_s = 10^2$Hz we find

$$h_+^{TT} \sim \frac{2\mu\omega_s^2 LA}{r} = 1.6 \times 10^{-39}$$

- as for the power, we compute first

$$Q_{ij} = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right) M_{33} \qquad (5)$$

hence, with the Einstein quadrupole radiation formula

$$\begin{aligned} P &= \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle = \frac{2}{15}\langle \dddot{M}_{33}^2\rangle \\ &= \mu^2 A^2 \omega_s^6 \frac{1}{T}\int_0^T (2L\cos\omega_s t + 32A\cos 2\omega_s t)^2\, dt \\ &= \frac{4}{15}\mu^2 A^2 \omega_s^6 \left(L^2 + 4A^2\right) \qquad (6) \end{aligned}$$

Circular orbit

- Consider a circular orbit in the $x - y$ plane

$$\mathbf{x}_{CM} = 0 \quad ; \quad \mathbf{x}_r = (R \sin \omega_s t \ , \ R \cos \omega_s t \ , \ 0)$$

Since $x_r^3 = 0$ we only obtain moments

$$\begin{aligned} M_{11}(t) &= \mu (x_r^1)^2 = \frac{\mu R^2}{2}(1 - \cos 2\omega_s t) \\ M_{22}(t) &= \mu (x_r^2)^2 = \frac{\mu R^2}{2}(1 + \cos 2\omega_s t) \\ M_{12}(t) &= \mu x_r^1 x_r^2 = -\frac{\mu R^2}{2} \sin 2\omega_s t \end{aligned}$$

hence

$$\ddot{M}_{11} = -\ddot{M}_{22} = 2\mu R^2 \omega_s^2 \cos 2\omega_s t \quad ; \quad \ddot{M}_{12} = 2\mu R^2 \omega_s^2 \sin 2\omega_s t$$

- Finally

$$\begin{aligned} h_{+}(t; \theta, \phi) &= \frac{4\mu\omega_s^2 R^2}{r} \left(\frac{1 + \cos^2 \theta}{2} \right) \cos(2\omega_s (t - r) + 2\phi) \\ h_{\times}(t; \theta, \phi) &= \frac{4\mu\omega_s^2 R^2}{r} (\cos \theta) \sin(2\omega_s (t - r) + 2\phi) \end{aligned}$$

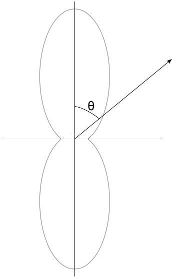
Notice that ϕ can be reabsorbed in a shift of the origin of $t \to t - \phi/\omega_s$.

Circular orbit

• The dependence $h_+ \sim (1+\cos^2\theta)$ and $h_\times \sim \cos\theta$ is generic for sources in a plane $M_{13} = M_{23} = M_{33} = 0$.

• The polarization is linear for $\theta = \pi/2$, circular for $\theta = 0$ and elliptic in between. Using $\langle\cos^2 2\omega_2 t\rangle = \langle\sin^2 2\omega_2 t\rangle = 1/2$

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{r^2}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \\ &= \frac{2\mu^2 R^4 \omega_s^6}{\pi}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right] \end{aligned}$$



Integrating over the angles for the total power

$$P = \frac{32}{5}\mu^2 R^4 \omega_s^6$$

Exercise: Recover this result from $dE/dt = \langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle/5$

Circular orbit

Comment: In the MKS the correct expression is

$$P = \frac{G}{c^5}\frac{32}{5}\mu^2 R^4 \omega_s^6$$

Exercise: Consider 2 bodies of $m = 1$ kg each, in circular motion at $R = 0.5$ m distance. Calculate the frecuency ω that they have to spin with, in order to produce one single graviton.

SUMMARY OF LESSON 2

- In the wave zone approximation we have a multipole expansion

$$h_{ij}^{TT}(t,\mathbf{x}) \;=\; \Lambda_{ij,pq}(\hat{\mathbf{n}})\,\frac{4}{r}\left[S_{pq}(t-r)+n_{i_1}\dot{S}_{pq,i_1}(t-r)+n_{i_1}n_{i_2}\frac{1}{2}\ddot{S}_{pq,i_1 i_2}(t-r)+\ldots\right]$$

- this is an expansion in powers of $v \ll 1$
- far field mass quadrupole approximation $S_{pq} = \frac{1}{2}\ddot{M}_{pq}$

$$\boxed{h_{ij}^{TT}(t,\mathbf{x})_{quad} = \frac{2}{r}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}_{kl}(t-r)}$$

- Radiated quadrupolar power $P_{quad} = \frac{dE}{dt} = \frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle$

- Radiation spectrum $\frac{dE}{d\omega} = \frac{1}{5\pi}\omega^6|\tilde{Q}_{ij}(\omega)|^2$

- Planar equatorial motion $h_+(t;\theta,\phi) \sim \left(\frac{1+\cos^2\theta}{2}\right)$; $h_\times(t;\theta,\phi) \sim \cos\theta$

Chapter 3: Examples

- Quasi-circular orbits

- Far field wave form

- Fourier transformed wave form

- Elliptic orbits

- Rotating Rigid Bodies

Circular Orbits

- consider a binary of system with total mass M and reduced mass μ

$$M = m_1 + m_2 \quad ; \quad \mu = \frac{m_1 m_2}{m_1 + m_2},$$

and positions $\mathbf{r}_{1,2}$. In the center of mass frame $\mathbf{r}_{CM} = 0$ with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$

- in the effective 1-body problem

$$\ddot{\mathbf{r}} = -\frac{M}{r^3}\mathbf{r}\,.$$

the radius R and orbital frequency $\omega_s = v/R$ and are related by $\omega_s^2 R = v^2/R = M/R^2$,

i.e.

Kepler's law

$$R = \left(\frac{M}{\omega_s^2}\right)^{1/3}$$

Circular Orbits

- Let us express everything in terms of the GW frequency $\boxed{\omega_{GW} = 2\omega_s}$

– the total energy

$$E = \frac{1}{2}\mu v^2 - \frac{\mu M}{R} = -\frac{\mu M}{2R} \quad \Longrightarrow \quad \boxed{E = -\left(\frac{M_c^5 \omega_{GW}^2}{32}\right)^{1/3}}$$

where M and μ enter through the *chirp mass*

$$M_c{}^5 = \mu^3 M^2 = \frac{(m_1 m_2)^3}{(m_1 + m_2)}$$

For example, for $m_1 = m_2 = m \;\Rightarrow\; \mu = \frac{m}{2}, M = 2m$ and $M_c = m/2^{1/5} = 0.87m = 0.43M$.

– the total radiated power from previous lesson

$$P = \frac{32}{5}\mu^2 R^4 \omega_s^6 \quad \Longrightarrow \quad \boxed{P = \frac{32}{5}\left(\frac{M_c \omega_{GW}}{2}\right)^{10/3}}$$

Exercise: For the PSR1913+16 system $m = 1.4 M_\odot$ and $T_s \simeq 8$h , obtain $P = 6 \times 10^{14}$ GW.

Frequency Shift

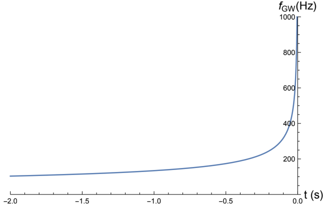
- Equating $P = -dE/dt$ yields a differential equation

$$\dot{\omega}_{GW} = \frac{48}{5}\left(\frac{M_c}{2}\right)^{5/3} \omega_{GW}^{11/3} \tag{7}$$

with solution $\omega_{GW}(t) \to f_{GW}(t) = 2\pi\, \omega_{GW}(t)$

$$f_{GW}(t) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{M_c^{5/8}} \frac{1}{(t_{coal} - t)^{3/8}} \tag{8}$$

with $-\infty < t < t_{coal}$.



- Rewrite in terms of a particular reference example.

Set $m_{1,2} = 1.4 M_\odot \Rightarrow M_c = 1.2 M_\odot = 1.2 \times 1.47\, 10^3 \mathrm{m}$
and $t = t_{coal} - 1\mathrm{s} \times (3 \times 10^8 \mathrm{m/s})$

$$f_{GW}(t_{coal} - 1\mathrm{s}; M_c = 1.2\, M_\odot) = 4.48 \times 10^{-7} \mathrm{m}^{-1} \sim 134\, \mathrm{Hz}$$

Hence,

$$f_{GW}(t) = 134\, \mathrm{Hz} \left(\frac{1.2\, M_\odot}{M_c}\right)^{5/8} \left(\frac{1\,\mathrm{s}}{t_{coal} - t}\right)^{3/8}$$

Frequency Shift

- Solving for $(t_{coal} - t)(f_{GW})$ and fixing $f_{GW} = 100$ Hz (the sensitivity of earth based interferometers)

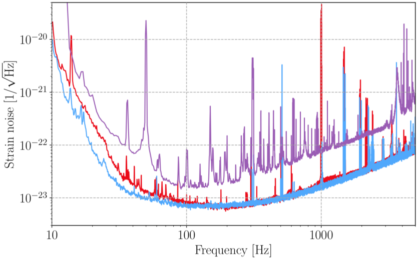
$$(t_{coal} - t) = 2.18\,\mathrm{s} \left(\frac{1.2\,M_\odot}{M_c}\right)^{5/3} \left(\frac{100\mathrm{Hz}}{f_{GW}}\right)^{8/3}$$

So when $M_c = 1.2\,M_\odot$ we get for

- $f_{GW} = 100$ Hz
the radiation in the last 2 seconds before coalescence

- $f_{GW} = 10\,\mathrm{Hz}$
the radiation 17 minutes before coalescence.

- $f_{GW} = 1/\mathrm{day}$
$t - t_{coal} \sim 10^{11}$year. This limits initial orbit f that coalesce by GW within the age of universe.



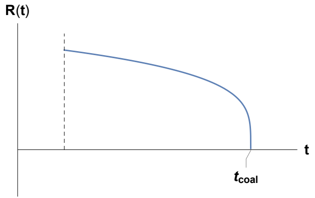
Quasi-circular orbit approximation

- taking the derivative of Kepler's Law $\omega_s = (m/R^3)^{1/2}$

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_s}{\omega_s} = -\frac{2}{3}\frac{\dot{f}_{GW}}{f_{GW}} = -\frac{1}{4(t - t_{coal})}$$

Integrating from an initial R_0 at time $t_0 \leq t \leq t_{coal}$

$$\boxed{R(t) = R_0 \left(\frac{t_{coal} - t}{t_{coal} - t_0}\right)^{1/4}}$$



- Quasi-circular approximation

$$\left|\frac{v_R}{v_\theta}\right| = \left|\frac{\dot{R}}{R\omega_s}\right| = \frac{2}{3}\frac{\dot{\omega}_s}{\omega_s^2} = \frac{4}{3}\frac{\dot{\omega}_{GW}}{\omega_{GW}{}^2} \ll 1 \quad \Longrightarrow \quad \boxed{\dot{\omega}_{GW} \ll \omega_{GW}^2}$$

Quasi-circular orbit approximation

• Using (7) this implies $\dot{\omega}_{GW} = \frac{48}{5}\left(\frac{M_c}{2}\right)^{5/3}\omega_{GW}^{11/3} \ll \omega_{GW}^2 \implies \omega_{GW} \ll \dfrac{(2^{7/3}3/5)^{3/5}}{M_c}$

For $M_c = 1.2M_\odot \Rightarrow f_{GW} \ll 13.7\,\mathrm{kHz}$ so in general, we can assume circular orbit as long as

$$\boxed{f_{GW} \ll 13.7\left(\frac{1.2\,M_\odot}{M_c}\right)\mathrm{kHz}}$$

• Non-linearity of GR entails the existence of an Innermost Stable Circular Orbit (ISCO) where strong inspiralling sets in

$$R_{ISCO} \geq 6M \quad \overset{Kepler}{\Longrightarrow} \quad \omega_s \leq \omega_{ISCO} = \left(\frac{M}{R_{ISCO}^3}\right)^{1/2} = \frac{1}{6\sqrt{6}}\frac{1}{M} \sim \frac{0.03}{M_c}$$

- For a BNS $M_c = 1.2\,M_\odot \implies f_{GW,ISCO} = 0.8\,\mathrm{kHz}$ then

$$f_{GW} \leq 800\left(\frac{1.2\,M_\odot}{M_c}\right)\mathrm{Hz}$$

- For a BBH of $M = 10M_\odot \Rightarrow M_c = 4.3M_\odot$ quasi-circular approx. valid for $f_{GW} \leq 200\mathrm{Hz}$

Far Field Wave-form

- The far field wave form of a circular binary system was (use Kepler's law and chirp mass)

$$h_+(t) = \frac{4\pi^{4/3} M_c^{5/3} f_{GW}^{2/3}}{r}\left(\frac{1+\cos^2\theta}{2}\right)\cos\left(2\pi f_{GW}(t-r)+2\phi\right)$$

$$h_\times(t) = \frac{4\pi^{4/3} M_c^{5/3} f_{GW}^{2/3}}{r}(\cos\theta)\sin(2\pi f_{GW}(t-r)+2\phi)$$

- In the quasi-circular approximation we can neglect $\dot{R}$ as long as $\dot{\omega}_s \ll \omega_s^2$. Now we have a time dependent frequency

$$f_{GW} \to f_{GW}(t) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8}\frac{1}{M_c^{5/8}}\frac{1}{(t_{coal}-t)^{3/8}}$$

Also we need to replace

$$2\pi f_{GW}\, t \to \Phi(t) = \int dt\, 2\pi f_{GW}(t) + \Phi_0$$

doing the integral

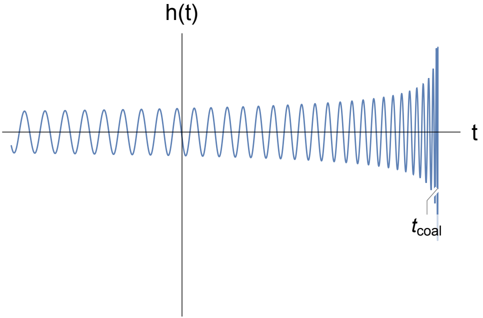
$$\Phi(t) = -\frac{2}{(5M_c)^{5/8}}(t_{coal}-t)^{5/8} + \Phi_{coal}$$

Far Field Wave-form

finally

$$h_+(t; M_c, t_{coal}) = \frac{M_c^{5/4}}{r}\left(\frac{5}{t_{coal}-t}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos[\Phi(t)]$$

$$h_\times(t; M_c, t_{coal}) = \frac{M_c^{5/4}}{r}\left(\frac{5}{t_{coal}-t}\right)^{1/4}\cos\theta\,\sin[\Phi(t)]$$



Fourier Transformed Signal

To compare the waveform with experimental signatures we need the Fourier transform

$$\tilde{h}(f) = \int_{-\infty}^{+\infty} dt\, h(t) e^{2\pi i f t} = \int_{-\infty}^{+\infty} dt\, A(t_{ret}, \theta) \left(e^{i\Phi(t_{ret})} + e^{-i\Phi(t_{ret})} \right) e^{2\pi i f t}$$

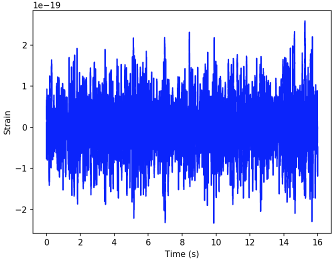
Exercise: using the method of stationary phase obtain the result

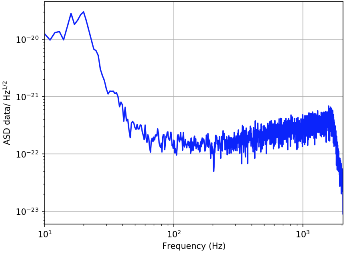
$$\begin{aligned} \tilde{h}_+(f; M_c, t_{coal}) &= \frac{e^{i\Psi(f)}}{r} a M_c^{5/6} \left(\frac{1 + \cos^2\theta}{2} \right) \frac{1}{f^{7/6}} \\ \tilde{h}_\times(f; M_c, t_{coal}) &= \frac{e^{i(\Psi(f) + \pi/2)}}{r} a M_c^{5/6} \cos\theta \frac{1}{f^{7/6}} \end{aligned}$$

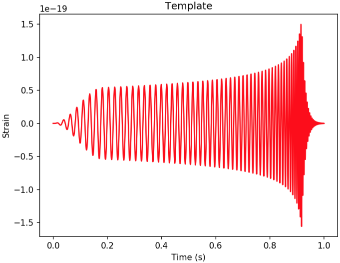
with $a = \frac{1}{\pi^{2/3}} \sqrt{\frac{5}{24}}$ and the phase given by

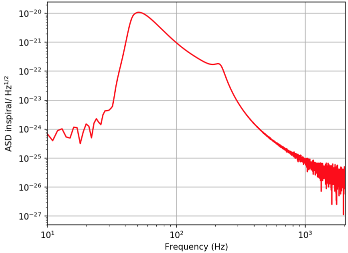
$$\Psi(f) = 2\pi(t_{coal} + r) f - \frac{3}{4\,(8\pi M_c)^{5/3}} \frac{1}{f^{5/3}} - \Phi_0 + \frac{\pi}{4}$$

Fourier Transformed Signal





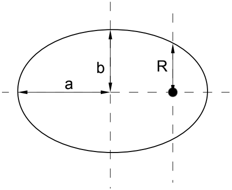




Elliptic orbits

- The excentricity e relates the major semi-axis to the radial scale

$$R = a(1 - e^2)$$



a and e are constans of motion related to E and L

$$a = -\frac{\mu M}{2E} \quad ; \quad (1 - e^2) = -\frac{2EL^2}{\mu^3 M^2} \,.$$

- Now ω_s is not constant (emision spectrum). Still the power is

$$\frac{dE}{dt} = -P \quad \overset{circular}{\Longrightarrow} \quad -\frac{32}{5}\mu^2 R^4 \omega_s^6 \quad \overset{Kepler}{=} \quad -\frac{32}{5}\frac{\mu^2 M^3}{R^5}$$

$$\overset{elliptic}{\Longrightarrow} \quad -\frac{32}{5}\frac{\mu^2 M^3}{a^5} f(e)$$

with

$$f(e) = \frac{1}{(1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right)$$

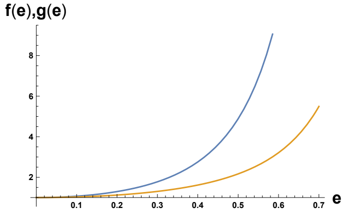
Elliptic orbits

- Also there is angular momentum emission

$$\frac{dL^i}{dt} = -\frac{2}{5}\epsilon^{ikl}\langle \ddot{Q}_{ka}\dddot{Q}_{la}\rangle \overset{elliptic}{\Longrightarrow} -\frac{32}{5}\frac{\mu^2 M^{5/2}}{a^{7/2}} g(e)$$

with now

$$g(e) = \frac{1}{(1-e^2)^2}\left(1+\frac{7}{8}e^2\right)$$



- this causes that both a and e decrease with time

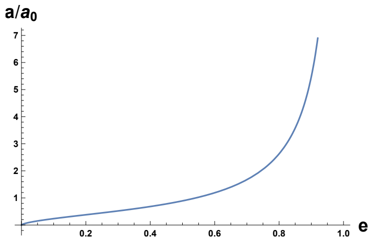
$$\frac{da}{dt} = -\frac{64}{5}\frac{\mu M^3}{a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)$$

$$\frac{de}{dt} = -\frac{304}{15}\frac{\mu m^2}{a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right)$$

Elliptic orbits

which, remarkably, can be integrated analytically

$$a(e) = c_0 \frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$



• Circularization: e decreases very fast with a.
For small $e \ll 1$

$$e = e_0 \left(\frac{a}{a_0}\right)^{19/12}$$

For example, the Hulse-Taylor binary pulsar PSR1913+16 has $a_0 = 2 \times 10^{19}$ m and quite large excentricity $e_0 = 0.617$.

By the time it reaches $a \sim 10^3$km we will have

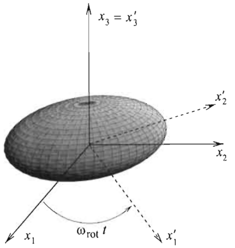
$$e = 0.617 \left(\frac{10^6}{2 \times 10^9}\right)^{19/12} = 3.6 \times 10^{-6}$$

Rotating Rigid Bodies

- Important problem for application to isolated pulsars.
- Assume an ellipsoidal body with semiaxes

a, b, c, uniform density ρ and total mass M, rotating about principal axis c with angular velocidy ω_r. The inertia tensor

$$I_{ij} = \int d^3x\rho(\mathbf{x})(r^2\delta_{ij} - x^i x^j)$$



In the body frame, with
the comoving axes x_i', aligned with the principal axes or inertia

$$I'_{ij} = \mathrm{diag}(I_1, I_2, I_3)$$

with principal moments

$$\begin{aligned} I_1 &= \int d^3x\rho(\mathbf{x}')(x_2'^2 + x_3'^2) = \frac{M}{5}(b^2 + c^2) \\ I_2 &= \int d^3x\rho(\mathbf{x}')(x_1'^2 + x_3'^2) = \frac{M}{5}(a^2 + c^2) \\ I_3 &= \int d^3x\rho(\mathbf{x}')(x_1'^2 + x_3'^2) = \frac{M}{5}(a^2 + b^2) \end{aligned}$$

I_{ij} in fixed coordinates x_i and I'_{ij} in are related by a rotation matrix $I = \mathcal{R}^T I' \mathcal{R}$ i.e.

$$I_{ij} = \begin{pmatrix} \cos\omega_r t & -\sin\omega_r t & 0 \\ \sin\omega_r t & \cos\omega_r t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \cos\omega_r t & \sin\omega_r t & 0 \\ -\sin\omega_r t & \cos\omega_r t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotating Rigid Bodies

$$
\begin{aligned}
I_{11} &= \frac{I_1 + I_2}{2} + \frac{I_1 - I_2}{2}\cos 2\omega t \\
I_{12} &= \frac{I_1 - I_2}{2}\sin 2\omega t \\
I_{22} &= \frac{I_1 + I_2}{2} - \frac{I_1 - I_2}{2}\cos 2\omega t \\
I_{33} &= I_3
\end{aligned} \tag{9}
$$

Notice that I_{ij} and M_{ij} are related by $I_{ij} = \delta_{ij} c - M_{ij}$. Hence since we only need $\ddot{M}_{ij}$

$$
\begin{aligned}
M_{11} &= -\frac{I_1 - I_2}{2}\cos 2\omega_r t + \mathrm{constant} \\
M_{22} &= -\frac{I_1 - I_2}{2}\sin 2\omega_r t + \mathrm{constant} \\
M_{33} &= +\frac{I_1 - I_2}{2}\sin 2\omega_r t + \mathrm{constant}
\end{aligned}
$$

with this we arrive at the strain

$$
\begin{aligned}
h_+ &= \frac{4\omega_r^2 (I_1 - I_2)}{r}\left(\frac{1 + \cos^2\theta}{2}\right)\cos(2\omega_r t) \\
h_x &= \frac{4\omega_r^2 (I_1 - I_2)}{r}\cos\theta\,\cos(2\omega_r t)
\end{aligned}
$$

Rotating Rigid Bodies

In terms of the ellipticity ϵ and GW frequency $f_{GW} = 2\pi(2\omega_r)$

$$\epsilon = \frac{I_1 - I_2}{I_3}$$

the amplitude is

$$\boxed{h_0 = \frac{4\pi^2}{r} I_3 f_{GW}^2 \epsilon}$$

since typical neutron stars have $M = 1.4M_\odot$ and $a \sim 10$km this gives

$$I_3 = \frac{2}{5}Ma^2 \simeq 10^{38}\mathrm{kg\,m}^2$$

Then fixing $r = 10$ kpc, $\epsilon = 10^{-6}$ and $f_{GW} = 1$ kHz we obtain

$$h_0 \simeq 10^{-25}\left(\frac{10\mathrm{kpc}}{r}\right)\left(\frac{I_3}{10^{38}\mathrm{kg\,m}^2}\right)\left(\frac{\epsilon}{10^{-6}}\right)\left(\frac{f_{GW}}{1\mathrm{kHz}}\right)^2$$

SUMMARY

- $f_{GW}(t) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{M_c^{5/8}} \frac{1}{(t_{coal} - t)^{3/8}}$
- Quasi-circular approximation $\dot{\omega}_{GW} \ll \omega_{GW}^2 \Rightarrow f_{GW} \ll 13.7 \left(\frac{1.2\, M_\odot}{M_c}\right) \mathrm{kHz}$
- ISCO cutoff $\Rightarrow f_{GW} \leq 0.8 \left(\frac{1.2\, M_\odot}{M_c}\right) \mathrm{kHz}$
- Fourier Transform $\Rightarrow \tilde{h}(f) \sim f^{-7/6}$
- Ellipticity, e, enhances energy and angular momentum emision, and causes fast circularization
- Bodies rotating around principal axis, x^3, emit iff transverse ellipticity $\epsilon \sim I_1 - I_2 \neq 0$.
  The amplitude is proportional to ω^2.

Chapter 3: Black Hole Quasi-Normal Modes

- Scalar field on a Schwarzschild Metric

- Metric perturbation in polar coordinates

- Boundary conditions

- The radiation field in the far zone

- Quasi Normal modes

Scalar field on a Scharzschild Metric

- On the Schwarzschild metric

$$d\bar{s}^2_{Schw} = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

with $A = \left(1 - \frac{R_S}{r}\right)$ and $R_S = 2M$, consider a free massless scalar field

$$\Box\phi = 0$$

The *most general* splitting $(t, r), (\theta, \phi)$ involves all scalar spherical harmonics $l = 0, 1, 2...$

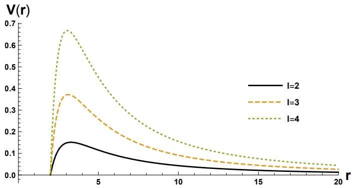
$$\phi(t, r, \theta, \phi) = \frac{1}{r}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} u_{lm}(t, r) Y_{lm}(\theta, \phi)$$

- Effective radial equation:
  using $L^2 Y_{lm} = l(l+1)Y_{lm}$ get

$$\left[A\partial_r(A\partial_r) - \partial_t^2 - V_l(r)\right] u_{lm}(t, r) = 0$$

where

$$V_l(r) = A(r)\left[\frac{l(l+1)}{r^2} + \frac{R_S}{r^3}\right]$$



Scalar field on a Schwarzschild Metric

- define the "tortoise coordinate" $r_*(r)$ by

$$d\bar{s}^2_{Schw} = A(r)(-dt^2 + dr_*^2) + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

hence $dr/A(r) = dr_*$ which integrates to

$$r_* = r + R_S \log \frac{r - R_S}{R_S} \quad \text{with} \quad r \in (R_S, \infty) \Longleftrightarrow r_* \in (-\infty, \infty)$$

Also from $A(r)\partial_r = \partial_{r_*}$ get

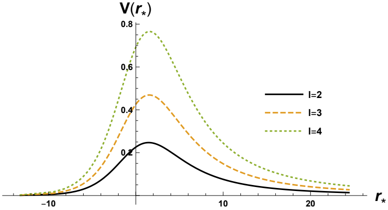
$$\left[\partial^2_{r_*} - \partial^2_t - V_l(r)\right] u_{lm} = 0$$

- Fourier transforming

$$u_{lm}(t, r) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \tilde{u}_{lm}(\omega, r) e^{-i\omega t}$$

arrive at a stationary Schrödinger equation

$$\boxed{\left[-\frac{d^2}{dr_*^2} + V_l(r)\right] \tilde{u}_{lm}(r_*) = \omega^2 \tilde{u}_{lm}(r_*)} \qquad (10)$$



Metric perturbation in general gauge

• Perturb the Black Hole by some external matter $T_{\mu\nu}$. The Schwarzschild metric will fluctuate $g_{\mu\nu}(x) = \bar{g}_{\mu\nu} + h_{\mu\nu}(x)$ and Einstein's equation will become $G^{(0)}_{\mu\nu}(\bar{g}) = 0$ plus a first order correction

$$G^{(1)}_{\mu\nu}(h) = 8\pi T_{\mu\nu}$$

• the most general expansion involves 10 tensor spherical harmonics

$$h_{\mu\nu}(t,\mathbf{x}) = \sum_{l,m}\left(\sum_{a\in\textbf{Polar}} H^{a}_{lm}(t,r)(\mathbf{T}^{a}_{lm})_{\mu\nu}(\theta,\phi) + \sum_{b\in\textbf{Axial}} H^{b}_{lm}(t,r)(\mathbf{T}^{b}_{lm})_{\mu\nu}(\theta,\phi)\right)$$

- The matrices $\{\mathbf{T}^{a}_{lm}, \mathbf{T}^{b}_{lm}\}$ are the Zerilli tensor harmonics.

- They form two groups:

$$\begin{array}{lll} \textbf{Polar} & a = & \{tt, Rt, Et, L0, T0, E1, E2\} \\ \textbf{Axial} & b = & \{Bt, B1, B2\} \end{array}$$

• The split comes from the behaviour under parity $\pi(x^i) = -x^i$

$$\begin{aligned} \pi(\mathbf{T}^{a}_{lm}) &= (-1)^{l}\,\mathbf{T}^{a}_{lm} \\ \pi(\mathbf{T}^{b}_{lm}) &= (-1)^{l+1}\mathbf{T}^{b}_{lm} \end{aligned} \qquad (11)$$

- The non-vanishing components are

$$(\mathbf{T}^{tt}_{lm})_{tt} = Y_{lm}(\theta,\phi)$$

$$(\mathbf{T}^{Rt}_{lm})_{0i} = \frac{1}{\sqrt{2}} n_i Y_{lm}(\theta,\phi)$$

$$(\mathbf{T}^{Et}_{lm})_{0i} = \frac{1}{\sqrt{2l(l+1)}} r\partial_i Y_{lm}$$

$$(\mathbf{T}^{Bt}_{lm})_{0i} = \frac{1}{\sqrt{2l(l+1)}} iL_i Y_{lm}$$

$$(\mathbf{T}^{L0}_{lm})_{ij} = n_i n_j Y_{lm}$$

$$(\mathbf{T}^{T0}_{lm})_{ij} = \frac{1}{\sqrt{2}}(\delta_{ij} - n_i n_j) Y_{lm}$$

$$(\mathbf{T}^{E1}_{lm})_{ij} = a_l (r/2)(n_i\partial_j + n_j\partial_i) Y_{lm}$$

$$(\mathbf{T}^{B1}_{lm})_{ij} = a_l (i/2)(n_i L_j + n_j L_i) Y_{lm}$$

$$(\mathbf{T}^{E2}_{lm})_{ij} = b_l r^2 \Lambda_{ij,i'j'}(\hat{\mathbf{n}})\partial_{i'}\partial_{j'} Y_{lm}$$

$$(\mathbf{T}^{B2}_{lm})_{ij} = b_l r (i/2)\Lambda_{ij,i'j'}(\hat{\mathbf{n}})(\partial_{i'} L_{j'} + L_{j'}\partial_{i'}) Y_{lm}$$

- Remark that these components are cartesian

• in the far wave zone $\tilde{g}_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}$ most convenient is the TT- gauge

Lemma

performing gauge transformations $h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\mu \xi_\nu)$ we can reach the TT- gauge , where $\implies \{a, b\} \in \{E2, B2\}$

$$h_{ij}^{TT}(t,x,y,z) = \frac{1}{r} \sum_{l \geq 2} \sum_{m=-l}^{l} \Big(u_{lm}(t-r)(\mathbf{T}_{lm}^{E2})_{ij}(\theta,\phi) + v_{lm}(t-r)(\mathbf{T}_{lm}^{B2})_{ij}(\theta,\phi) \Big)$$

and $u_{lm}(t-r), v_{lm}(t-r)$ carry the physical polarizations of the wave in the radiation zone.

• in the near horizon zone, $\tilde{g}_{\mu\nu} \to g_{\alpha\beta}^{Schw} + h_{\alpha\beta}$ and, hence, polar $x^\alpha = (t, r, \theta, \phi)$ coordinates are preferred

$$h_{\alpha\beta}(t,r,\theta,\phi) = \sum_{l,m} \left(\sum_{a \in \mathbf{Polar}} h_{lm}^{a}(t,r)(\mathbf{t}_{lm}^{a})_{\alpha\beta}(\theta,\phi) + \sum_{b \in \mathbf{Axial}} h_{lm}^{b}(t,r)(\mathbf{t}_{lm}^{b})_{\alpha\beta}(\theta,\phi) \right)$$

most convenient gauge is RW-gauge where $\{a, b\} \in \{tt, Rt, \cancel{Et}, L0, T0, \cancel{E1}, \cancel{E2}, Bt, B1, \cancel{B2}\}$.

Lemma:

performing gauge transformations $h'_{\alpha\beta} = h_{\alpha\beta} - (\bar{D}_\alpha \xi_\beta + \bar{D}_\beta \xi_\alpha)$ we may reach the Regge-Wheeler (RW) gauge , where $\Rightarrow \{a, b\} \in \{tt, Rt, L0, T0, Bt, B1\}$

- write the metric perturbation $g_{\alpha\beta} = \bar{g}^{Schw}_{\alpha\beta} + h_{\alpha\beta}(t, r, \theta, \phi)$ in the *RW* gauge

$$h^{RW}_{\alpha\beta} = \sum_{l \geq 0,1,2} \sum_{m=-l}^{l} \begin{pmatrix} h^{tt}_{lm}(t,r) & h^{Rt}_{lm}(t,r) & -\frac{1}{\sin\theta} h^{Bt}_{lm}(t,r)\partial_\phi & \sin\theta\, h^{Bt}_{lm}(t,r)\,\partial_\theta \\ - & h^{L0}_{lm}(t,r) & -\frac{1}{\sin\theta} h^{B1}_{lm}(t,r)\partial_\phi & \sin\theta\, h^{B1}_{lm}(t,r)\,\partial_\theta \\ - & - & h^{T0}_{lm}(t,r) & 0 \\ - & - & - & \sin^2\theta\, h^{T0}_{lm}(t,r) \end{pmatrix} Y_{lm}(\theta, \phi)$$

- expand as well the perturbing Energy-Momentum tensor

$$T_{\alpha\beta}(t, r, \theta, \phi) = \sum_{l,m} \left(\sum_{a \in \mathbf{Polar}} s^a_{lm}(t,r)(\mathbf{t}^a_{lm})_{\alpha\beta}(\theta,\phi) + \sum_{b \in \mathbf{Axial}} s^b_{lm}(t,r)(\mathbf{t}^b_{lm})_{\alpha\beta}(\theta,\phi) \right)$$

with $\{a, b\} \in \{tt, Rt, Et, L0, T0, E1, E2, Bt, B1, B2\}$.

- Plug $h^{RW}_{\alpha\beta}$ and $T_{\alpha\beta}$ into the linearized Einstein equation,

$$G^{(1)}_{\alpha\beta}(h) = 8\pi T_{\alpha\beta}$$

after some tedious algebra

1.- equations for axial and polar perturbations *decouple*

2.- in each sector, a clever combination of perturbations (master field) satisfies a fully decoupled equation

3.- all the other perturbations can be derived from these master fields.

4.- this is only true if we first Fourier transform the fields

$$\tilde{h}^{a,b}_{lm}(\omega, r) = \int dt\ h^{a,b}_{lm}(t, r) e^{i\omega t}$$

Axial perturbations

- the RW-master field for axial perturbations

$$\tilde{Q}_{lm}(\omega, r) = -\frac{A(r)}{r}\,\tilde{h}^{B1}_{lm}(\omega, r)$$

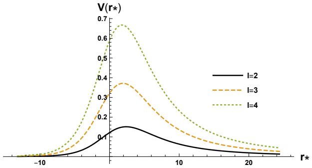
satifies the following decoupled master equation

Regge-Wheeler equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l^{RW}(r)\right]\tilde{Q}_{lm} = \tilde{S}^{axial}_{lm}$$

where

$$V_l^{RW}(r) = \left(1 - \frac{R_S}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{3R_S}{r^3}\right]$$



and

$$\tilde{S}^{axial}_{lm} = i\,\frac{16\pi A(r)}{r}\left(A(r)\tilde{s}^{B1}_{lm}(\omega, r) + \left(\partial_r - \frac{2}{3}\right)\left[A(r)s^{B2}_{lm}(\omega, r)\right]\right)$$

Polar perturbations

- the Zerilli master field for polar perturbations

$$\tilde{Z}_{lm}(\omega, r) = \frac{1}{\lambda r + 3M} \tilde{h}^{T0}_{lm}(\omega, r) + \frac{rA(r)}{i\omega(\lambda r + 3M)} \tilde{h}^{Rt}(\omega, r)$$

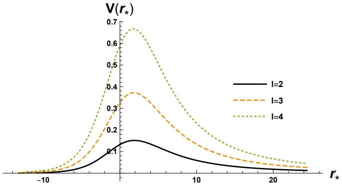
it satifies the decoupled Zerilli master equation

Zerilli equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V^Z_l(r)\right] \tilde{Z}_{lm} = \tilde{S}^{polar}_{lm}$$

where $\lambda = (l-1)(l+2)/2$ and $R_S = 2M$

$$V^Z_l(r) = A(r) \frac{2\lambda^2(\lambda+1)r^3 + 12\lambda^2 M r^2 + 18\lambda M^2 r + 18M^3}{r^3(\lambda r + 3M)^2}$$



Boundary conditions

- In summary, both $\tilde{\Phi} = \{\tilde{Q}_{lm}, \tilde{Z}_{lm}\}$ satisfy a similar equation

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right]\tilde{\Phi} = \tilde{S}$$

with $V(r_* \to \pm\infty) = 0$ and (assumption) $S(r_* \to \pm\infty) = 0$ this asymptotes to

$$\left[\frac{d^2}{dr_*^2} + \omega^2\right]_{r_* \to \pm\infty} \tilde{\Phi} = 0 \qquad ; \qquad \tilde{\Phi}(\omega, r) \stackrel{r_* \to \pm\infty}{\longrightarrow} e^{\pm i\omega r_*} \, .$$

- They seed the ingoing and outgoing solutions $\Phi(t, r) = \int_{-\infty}^{\infty} d\omega \, \tilde{\Phi}(\omega, r) e^{-i\omega t}$

$$\Phi(t, r \to \infty) \quad \stackrel{r_* \to +\infty}{\longrightarrow} \quad \int_{-\infty}^{\infty} d\omega \left[\Phi_\infty^{\rm out}(\omega) e^{-i\omega(t - r_*)} + \Phi_\infty^{\rm in}(\omega) e^{-i\omega(t + r_*)}\right]$$
$$\Phi(t, r \to R_S) \quad \stackrel{r_* \to -\infty}{\longrightarrow} \quad \int_{-\infty}^{\infty} d\omega \left[\Phi_S^{\rm out}(\omega) e^{-i\omega(t - r_*)} + \Phi_S^{\rm in}(\omega) e^{-i\omega(t + r_*)}\right]$$

Select behaviour: outgoing at $r \to \infty$ and infalling at the horizon $r \to R_S$

Boundary conditions

- In summary, both $\tilde{\Phi} = \{\tilde{Q}_{lm}, \tilde{R}_{lm}\}$ satisfy a similar equation

$$\frac{d^2}{dr_*^2}\tilde{\Phi} + \left[\omega^2 - V(r)\right]\tilde{\Phi} = \tilde{S}_{lm}$$

with $V(r_* \to \pm\infty) = 0$ and (assumption) $S_{lm}(r_* \to \pm\infty) = 0$ this asymptotes to

$$\left[\frac{d^2}{dr_*^2} + \omega^2\right]\tilde{\Phi} = 0 \qquad ; \qquad \tilde{\Phi}(\omega, r) \overset{r_* \to \pm\infty}{\longrightarrow} e^{\pm i\omega r_*} \,.$$

- They seed the ingoing and outgoing solutions $\Phi(t, r) = \int_{-\infty}^{\infty} d\omega\, \tilde{\Phi}(\omega, r) e^{-i\omega t}$

$$\Phi(t, r \to \infty) \overset{r_* \to +\infty}{\longrightarrow} \int_{-\infty}^{\infty} d\omega \left[\Phi_{\infty}^{\rm out}(\omega) e^{-i\omega(t - r_*)} + \cancel{\Phi_{\infty}^{\rm in}(\omega)} e^{-i\omega(t + r_*)}\right]$$

$$\Phi(t, r \to R_S) \overset{r_* \to -\infty}{\longrightarrow} \int_{-\infty}^{\infty} d\omega \left[\cancel{\Phi_{S}^{\rm out}(\omega)} e^{-i\omega(t - r_*)} + \Phi_{S}^{\rm in}(\omega) e^{-i\omega(t + r_*)}\right]$$

Select behaviour: outgoing at $r \to \infty$ and ingoing at the horizon $r \to R_S$ Hence, in both limits.

Boundary conditions

$$\tilde{\Phi}(\omega, r) \overset{r_* \to \pm\infty}{\longrightarrow} e^{i\omega |r_*|}$$

Boundary conditions

- So all we have to do is solve Zerilli and RW-master equations with asymptotic behaviour at infinity $r \to \infty$

$r_* \to +\infty$

$$\tilde{Z}_{lm}(t,\omega) \longrightarrow A^{\rm out}_{lm}(\omega) e^{i\omega r_*} \quad ; \quad \tilde{Q}_{lm}(t,\omega) \longrightarrow B^{out}_{lm}(\omega) e^{i\omega r_*}$$

as well as near horizon $r \to R_S$

$r_* \to -\infty$

$$\tilde{Z}_{lm}(t,\omega) \longrightarrow A^{\rm in}_{lm}(\omega) e^{-i\omega r_*} \quad ; \quad \tilde{Q}_{lm}(t,r) \longrightarrow B^{\rm in}_{lm}(\omega) e^{-i\omega r_*}$$

- **Two questions to answer:**

  1.- Can we reconstruct the radiation field in the far zone out of $A^{\rm out}_{lm}(\omega)$ and $B^{out}_{lm}(\omega)$?

  2.- How does the spectrum of solutions to the Schrödinger equations above look like ?

The radiation field in the far zone

- In the radiation zone $r \to \infty$, in the TT gauge

$$h_{\mu\nu}^{TT}(t, x^1, x^2, x^3) = \frac{1}{r} \sum_{l \geq 2} \sum_{m=-l}^{l} \left[u_{lm}(t-r)(\mathbf{T}_{lm}^{E2})_{\mu\nu} + v_{lm}(t-r)(\mathbf{T}_{lm}^{B2})_{\mu\nu} \right]$$

how can we connect $(u_{lm}, v_{lm}) \iff (A_{lm}^{out}, B_{lm}^{out})$?

Answer

a gauge transformation plus the change $(r, \theta, \phi)^{RW} \to (x^1, x^2, x^3)^{TT}$ triggers the miracle

$$\begin{aligned} u_{lm}(t-r) &= c_l \int d\omega\, A_{lm}^{\rm out}(\omega) e^{-i\omega(t-r)} \\ v_{lm}(t-r) &= c_l \int d\omega\, B_{lm}^{\rm out}(\omega) e^{-i\omega(t-r)} \end{aligned}$$

with $c_l = \frac{1}{\sqrt{2}} \left[\frac{(l+2)!}{(l-2)!} \right]^{1/2}$,

Quasi-Normal Modes (QNM)

- Let us solve the **master field equations** for $\tilde{\Phi} = \{\tilde{Q}_{lm}, \tilde{Z}_{lm}\}$ without source

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V(r)\right] \tilde{\Phi}(\omega, r) = 0 \qquad (12)$$

with $V = V_l^{RW}, V_l^{Z}$.

Theorem

The equation (12) admits a solution $\tilde{\Phi}(\omega, r_*)$ with the boundary behaviour

$$\tilde{\Phi}(\omega, r_* \to \pm\infty) \;\propto\; e^{i\omega|r_*|}\,.$$

only for a discrete and complex set of frequencies

$$\omega_n = \omega_{R,n} + i\omega_{I,n} \qquad (n = 1, 2, 3, ..)$$

- Solutions $\tilde{\Phi}(\omega_n, x)$ are termed quasi-normal modes.

- Remarkably V_l^{RW} and V_l^{Z} are isospectral

Quasi-normal modes

- ω_n (in units of c/R_S)

	l=2	l=3
n	$\omega_R + i\omega_I$	$\omega_R + i\omega_I$
1	0.747343 - i 0.177925	1.198887 - i 0.185406
2	0.693422 - i 0.547830	1.165288 - i 0.562596
3	0.602107 - i 0.956554	1.103370 - i 0.958186
⋮		

- the full solution is a combination that decays exponentially with time, since $\omega_I < 0$.

$$\begin{aligned} \Phi(t,x) &= \int_{-\infty}^{\infty} \tilde{\Phi}(\omega,x) e^{-i\omega t} d\omega \hookrightarrow \sum_n \tilde{\Phi}^{(n)}(x) e^{-i\omega_n t} \\ &= \sum_n \tilde{\Phi}^{(n)}(x) e^{-i\omega_{R,n} t + \omega_{I,n} t} \end{aligned} \quad (13)$$

- The least damped mode emits GW at a frequency f_1 and decay time τ_1 such that

$$f_1 = \frac{\omega_{R,1}}{2\pi}\frac{c}{R_S} = \frac{0.747343}{2\pi}\frac{3\times 10^8}{2M} \simeq 12\,kHz\left(\frac{M_\odot}{M}\right)\mathrm{Hz}$$

$$\tau \simeq 1/|\omega_{I,1}| = \frac{R_S}{0.177925c} \simeq 5.5\times 10^{-5}\left(\frac{M}{M_\odot}\right)\mathrm{s}$$

Quasi-normal modes

- The spectrum reveals a rather weird structure

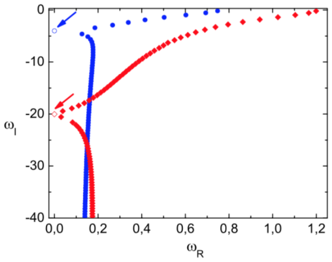


Figure: QNM modes ω_n for $l = 2$ (dots) and $l = 3$ (diamonds) (taken from Berti E. arXiv:0411025)

- $\omega_{I,n}$ decrease monotonically
- $\omega_{R,n}$ has two branches, separated by a mode with $\omega_{R,n_0} = 0$.
- for large $n \gg 1$ and fixed l $\omega_{R,n}$ saturates

$$\omega_n \sim \frac{\log 3}{4\pi} - \frac{i}{2}\left(n + \frac{1}{2}\right)$$

- for large $l \gg 1$ and fixed (large) $n \gg 1 \Rightarrow \omega_{n,l} \sim (2l+1) - i(2n+1)$

Radial infall into a black hole

- consider a particle in free radial infall in the Schwarzschild metric

$$x_0^\mu(t) = (t, r_0(t), \theta_0, \phi_0)$$

then

$$T^{\mu\nu}(t, r, \theta, \phi) = m\gamma \frac{dx_0^\mu}{dt} \frac{dx_0^\nu}{dt} \frac{\delta[r - r_0(t)]}{r^2} \delta[\cos(\theta) - \cos(\theta_0)] \delta[\phi - \phi_0]$$

- project to get the source tensors harmonics

$$s_{lm}^{a,b}(t, r) = c^a(r)^2 \int d\Omega (\mathbf{t}_{lm}^{a,b})^{*\mu\nu} T_{\mu\nu}(t, r, \theta, \phi)$$

- As a consequence of cylindrical symmetry (let $\theta = 0$)

$\Rightarrow s_{lm}^b = 0$ for $= B1, B2$. Hence the RW equation is not excited $\Rightarrow B_{lm}^{out} = 0$ (no B modes)

$\Rightarrow m = 0$ hence only $s_l^a \equiv s_{lm=0}^a \neq 0$.

- Integrating the Zerilli equation $\left[\frac{d^2}{dx^2} + \omega^2 - V_l^Z(r)\right] \tilde{Z}_l(\omega, r) = \tilde{S}_l(\omega, r)$ obtain the $A_l^{out}(\omega)$ modes.

Radial infall into a black hole

- Reconstruct the far field wave-form

$$u_l(t-r) = c_l \int d\omega A_l^{\text{out}}(\omega) e^{-i\omega(t-r)}$$

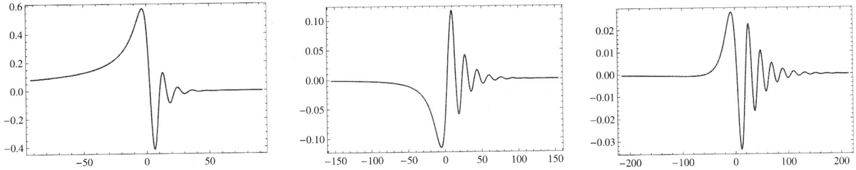


Figure: The gravitational wave form $u_l(t)$ for $l = 2, 3, 4$ (from Maggiore M., numerical data courtesy of Ermis Mitsou)

SUMMARY

• Perturbations of Schwarzschild by some infalling matter $T_{\mu\nu}$ split in two sectors: (polar and axial)
$h_{\alpha\beta}(t,r,\theta,\phi) = \sum_{l,m}\left(\sum_{a\in\textbf{Polar}} h^a_{lm}(t,r)(\mathbf{t}^a_{lm})_{\alpha\beta}(\theta,\phi) + \sum_{b\in\textbf{Axial}} h^b_{lm}(t,r)(\mathbf{t}^b_{lm})_{\alpha\beta}(\theta,\phi)\right)$

• Each sector has a master field $\tilde{\Phi}(\omega) = (\tilde{Z}_{lm}(\omega), \tilde{Q}_{lm}(\omega))$ for which Einstein equations becomes a Schrödinger like equation

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\tilde{\Phi}(\omega,x) = 0$$

• The physical boundary conditions are outgoing at $r \to \infty$ and infalling at the horizon. They entail $\tilde{\Phi} \sim e^{i\omega|r_*|}$ for $r_* \to \pm\infty$

• They are only possible for a discrete spectrum of complex frequencies $\omega_n = \omega_{R,n} + i\omega_{I,n}$.

• $\omega_{I,n}$ are negative and monotonically decreasing with n. They entail an exponential damping of the initial perturbation.

• Perturbing a BH of mass $M_\odot$, it will ring at $f \sim 10$kHz and relax in $\tau \sim 5 \times 10^{-5}$