Transverse Beam Dynamics

JUAS 2019 - Tutorial 3 (solutions)

1 Exercise: Basics of lattice design

Design a FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

Answer.

Lattice parameters: $L = 10$ m, $\theta = 5$ degrees$= 0.087266$ rad, $f = \frac{1}{\sqrt{2}}\frac{L}{2} = 3.535$ m.

Maximum and minimum betatron functions:

$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin\mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin\mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta\left(1 + \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f + \frac{L}{2}\right)\theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta\left(1 - \frac{1}{2}\sin\frac{\mu}{2}\right)}{4\sin^2\frac{\mu}{2}} = \frac{f}{L}\left(4f - \frac{L}{2}\right)\theta = 0.28207 \text{ m}$$

2 Exercise: Bump and Orbit Control

Given two kickers located at the two ends of a FODO cell with phase advance 45 degrees (the two kickers are located at L_{cell} distance from each other), compute the strengths of such kickers (in radians) in order to give the beam, initially at $(x_i,\, x_i') = (0,\, 0)$, an arbitrary offset at the end of the cell while preserving its angle, $\left(x_f,\, x_f'\right) = (x_{\text{arbitrary}},\, 0)$.

Solution

The transfer matrix of a periodic cell is:

$$M = \begin{pmatrix} \cos\varphi + \alpha\sin\psi & \beta\sin\varphi \\ -\gamma\sin\varphi & \cos\varphi - \alpha\sin\varphi \end{pmatrix}$$

Substituting the value for the phase advance we get the matrix to apply to the beam right after the first kick k_1:

$$\begin{pmatrix} x_f \\ x_f' \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1+\alpha & \beta \\ -\gamma & 1-\alpha \end{pmatrix}\begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} \beta k_1 \\ (1-\alpha)k_1 \end{pmatrix}$$

From this we see that to achieve an arbitrary x_f we need:

$$k_1 = \frac{\sqrt{2}x_f}{\beta}$$

The second kick, k_2, has only to remove the final tilt:

$$k_2 = -x_f' = -\frac{(1-\alpha)}{\sqrt{2}}k_1$$

Notice that one can reduce the strength of the kickers by placing them close to a focusing quadrupole, where β has its maximum.

3 Exercise: Measurement of Twiss parameters

One of the possible ways to determine experimentally the Twiss parameters at a given point makes use of a so-called quadrupole scan. One can measure the transverse size of the beam in a profile monitor, called Wire Beam Scanner (WBS), located at a distance L downstream a focusing quadrupole, as a function of the normalised gradient in this quadrupole. This allows to compute the emittance of the beam, as well as the β and the α functions at the entrance of the quadrupole.

Let's consider a quadrupole Q with a length of $l = 20$ cm. This quadrupole is installed in an electron transport line where the particle momentum is 300 MeV$/c$. At a distance $L = 10$ m from the quadrupole the transverse beam size is measured with a WBS, for various values of the current I_Q. The maximum value of the quadrupole gradient G is obtained for a current of 100 A, and is $G = 1$ T/m.

Hint: G is proportional to the current. **Advice:** use thin-lens approximation.

1. How does the normalised focusing strength K vary with I_Q?

   Answer. The quadrupole gradient G is proportional to the current flowing through the coils I_Q

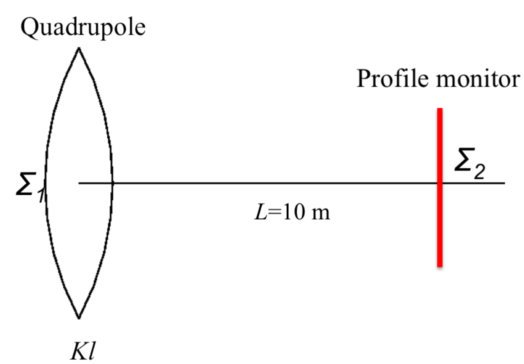
   $$G = C \cdot I_Q,$$

   C is the proportionality coefficient. We know that $G = 1$ T/m when $I_Q = 100$ A, therefore $C = 0.01$ T/(A·m). The normalised focusing strength is

   $$K = \frac{G}{P/q} \quad \text{therefore} \quad K = \frac{C \cdot I_Q}{P/q}$$

2. Give the expression Σ_2 as function of α_1, β_1, and γ_1

   Answer. Let Σ_1 and Σ_2 be the 2×2 matrices with the twiss parameters, $\Sigma = \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$, at the quadrupole entrance and at the wire scanner, respectively.



It is worth explaining that the matrix Σ multiplied by the emittance ϵ is the covariance matrix of the beam distribution:

$$\Sigma\epsilon = \begin{pmatrix} \beta\epsilon & -\alpha\epsilon \\ -\alpha\epsilon & \gamma\epsilon \end{pmatrix} = \begin{pmatrix} \langle x^2 \rangle & \langle xx' \rangle \\ \langle x'x \rangle & \langle x'^2 \rangle \end{pmatrix}$$

The transverse beam size of the beam is given by $\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta_x \epsilon_x}$ (horizontal beam size), and $\sigma_y = \sqrt{\langle y^2 \rangle} = \sqrt{\beta_y \epsilon_y}$ (vertical beam size). Here we will simply use the following notation: $\sigma_1 = \sqrt{\beta_1 \epsilon}$ for the beam size (horizontal or vertical) at position 1, and $\sigma_2 = \sqrt{\beta_2 \epsilon}$ for the beam size (horizontal or vertical) at position 2.The matrix Σ propagates from position 1 to position 2 as follows:

$$\Sigma_2 = M \Sigma_1 M^T$$

where M is the transfer matrix of the system and M^T its transpose. We have:

$$\begin{aligned} \Sigma_2 = \begin{pmatrix} \beta_2 & -\alpha_2 \\ -\alpha_2 & \gamma_2 \end{pmatrix} &= \begin{pmatrix} 1 - KlL & L \\ -Kl & 1 \end{pmatrix} \begin{pmatrix} \beta_1 & -\alpha_1 \\ -\alpha_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} 1 - KlL & -Kl \\ L & 1 \end{pmatrix} \\ &= \begin{pmatrix} \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1) Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2 & \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 \\ \beta_1 L (Kl)^2 + (2\alpha_1 L - \beta_1) Kl + \gamma_1 L - \alpha_1 & \beta_1 (Kl)^2 + 2\alpha_1 Kl + \gamma_1 \end{pmatrix} \end{aligned} \tag{1}$$

3. Show that β_2 can be written in the form: $\beta_2 = A_2 (Kl)^2 + A_1 (Kl) + A_0$, and express A_0, A_1, and A_2 as a function of L, α_1, β_1, and γ_1.

   Answer. We can see from Eq. (1) that:

$$\beta_2 = \beta_1 L^2 (Kl)^2 + 2L(\alpha_1 L - \beta_1)Kl + \beta_1 - 2\alpha_1 L + \gamma_1 L^2$$

   and therefore:

$$\begin{aligned} A_2 &= \beta_1 L^2 \\ A_1 &= 2L(\alpha_1 L - \beta_1) \\ A_0 &= \beta_1 - 2\alpha_1 L + \gamma_1 L^2 \end{aligned}$$

   Hint for the next questions: show that if one expresses β_2 as

$$\beta_2 = B_0 + B_1 (Kl - B_2)^2$$

   one has:

$$\begin{aligned} B_0 &= A_0 - A_1^2/4A_2^2 = L^2/\beta_1 \\ B_1 &= A_2 = L^2\beta_1 \\ B_2 &= -A_1/A_2 = 1/L - \alpha_1/\beta_1 \end{aligned}$$

4. Express the final beam size, σ_2, as a function of Kl, and find its minimum, which will correspond to $(Kl)_{\mathrm{min}}$.

   Answer. The transverse r.m.s. beam size is $\sigma = \sqrt{\epsilon\beta}$, where ϵ is the transverse (geometric) emittance. As we have seen in the previous questions β_2 depends quadratically on Kl: $\beta_2 = B_0 + B_1 (Kl - B_2)^2$. Since ϵ is constant, if we want to minimise σ_2, we have to minimise β_2:

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}(Kl)} = 0 \longrightarrow 2B_1(Kl - B_2) = 0 \longrightarrow (Kl)_{min} = B_2 = \frac{1}{L} - \frac{\alpha_1}{\beta_1} \tag{2}$$

   We can write:

$$\sigma_2^2 = \beta_2\epsilon = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon$$

   Why is this useful? By means of a quadrupole scan (i.e. changing the quadrupole strength) we identify the strength Kl which minimises the value σ_2^2. We fit a parabola to the measurements σ_2^2 vs. Kl, and select then $\sigma_2^2((Kl)_{min})$. The minimum beam size is given by:

$$\mathrm{Min}(\sigma_2) = L\sqrt{\frac{\epsilon}{\beta_1}} = \sqrt{B_0\epsilon} \tag{3}$$

5. How does σ_2 vary with Kl when $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$?

   Answer. Under this condition:

$$\sigma_2^2 = \frac{L^2}{\beta_1}\left(1 + \beta_1^2(Kl - (Kl)_{min})^2\right)\epsilon \longrightarrow \sigma_2 \simeq L\sqrt{\beta_1\epsilon}(Kl - (Kl)_{min})$$

   For $|Kl - (Kl)_{\mathrm{min}}| \gg 1/\beta_1$, σ_2 depends linearly on Kl, with slope

$$\frac{\mathrm{d}\sigma_2}{\mathrm{d(Kl)}} = \frac{L^2\beta_1}{\sigma_2}(Kl - (kl)_{min})\epsilon.$$

6. Deduce the values of α_1, β_1, and γ_1 from the measurement σ_2, as a function of the quadrupole current I_Q.

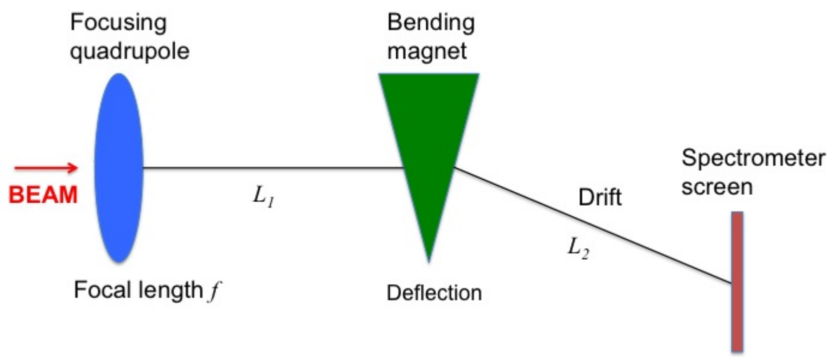
   Answer. We know that

$$Kl = \frac{G \cdot l}{p/e} = \frac{C \cdot l \cdot I_Q}{p/e} = \frac{0.01[\mathrm{T/(Am)}]\cdot 0.2[\mathrm{m}]}{(0.3[\mathrm{GeV}]/0.3)[\mathrm{Tm}]} \cdot I_Q = 2 \times 10^{-3} \cdot I_Q$$

   If we measure σ_2 as a function of the quadrupole current I_Q, from the minimum value we can get β_1 (Eq. (3)), and since from the measurement we obtain $(Kl)_{min} = 2 \times 10^{-3} (I_Q)_{min}$, using Eq. (2) we can calculate α_1. Once we know β_1 and α_1, it is then straightforward to calculate $\gamma_1 = (1 + \alpha_1^2)/\beta_1$.

4 Exercise: The spectrometer line of CTF3

The CTF3 (CLIC Test Facility 3) experiment at CERN consists of a linac that injects very short electron bunches into an isochronous ring. A spectrometer line made of one quadrupole and one bending magnet is located at the end of the linac where the particle momentum is 350 MeV/c. The goal of the spectrometer is to measure the energy before injecting the electrons in the ring. The spectrometer line is sketched on the figure below. It is made of a focusing quadrupole of focal length f, a drift space of length L_1, a bending magnet of deflection angle θ in the horizontal plane, and a drift space of length L_2. We assume that the spectrometer line starts at the quadrupole and ends at the end of the second drift. We neglect the focusing effect of the dipole.



1. If the effective length of the dipole is $l_B = 0.43$ m, what should be the magnetic field (in Tesla) inside the dipole to deflect the electrons by an angle of 35 degrees?

   Answer. One has $\theta = \frac{l}{\rho}$ and $B\rho = 3.356\,p$: $B = \dfrac{3.356\,p\,\theta}{l} = 1.66$ T.

2. Starting from the general horizontal 3×3 transfer matrix of a sector dipole of deflection angle θ, show that the transfer matrix of a dipole in the thin-lens approximation is

$$M_{dipole} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

   Which approximations are done?

   Hint: A sector dipole has the following 3×3 transfer matrix:

$$M_{\mathrm{dipole}} = \begin{pmatrix} \cos\theta & \rho\sin\theta & \rho(1-\cos\theta) \\ -\frac{\sin\theta}{\rho} & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

   Answer. We need to compute the limit for $l \to 0$ while keeping $\theta = \frac{l}{\rho} = \mathrm{const}$. Remember that, if θ is a small angle, $\cos\theta \approx 1$, $\sin\theta \approx \theta$. Besides the trivial elements, such as m_{11}, m_{22}, and m_{23}, the others read:

$$m_{12}: \qquad \lim_{l\to 0}\rho\sin\theta = \lim_{l\to 0}\frac{\sin\theta}{\frac{1}{\rho}} = \lim_{l\to 0} l\cdot\underbrace{\frac{\sin\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{13}: \qquad \lim_{l\to 0}\rho\,(1-\cos\theta) = \lim_{l\to 0}\rho\left(1-\cos\frac{l}{\rho}\right) = \lim_{l\to 0} l\cdot\underbrace{\frac{1-\cos\frac{l}{\rho}}{\frac{l}{\rho}}}_{\text{const}} = 0$$

$$m_{21}: \qquad \lim_{l\to 0} -\frac{\sin\theta}{\rho} = \lim_{\rho\to\infty} -\frac{\sin\theta}{\rho} = 0$$

therefore, in thin-lens approximation the matrix of a dipole magnet, is

$$M_{\text{dipole}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}.$$

3. In the thin-lens approximation, derive the horizontal extended 3×3 transfer matrix of the spectrometer line. Show that it is:

$$M_{spectro} = \begin{pmatrix} \frac{f-L_1-L_2}{f} & L_1+L_2 & L_2\theta \\ -\frac{1}{f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}$$

Answer. For the spectrometer line, one has

$$M_{\text{spectro}} = M_{\text{Drift2}}\cdot M_{\text{Dipole}}\cdot M_{\text{Drift1}}\cdot M_{\text{Quad}}$$

therefore:

$$M_{\text{spectro}} = \begin{pmatrix} 1 & L_2 & L_2\theta \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1-\frac{L_1}{f} & L_1 & 0 \\ -\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Assuming $D = D' = 0$ at the entrance of the quadrupole, what is the dispersion and its derivative at the end of the spectrometer line? Give the numerical value of D' at the end of the spectrometer line for the angle of 35 degrees.

Answer. At the entrance of the line, $D = 0$ and $D' = 0$. If M is the transfer matrix of a system the dispersion D at exit is the element m_{13} of M, whereas D' is the element m_{23}:

$$D = L_2\theta,$$
$$D' = \theta = 35 \text{ degrees} = 0.61.$$

5. What is the difference between a periodic lattice and a beam transport lattice (or transfer line) as concerns the betatron function ?

Answer. In a periodic lattice the β-functions are periodic and contained in the (periodic) transfer matrix of the lattice. In transfer line one needs to know the initial conditions in order to calculate the β-functions at any point (using the transfer matrix).

6. Derive the betatron function β_2 at the end of the spectrometer line in terms of L_1, L_2, f and β_1, assuming $\alpha_1 = 0$.

Hint 1. Remember from the lecture:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & SC'+S'C & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

An alternative way to transport the Twiss parameters is through the σ matrix:

$$\sigma_i = \begin{pmatrix} \beta_i & -\alpha_i \\ -\alpha_i & \gamma_i \end{pmatrix}$$

This matrix multiplied by the emittance ϵ gives the so-called beam matrix (which has already been introduced during the lecture):

$$\Sigma_i = \begin{pmatrix} \beta_i\epsilon & -\alpha_i\epsilon \\ -\alpha_i\epsilon & \gamma_i\epsilon \end{pmatrix}$$

If σ_1 is the matrix at the entrance of the transfer line, the matrix σ_2 at the exit of the transfer line is given by

$$\sigma_2 = M\sigma_1 M^T$$

where M is the 2×2 transfer matrix of the line extracted from the extended 3×3 transfer matrix (see question 3), and M^T the transpose matrix of M.

Hint 2. For the calculations, write M as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ and replace the values of the matrix elements only at the end.

Answer. One has $\sigma_2 = M\sigma_1 M^T$. If $\alpha_1 = 0$, then $\sigma_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix}$

$$\sigma_2 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} \beta_1 & 0 \\ 0 & 1/\beta_1 \end{pmatrix} \begin{pmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} \beta_1 m_{11}^2 + m_{12}^2/\beta_1 & \beta_1 m_{11}m_{21} + m_{12}m_{22}/\beta_1 \\ \beta_1 m_{11}m_{21} + m_{12}m_{22}/\beta_1 & \beta_1 m_{21}^2 + m_{22}^2/\beta_1 \end{pmatrix}$$

Therefore:

$$\beta_2 = \beta_1 m_{11}^2 + m_{12}^2/\beta_1$$

Since $m_{11} = \frac{f-L_1-L_2}{f}$ and $m_{12} = L_1 + L_2$, one has:

$$\beta_2 = \beta_1 \left(1 - \frac{L_1 + L_2}{f}\right)^2 + \frac{(L_1 + L_2)^2}{\beta_1}.$$

7. Given the numerical values $L_1 = 1$ m, $L_2 = 2$ m, $\beta_1 = 10$ m, $\alpha_1 = 0$, compute the value of the focal length f such that the betatron function at the end of the spectrometer line is minimum.

   Answer. If $L_1 = 1$ m, $L_2 = 2$ m, and $\beta_1 = 10$ m, then $\beta_2 = 0.9 + 10\left(1 - \frac{3}{f}\right)^2$. To have β_2 minimum one needs $\left(1 - \frac{3}{f} = 0\right)$.Therefore, $f = 3$ m.

8. For an off-momentum particle, compute the deviation from the design orbit? Why is it important to minimise the β function in the spectrometer?

   Answer. With dispersion, the deviation from the design orbit is $\Delta x = D\frac{\Delta P}{P_0}$. Measuring Δx allows to determine ΔP and therefore P, if one has calibrated the spectrometer at P_0. It is important to minimise β_2 (at the screen location) in order to have the best possible resolution for Δx: a smaller β_2 will result in a smaller transverse beam size on the screen, which favours an accurate measurement of the momentum.

5 Exercise: Transfer matrix of a dipole magnet

- Remember weak focusing:
  $K = \frac{1}{\rho^2}$:

$$M_{\text{Dipole}} = \begin{pmatrix} \cos\left(\sqrt{K}L\right) & \frac{1}{\sqrt{K}}\sin\left(\sqrt{K}L\right) \\ -\sqrt{K}\sin\left(\sqrt{K}L\right) & \cos\left(\sqrt{K}L\right) \end{pmatrix} = \begin{pmatrix} \cos\frac{L}{\rho} & \rho\sin\frac{L}{\rho} \\ -\frac{1}{\rho}\sin\frac{L}{\rho} & \cos\frac{L}{\rho} \end{pmatrix}$$

- Compute the 3×3 matrix of a sector dipole including the dispersion terms.

  Remembering that:

$$D(s) = S(s) \int_0^s \frac{1}{\rho(t)} C(t)\,\mathrm{d}t - C(s) \int_0^s \frac{1}{\rho(t)} S(t)\,\mathrm{d}t$$

  one can easily find that:

$$D(L) = \rho \left(1 - \cos\frac{L}{\rho}\right)$$

$$D'(L) = \sin\frac{L}{\rho}$$

  which allows to write M_{dipole} as 3×3 matrix in the form:

$$M_{\text{Dipole}} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix}$$