TWISS FUNCTIONS

Lecture 1
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Introduction

- **These lectures assume knowledge of :**
  - **The 2nd order differential equations of motion in 'hard-edge' field models for various elements with a momentum deviation,**
  - **The solutions to the differential equations,**
  - **The solutions expressed in terms of matrices,**
  - **The use of matrices to track ions though a lattice with respect to a curvilinear co-ordinate system that follows the central orbit thus providing physical co-ordinates for individual ions that are easy to understand,**
  - **Thin (zero thickness) quadrupole and higher order lenses**
  - **The closed-orbit perturbation equation for small dipole errors.**
  - **Skew quadrupole lenses.**

'Twiss' functions

- One of the historical mysteries in accelerators is how the 'Twiss' functions got their name. Twiss was once asked to elucidate this problem and he claimed there was no paper that made the link to him.
- There are two ways of looking at Twiss functions:
  - The first is to regard them as a parametric way of expressing the motion equation and its solution. This interpretation makes a bridge from tracking single ions to the wider view of calculating beam envelopes.
  - The second is to regard them as purely geometric parameters for defining ellipses and hence beam envelopes. Dropping the strict correspondence to individual particles can lead to some interesting extensions such as the inclusion of scattering.

Twiss and the transverse motion equation

- **The general motion equation has the form,**

$$\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + K_z(s) z = 0 \qquad (1.1)$$

where z can be either x or y.

- **Start by parameterising the coordinate z as,**

$$z(s) = A\beta_z(s)^{1/2} \cos\left[\int_0^s \frac{\mathrm{d}\sigma}{\beta_z(\sigma)} + B\right] \qquad (1.2)$$

where s is the distance along the equilibrium orbit, A and B are constants depending on the starting conditions, $\beta(s)$ is the betatron amplitude function and σ is an integration variable representing distance.

- **The phase, $\mu(s)$ of the pseudo oscillation is given by,**

$$\mu_z(s) = \int_0^s \frac{1}{\beta_z(\sigma)} \mathrm{d}\sigma \qquad (1.3)$$

The above parameterisation is done with hindsight. Refs [1.1] and [1.2] give the historical background and explain how to come to this point. However, a separate derivation/definition of α_z, β_z, and γ_z will be given in Lecture 6.

Twiss & transverse motion continued

- To complete this description, new variables related to the derivative of $\beta(s)$ are defined,

$$\alpha_z(s) = -\frac{1}{2}\frac{\mathrm{d}\beta_z}{\mathrm{d}s} \qquad (1.4)$$

and

$$\gamma_z(s) = \frac{1+\alpha_z^{\,2}}{\beta_z} \qquad (1.5)$$

- **Notes:**
  - The phase shift for 1 turn in a ring divided by 2π is known as the *tune, Q*:

$$Q_z = \frac{\mu_{z,1\,\mathrm{Turn}}}{2\pi} = \frac{1}{2\pi}\oint_{\mathrm{Circ.}} \frac{1}{\beta_z(\sigma)}\,\mathrm{d}\sigma \qquad (1.6)$$

  - α, β and γ are distinguished from the relativistic parameters by a suffix for the plane, but this is later dropped for brevity.
  - *Equations (1.1) to (1.6) are so widely used that they need to be committed to memory.*

Basic Twiss equation

- **Substitution of equation (1.2) into (1.1) yields a differential equation for $\sqrt{\beta_z(s)}$ that is more complicated than the original motion equation, which at first sight seems a poor deal,**

$$\frac{\mathrm{d}^2\sqrt{\beta_z}}{\mathrm{d}s^2} + K_z(s)\sqrt{\beta_z} = \left(\sqrt{\beta_z}\right)^{-3} \qquad (1.7)$$

(To derive this you will need the α_z function.)

- **Equation (1.7) is rarely used, but it is necessary to know that it exists. There will be two applications given in Lectures 2 and 6.**

- **Today, we will take the approach of comparing the matrix equations to the equivalent Twiss equations. This leads to a whole battery of new equations, but they appear so often that they eventually become familiar.**

General Twiss transfer matrix

- **Re-express equation (1.2) as,**

$$z(s) = A\beta^{1/2}\cos\mu + B\beta^{1/2}\sin\mu$$

where *A* and *B* are different constants and the suffix 'z' has been dropped for brevity.

Differentiation gives,

$$z'(s) = -A\beta^{-1/2}(\alpha\cos\mu + \sin\mu) + B\beta^{-1/2}(\cos\mu - \alpha\sin\mu)$$

- **The constants *A* and *B* can be replaced using the initial conditions at $s = s_1, \mu = 0$,**

$$A = z_1\beta_1^{-1/2} \quad \text{and} \quad B = z_1'\beta_1^{1/2} + z_1\alpha_1\beta_1^{-1/2}$$

- **To get the general transfer matrix from position s_1 to position s_2, write the phase advance from s_1 to s_2 as $\Delta\mu$, so that,**

$$\boldsymbol{M}(s_1 \to s_2) = \qquad (1.8)$$

$$\begin{pmatrix} \left(\frac{\beta_2}{\beta_1}\right)^{1/2}(\cos\Delta\mu + \alpha_1\sin\Delta\mu) & (\beta_1\beta_2)^{1/2}\sin\Delta\mu \\ -(\beta_1\beta_2)^{-1/2}[(1+\alpha_1\alpha_2)\sin\Delta\mu + (\alpha_2 - \alpha_1)\cos\Delta\mu] & \left(\frac{\beta_1}{\beta_2}\right)^{1/2}(\cos\Delta\mu - \alpha_2\sin\Delta\mu) \end{pmatrix}$$

Twiss transfer matrix for a single turn in a ring or for a matched cell

- When equation (1.8) is applied to a full turn in a ring or to a matched cell, the input conditions equal the output conditions, that is $\alpha=\alpha_1=\alpha_2$, $\beta=\beta_1=\beta_2$ and $\Delta\mu = 2\pi Q$, so that,

$$\boldsymbol{M}_{1\,\text{turn}} = \begin{pmatrix} (\cos 2\pi Q + \alpha \sin 2\pi Q) & \beta \sin 2\pi Q \\ -\gamma \sin 2\pi Q & (\cos 2\pi Q - \alpha \sin 2\pi Q) \end{pmatrix} \quad (1.9)$$

- Remember Q is known as the tune and is the number of betatron oscillations around a ring.
- We will see in the next section that equation (1.9) allows us to unambiguously solve for α, β and γ in terms of the matrix coefficients, at least for a ring.
- We will treat transfer lines much later because they require some further thought.

Solving Twiss in a ring

- **A lattice program can proceed as follows:**
  - **List all the elements in the lattice.**
  - **Calculate the transfer matrices of all elements.**
  - **Multiply all the matrices to obtain the single-turn matrix.**
  - **Compare this matrix to equation (1.8) and solve for α, β and γ using,**

$$\sin 2\pi Q = \frac{m_{12}}{|m_{12}|}\sqrt{1-\left(\frac{m_{11}+m_{22}}{2}\right)^2} \qquad (1.10)$$

$$\beta = \frac{m_{12}}{\sin 2\pi Q} \qquad (1.11)$$

$$\alpha = \frac{(m_{11}-m_{22})}{2\sin 2\pi Q} \qquad (1.12)$$

  - **To step round the lattice, pre-multiply by the matrix of the next element after the observation point and post multiply by the inverse of the same matrix.**

$$M(s_n \rightarrow s_{n-1}) = M_n^{-1} M(s_1 \rightarrow s_n)\, M_n$$

Solving for Q and μ

- Let $Q = 2n\pi + q$, where n is an integer.
- Equation (1.10) allows you to solve for $\sin(2\pi Q)$.
- $\mathrm{Tan}(2\pi Q)$ or $\cos(2\pi Q)$ can also be found easily.
- This allows you to find q (the fractional part), but NOT n (the integer part).
- To find the integer number of oscillations around a ring, or the total phase shift through a long line, it is necessary to step through the lattice with steps of less than 2π and to sum up for the total.
- This can be done in a number of ways but, with the information given so far, use the previous slide to find α, β and γ at all elements in the ring and then use equation (1.8) to cross each element to find the $\Delta\mu$ values.

Dispersion

- To complete the parameterisation of the particle motion we need to include the motion of off-momentum ions using,

$$z(s) = \underbrace{A\beta(s)^{1/2}\cos\left[\int_0^s \frac{\mathrm{d}\sigma}{\beta(\sigma)} + B\right]}_{\text{Betatron motion}} + \underbrace{D(s)\frac{\Delta p}{p}}_{\text{Dispersion motion}} \quad (1.13)$$

- $D(s)$ is known as the dispersion function.
- An analytic derivation of the dispersion function is possible, but it is usual to rely on lattice programs for numerical listings of $D(s)$ and its derivative with distance $D'(s)$.
- The dispersion function is found in much the same way as was done for α and β.
- For rings, the cyclic condition is imposed,

$$\begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ 0 & 0 & 1 \end{pmatrix}_{1\,\text{Turn}} \begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix}$$

where the matrix is for one turn and the input and output values of $D(s)$ and $D'(s)$ are equated.

Dispersion continued

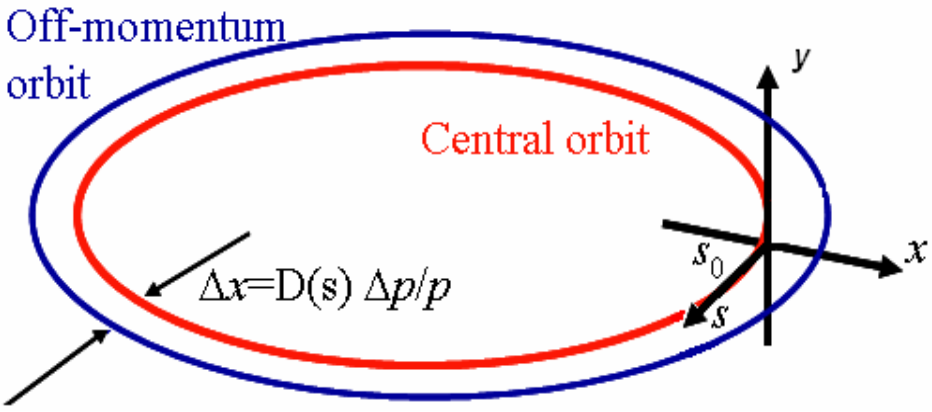
- The dispersion and its derivative at the point of evaluation of the matrix can be solved as,

$$D(s_0) = \frac{(1-m_{22})m_{13} + m_{12}m_{23}}{(2-m_{11}-m_{22})}$$

$$D'(s_0) = \frac{(1-m_{11})m_{23} + m_{21}m_{13}}{(2-m_{11}-m_{22})}$$

- Having found the dispersion vector at one point, s_0, it is simple to tabulate the values at all intermediate points in the ring by either stepping the single-turn matrix round as was already described, or by tracking the vector through the structure from the known point, s_0, to a new point, s_1, by,

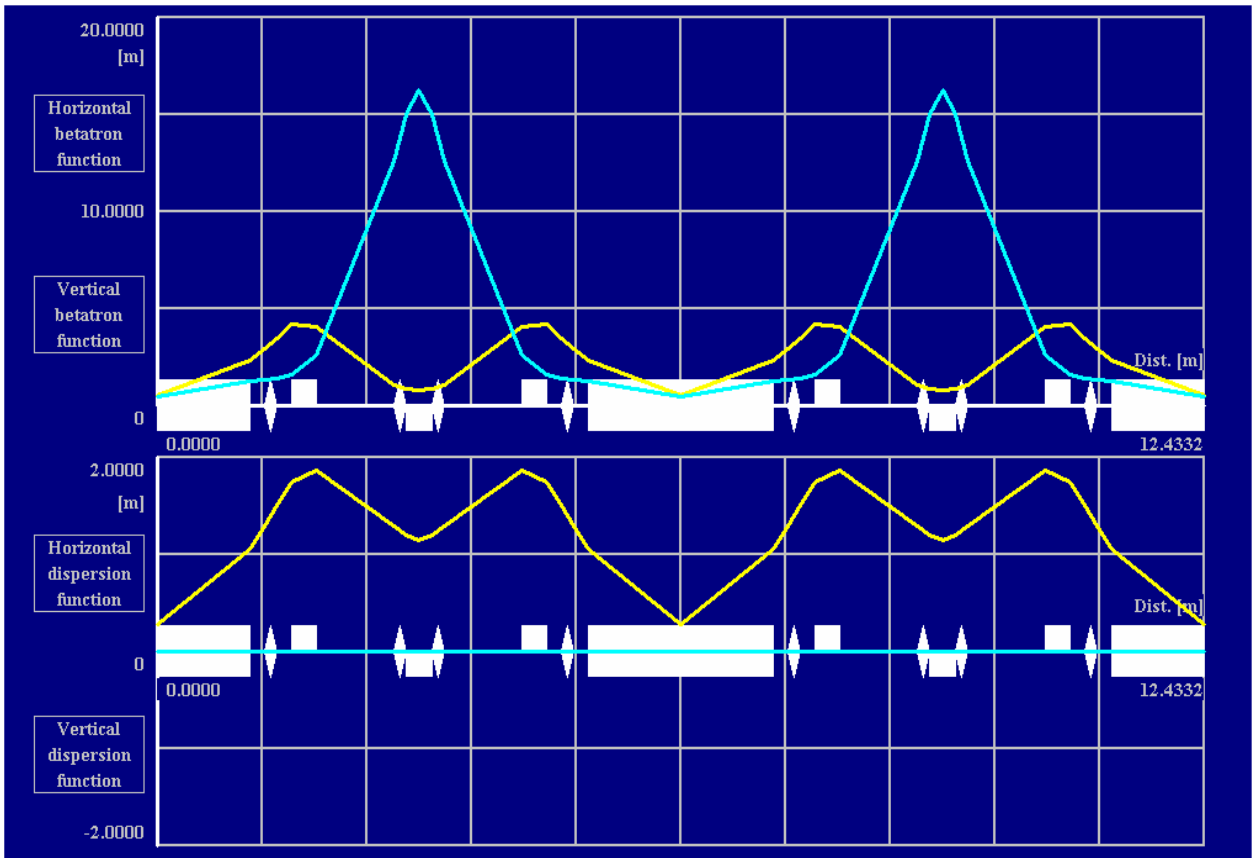
$$\begin{pmatrix} D(s_1) \\ D'(s_1) \\ 1 \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ 0 & 0 & 1 \end{pmatrix}_{s_0 - s_1} \begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix}$$



Typical output from a lattice program

Unit no. 1	Name 2	Type 3	Length [m] 4	Beta-x [m] 5	Alpha-x 6	Mu-x [rad] 7	Dx [m] 8	dDx/ds 9
1	Mb	SBEND	1.0904	0.519	0.0000	0.0000	0.2743	0.00000
2	D1	DRIFT	0.1750	2.365	-1.4757	1.1776	1.0542	1.38548
3	SF	SEXTU	0.1500	2.923	-1.7108	1.2442	1.2967	1.38548
4	D1	DRIFT	0.1750	3.466	-1.9124	1.2914	1.5045	1.38548
5	QF	QUADR	0.3000	4.177	-2.1475	1.3374	1.7469	1.38548
6	D2	DRIFT	0.9179	4.096	2.3884	1.4061	1.8642	-0.62515
7	SD	SEXTU	0.1500	1.090	0.8858	1.8555	1.2904	-0.62515
8	QD	QUADR	0.1500	0.861	0.6403	2.0109	1.1966	-0.62515
9	QD	QUADR	0.1500	0.768	0.0000	2.1989	1.1500	-0.00000
10	SD	SEXTU	0.1500	0.861	-0.6403	2.3870	1.1966	0.62515

The dispersion function (*D*) and the derivative of the dispersion function (d*D*/d*s*) are usually listed and included graphically with the Twiss parameters to give a complete description of the beam.



Phase space

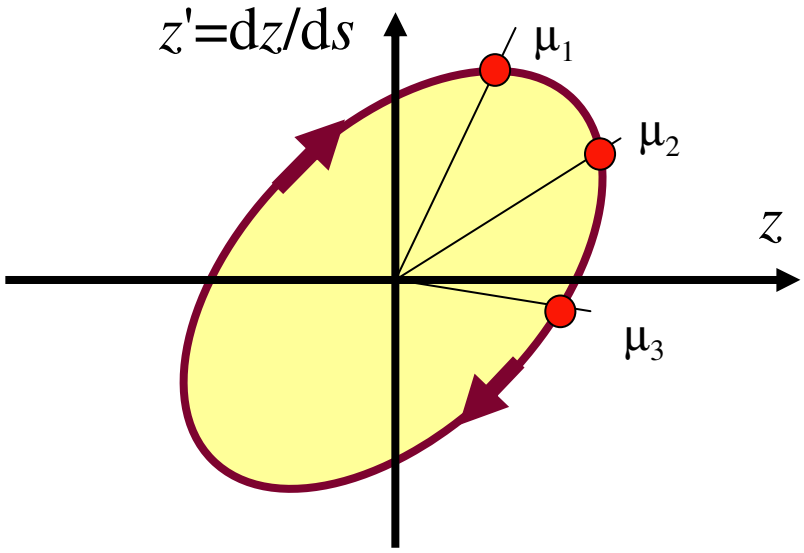
- Returning to equation (1.2) substituted with (1.3),

$$z(s) = A\beta^{1/2}\cos(\mu + B) \qquad (1.14)$$

- Differentiating gives,

$$z'(s) = -A\alpha\beta^{-1/2}\cos(\mu + B) - A\beta^{-1/2}\sin(\mu + B) \quad (1.15)$$

- If these two equations are used to plot a graph for (z, z') for $\mu = 0$ to 2π, one gets an ellipse.



- In the case of a ring or matched cell, the periodicity imposes equality on the input and output α and β values. This means that the particle returns after each turn or transit to the same ellipse but at phases $\mu_1 = B$, $\mu_2 = B + 2\pi Q$, $\mu_3 = B + 4\pi Q$, ….., $B + n2\pi Q$ and so on.

Motion invariant

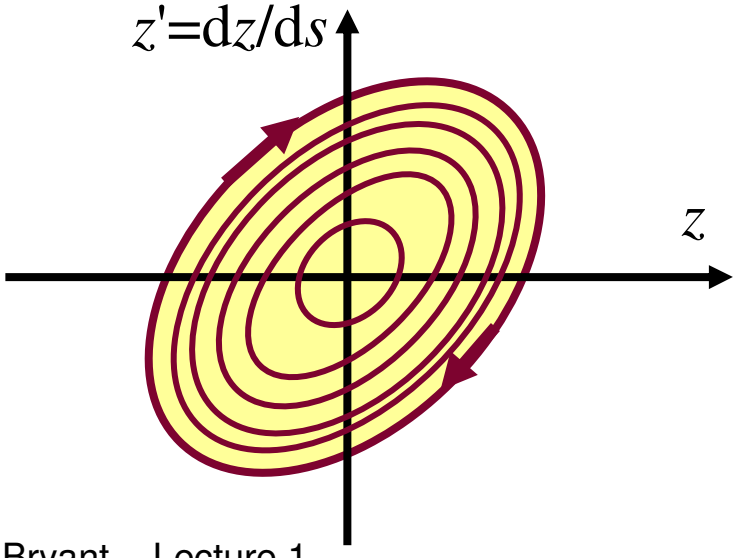
- The elimination of the phase advance from equations (1.14) and (1.15) yields an invariant of the motion,

$$A^2 = \gamma z^2 + 2\alpha z z' + \beta z'^2 \qquad (1.16)$$

- This is known as the *Courant & Snyder Invariant*.

- The motion invariant, A^2, equals the (area/π) of the ellipse described by the betatron motion in phase space. When referring to a single ion, this area is sometimes called the *single-particle emittance,* although this is strictly incorrect (see later slide on emittance).

- All ions in the beam will have a value for this invariant (area/π) and follow similar ellipses.



Liouville's theorem

- **Liouville states that phase space is conserved.**

- **Primarily, this refers to 6-dimensional phase space (*x*-*x*´, *y*-*y*´ and *s*-d*p*/*p*). When the component phase spaces are uncoupled, the phase space is conserved within the 2-dimensional and/or 4-dimensional spaces.**

- **The invariant of the motion in the uncoupled *x*-*x*´or *y*-*y*´ spaces is another way of saying the phase space is conserved.**

- **Phase space is not conserved if ions change, e.g. by stripping or nuclear fragmentation, or if non-Hamiltonian forces appear e.g. scattering or photon emission.**

Transferring Twiss functions

- We have calculated the Twiss functions from the single-turn matrix of a ring and shown how to step round the ring to make a table of the functions.
- We have shown that the Twiss functions define an ellipse in phase space and the area of this ellipse is a constant of the motion.
- Thus, between two points,

$$\begin{aligned} A^2 &= \gamma_2 z_2^{\,2} + 2\alpha_2 z_2 z_2' + \beta_2 z_2'^{\,2} \\ &= \gamma_1 z_1^{\,2} + 2\alpha_1 z_1 z_1' + \beta_1 z_1'^{\,2} \end{aligned} \qquad (1.17)$$

Note this trick of equating the invariant at 2 points for examination questions.

- A trajectory at the two points is related by the transfer matrix $T(s_1 \rightarrow s_2)$, which on this occasion is more conveniently written in the inverse form from points 2 to 1, as,

$$\begin{pmatrix} z \\ z' \end{pmatrix}_1 = \begin{pmatrix} t_{22} & -t_{12} \\ -t_{21} & t_{11} \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix}_2 \qquad (1.18)$$

Note that the modulus is unity so that the inverse is simplified. Remember this for questions.

Transferring Twiss functions continued

- Equation (1.18) can be used to substitute for (z_1, z'_1) on the right hand side of (1.17). After regrouping the terms, expressions for α_2, β_2 and γ_2 can be found in terms of α_1, β_1 and γ_1. These results are usually written in the form of a 3 ×3 matrix (Ref. [1.3]),

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_2 = \begin{pmatrix} t_{11}^{\,2} & -2t_{11}t_{12} & t_{12}^{\,2} \\ -t_{11}t_{21} & [t_{11}t_{22}+t_{12}t_{21}] & -t_{12}t_{22} \\ t_{21}^{\,2} & -2t_{21}t_{22} & t_{22}^{\,2} \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_1 \qquad (1.19)$$

- Special case. In a drift space, $t_{11} = t_{22} = 1$, $t_{12} = \ell$ and $t_{21} = 0$, so that,

$$\beta_2 = \beta_1 - 2\ell\alpha_1 + \ell^2\gamma_1$$

Starting from the centre of a low-β insertion α_1=0 , β_1=β* and γ_1=1/β_1

$$\beta_2 = \beta^* + \ell^2 / \beta^* \qquad (1.20)$$

Thus low-β insertions are associated with high β-values on each side because ℓ must provide sufficient space for a physics experiment.

Note this is often exploited for questions.

Emittance

- The emittance of a beam is related to the phase-space area that it occupies and is therefore related to the motion invariants of the constituent ions (see earlier slide).

- A practical definition of emittance requires a choice for the limiting ellipse that defines the phase-space area of the beam. Usually this is related to some number of standard deviations of the beam distribution, but it could also be the overall ellipse that includes all ions or some fraction of the ions. The definition is best included in the name e.g. 'the 95% emittance equals…' or 'the 1-sigma emittance is…'.

- A further problem of definition is whether the emittance is the phase-space area or the phase-space area divided by π. Since the literature mixes these two definitions, it is better to express the emittance with the π apparent, that is $30\pi \times 10^{-6}$ [m rad] or 30×10^{-6} [π m rad] . In this way, the user sees that the π is included, but can easily remove it, if desired.

- **In these lectures:**

  Geometric emittance, ε = Phase-space area

  BUT π will be apparent in the numerical values or definitions. We write *geometric emittance* to distinguish it from the *normalised emittance* that comes later.

Beam envelopes and Acceptance

- **Referring back to equation (1.14),**

$$z(s) = A\beta^{1/2}(s)\cos(\mu(s) + B)$$

- **The amplitude of the oscillation of an ion is given by,**

$$w(s) = A\beta^{1/2}(s)$$

- **If we talk of a beam, then the envelope or beam half-width is given by,**

$$w(s) = \sqrt{\frac{\beta(s)\varepsilon}{\pi}} \qquad (1.21)$$

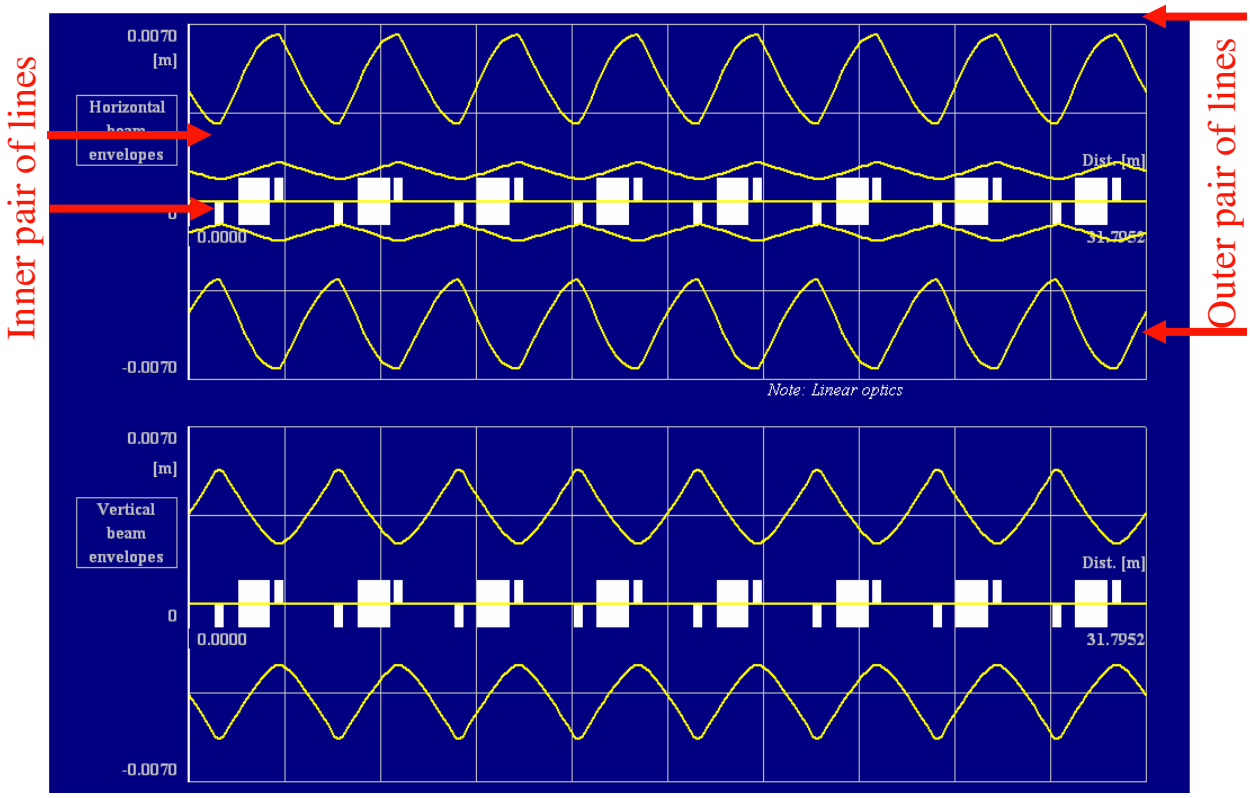
where ε is the emittance. This is a useful formula.

- **The beam envelope or width is subject to the same definition problems as the emittance. So, for example, the beam envelope calculated with the 1-sigma emittance will be the 1-sigma envelope, the beam envelope calculated with the 95% emittance will be the 95% beam envelope and so on.**
- **The area (or area/π) of the largest phase-space ellipse that can pass through a lattice is know as the *acceptance*. *This is a description of the lattice and not the beam.***

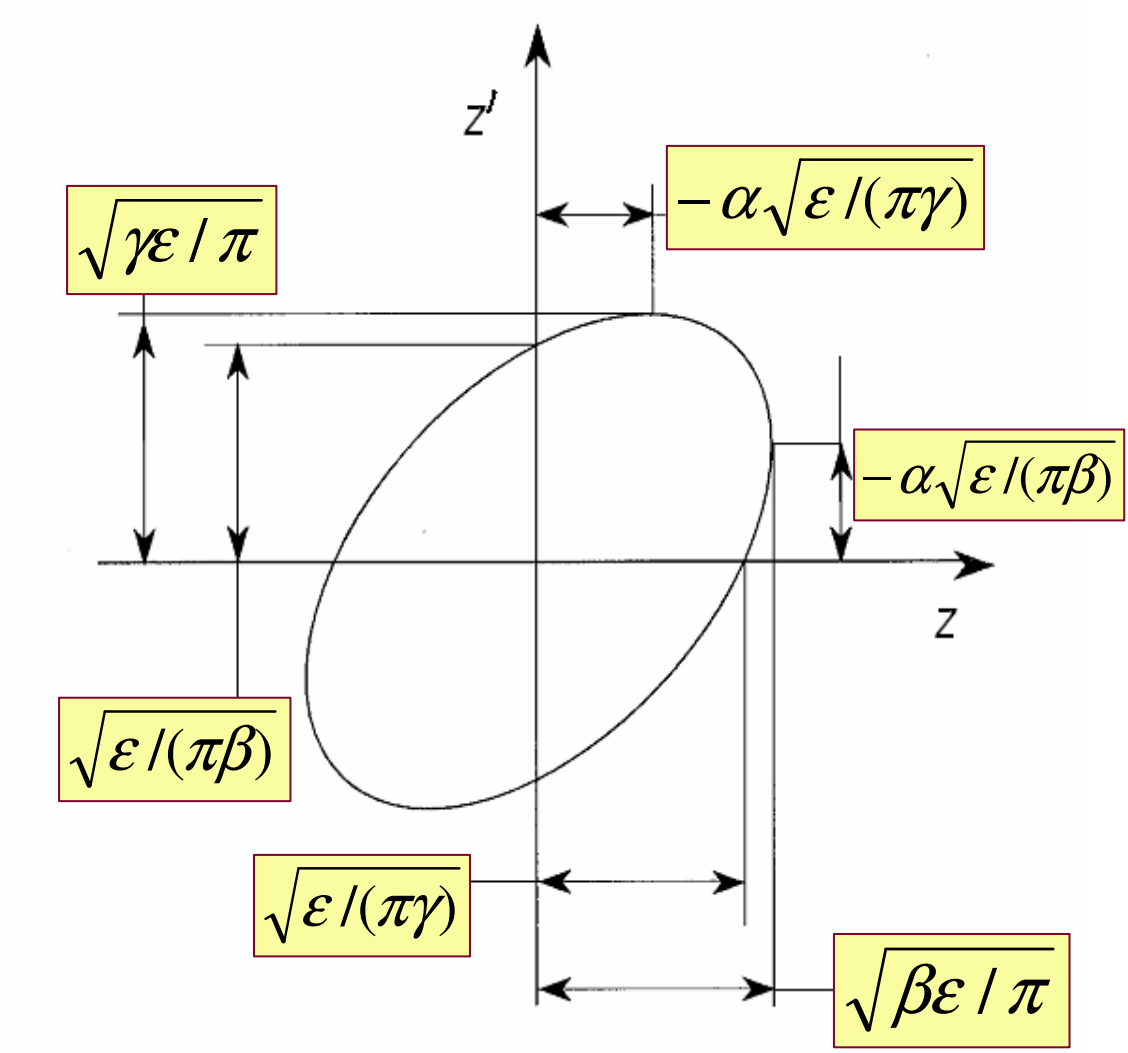
Typical output from a lattice program

Unit no.	Name	Type	Length [m]	Sigma-x [mm]	x-inner [m]	xp-inner [m]	x-Track [m]	xp-outer [m]	x-outer [m]
1	Mb	SBEND	1.0904	0.720	-0.0023	-0.0007	0.0000	0.0007	0.0023
2	D1	DRIFT	0.1750	1.538	-0.0061	-0.0026	0.0000	0.0026	0.0061
3	SF	SEXTU	0.1500	1.710	-0.0071	-0.0032	0.0000	0.0032	0.0071
4	D1	DRIFT	0.1750	1.862	-0.0079	-0.0038	0.0000	0.0038	0.0079
5	QF	QUADR	0.3000	2.044	-0.0089	-0.0044	0.0000	0.0044	0.0089
6	D2	DRIFT	0.9179	2.024	-0.0092	-0.0047	0.0000	0.0047	0.0092
7	SD	SEXTU	0.1500	1.044	-0.0056	-0.0032	0.0000	0.0032	0.0056
8	QD	QUADR	0.1500	0.928	-0.0051	-0.0030	0.0000	0.0030	0.0051
9	QD	QUADR	0.1500	0.876	-0.0048	-0.0029	0.0000	0.0029	0.0048
10	SD	SEXTU	0.1500	0.928	-0.0051	-0.0030	0.0000	0.0030	0.0051
11	D2	DRIFT	0.9179	1.044	-0.0056	-0.0032	0.0000	0.0032	0.0056
12	QF	QUADR	0.3000	2.024	-0.0092	-0.0047	0.0000	0.0047	0.0092
13	D1	DRIFT	0.1750	2.044	-0.0089	-0.0044	0.0000	0.0044	0.0089

In the horizontal plane, the inner pair of lines define the dispersion width and the outer lines the betatron width. In the vertical plane, the dispersion is zero and only the betatron width is visible.



Geometry of the phase-space ellipse



$$\text{Area} = \varepsilon = \pi\left(\gamma z^2 + 2\alpha z z' + \beta z'^2\right)$$

Practical emittance definition that defines the ellipse:

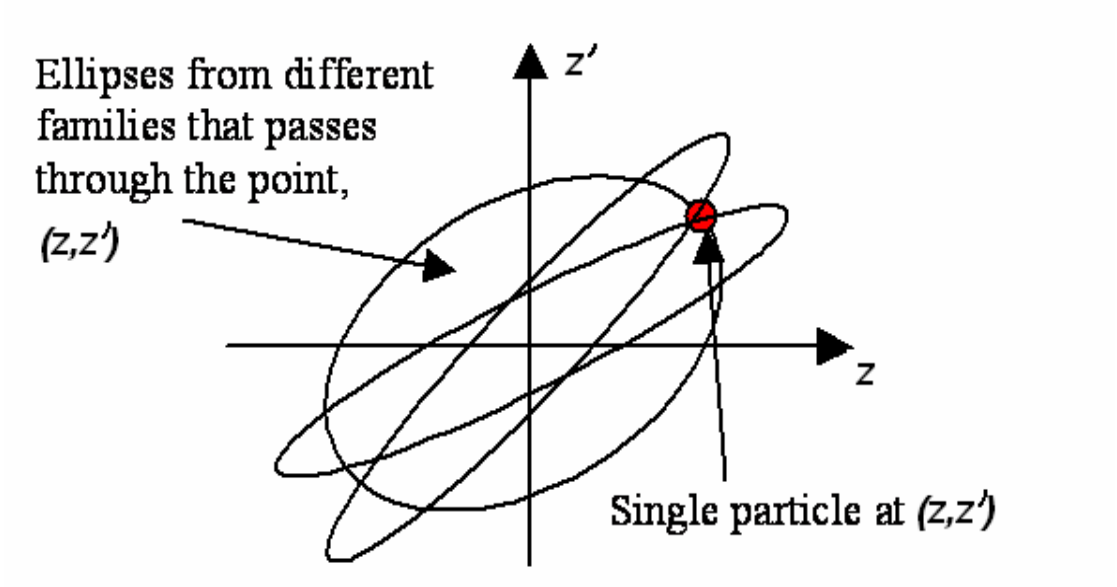
$$\varepsilon = \pi\left(\gamma_y z^2_{1\text{-}\sigma} + 2\alpha_y z_{1\text{-}\sigma} z'_{1\text{-}\sigma} + \beta_y z'^2_{1-\sigma}\right)$$

Transfer lines

- **It was mentioned earlier that transfer lines were in some way different to rings.**
- **The lack of periodicity in a transfer line removes the constraint that the Twiss functions at the exit must equal those at the entry and consequently the Twiss functions are undefined unless the user supplies the values at some reference point, e.g. at the exit from a ring (where the functions are known) which is the entry to the transfer line.**
- **Understanding this difference and the implications can take some time, so be patient.**

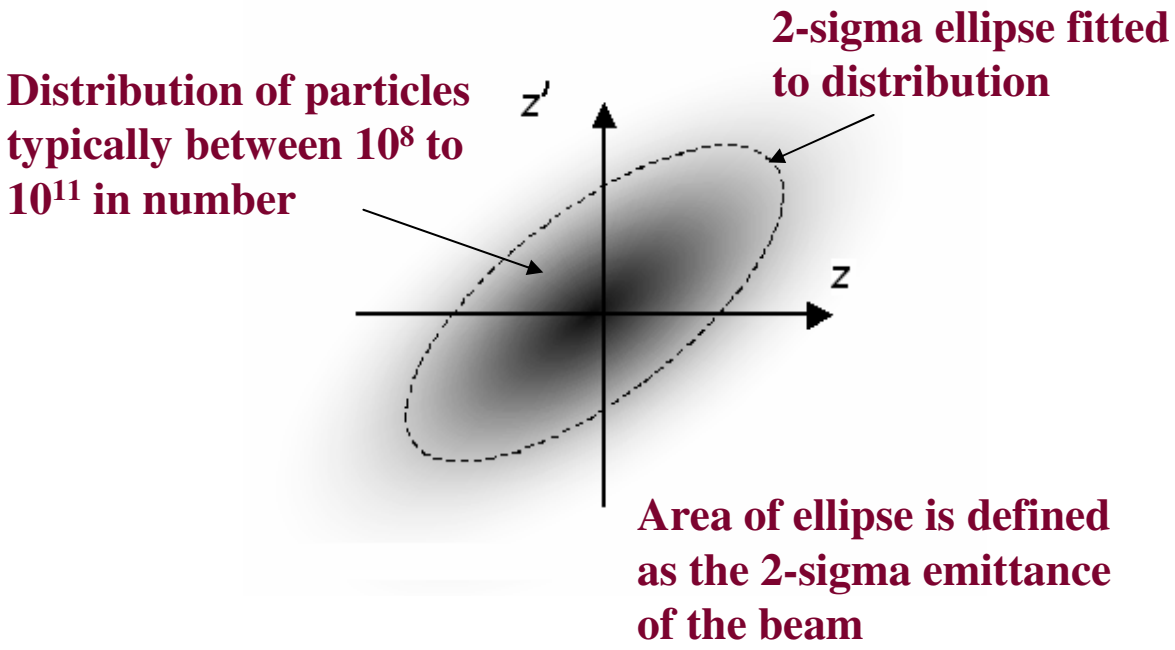
Transfer lines continued

A single ion in phase space provides insufficient information to associate it with one unique set of Twiss functions (see Figure). Without additional information, a single point can be equally well represented by any of an infinite number of sets of Twiss functions (i.e. families of ellipses). Once an arbitrary choice has been made for the Twiss functions, a unique emittance can always be found that places the single ion on just one ellipse in that family. This arbitrary set of parameters can then be tracked through the lattice and will always represent the ion's motion correctly.



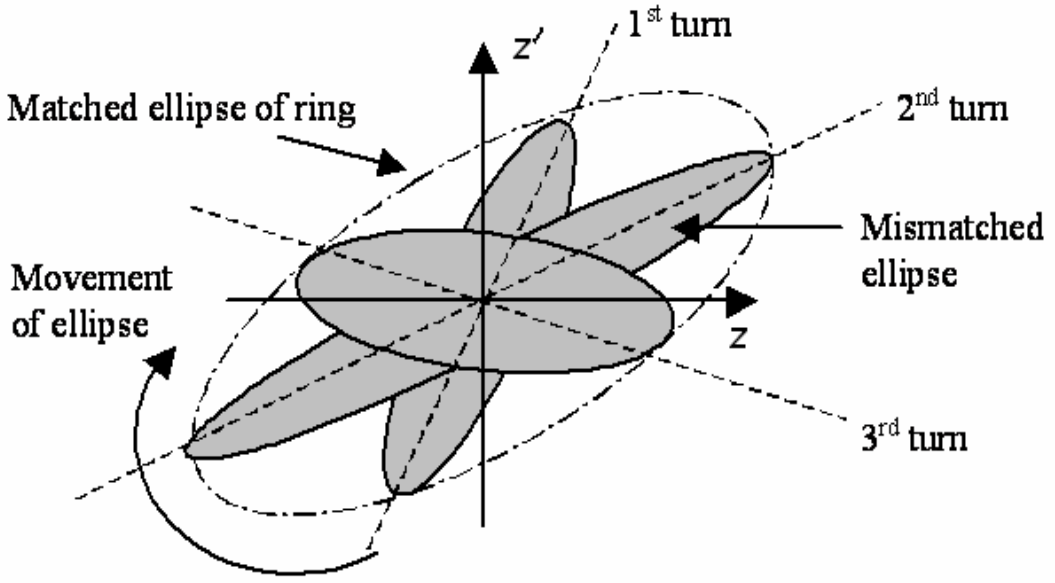
Transfer lines continued

- A collection of ions in phase space will, 'subject to interpretation', define a unique set of Twiss parameters and an emittance that together define the beam.
- One can always impose a statistical solution on the phrase 'subject to interpretation' by making a least squares fit of an ellipse to the ion distribution.



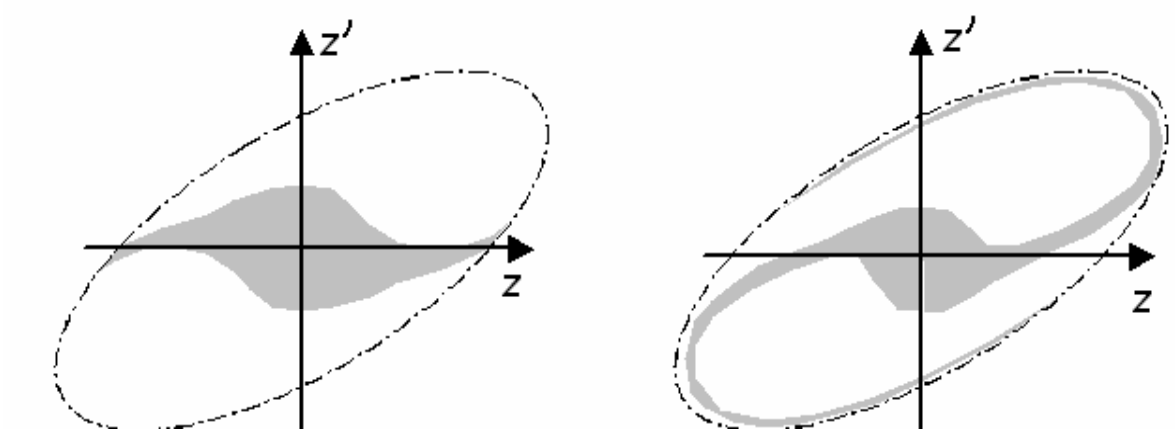
What happens when a transfer line meets a ring?

- ***In transfer lines, the ellipse always "belongs" to the beam*, or at least the user's interpretation of what the beam should be. The Twiss parameters should be marked in some way to show this, but this is rarely done.**
- ***In a ring, the matched ellipse "belongs" to the lattice* because it is defined by the periodicity.**
- **If now a beam ellipse, that is not equal to the matched ellipse, is injected into a ring and observed at the same position in the ring over several turns, it will turn with regular angular steps inside the *matched ellipse* (see Figure below).**
- **In this situation, the beam has a *mismatched ellipse* and the ring is effectively behaving like a long transfer line that has a repeating structure.**



Debunching and filamentation

- After a few thousand turns the structure of the *mismatched ellipse* will start to be lost and the beam will fill the *matched ellipse*. The matched ellipse corresponds to the Twiss parameters as derived earlier for a ring. These parameters "belong" to the lattice and always impose themselves on any beam that circulates in the ring for many turns.
- Two processes spread out the ions in the mismatched ellipse to fill the matched ellipse.
  - A momentum spread, however small, introduces a spread in the revolution frequency that destroys the initial distribution. *This is a debunching effect*.
  - There is always some non-linearity that correlates tune value with amplitude. This effect, called *filamentation*, distorts the initial ellipse into an "S" (see Figure). As the tails grow longer they grow narrower. From the mathematician's viewpoint, phase-space area is conserved (Liouville's theorem), but for all practical purposes filamentation is a loss of phase-space density and an increase in emittance.



Chromaticity

- ***Chromaticity*** **refers to effects caused by a momentum dependence. The name arises because the momentum/energy of an ion is closely analogous to the frequency (and hence the colour) of light in classical optics.**
- **The dispersion function that arises from the differential bending in dipoles for ions of different momenta is strictly a chromaticity effect, but it is not referred to as such.**
- **The effect arising from the differential focusing with momentum causes the betatron phase advance or tune in a ring to change with momentum. This is generally known as the chromaticity and can be defined in two ways:**

$$Q' = \frac{\Delta Q}{\Delta p / p} \quad \text{or} \quad \zeta = \frac{\Delta Q / Q}{\Delta p / p} \qquad (1.22)$$

- **The first definition is the more widely used, but the second definition is liked for its symmetry.**
- **The next level of chromaticity is the variation of the Twiss α and β with momentum. This is treated by formulating a so-called *w-vector* (see last lecture). There is also a suite of routines in the lattice program WinAGILE on the CD-ROM.**