PAUL SCHERRER INSTITUT

Rasmus Ischebeck

Machine Physics 2

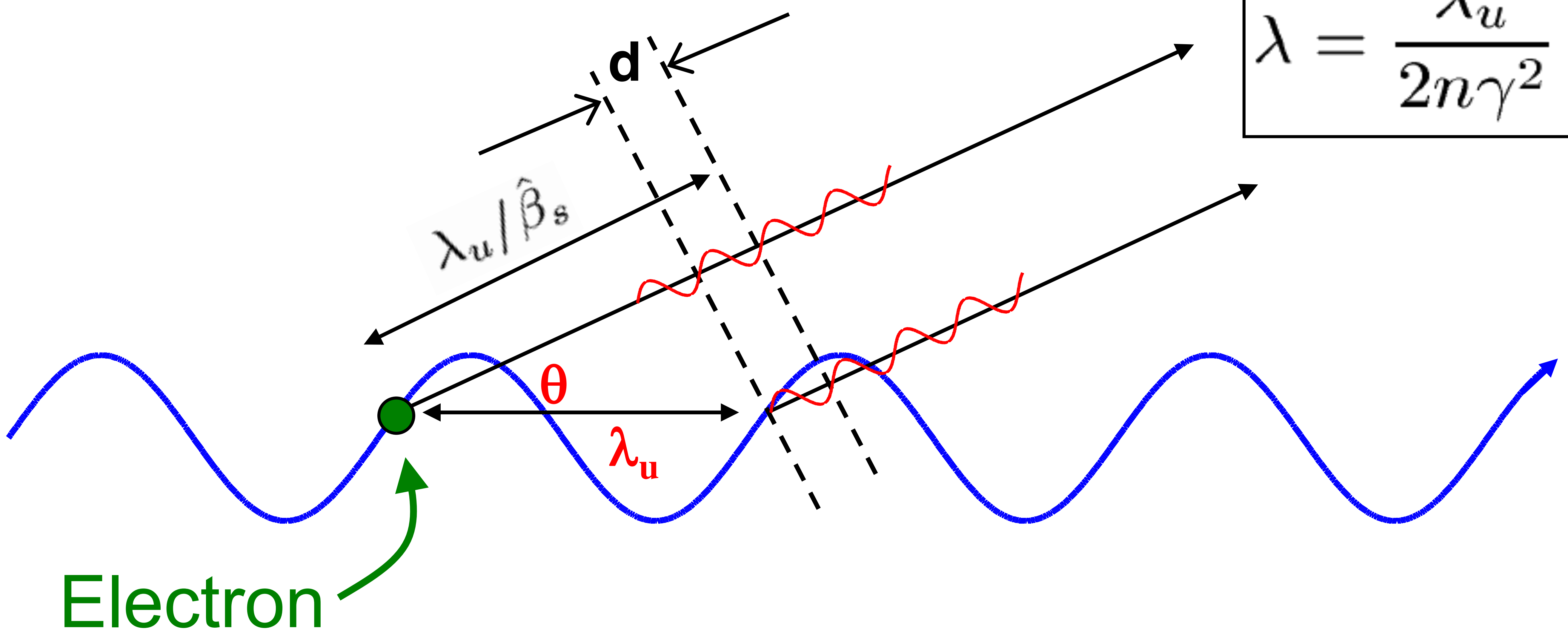
Joint Universities Accelerator School

Undulators

Multipole wigglers are periodic, high field devices, used to generate enhanced flux levels (proportional to the number of poles)

Undulators are periodic, relatively low field, devices which generate radiation at **specific harmonics**

$$\lambda = \frac{\lambda_u}{2n\gamma^2}\left(1 + \frac{K^2}{2} + \theta^2\gamma^2\right)$$



Undulators

- Undulator Parameter

$$K = \frac{eB_0}{m_e c k_u} = \frac{eB_0 \lambda_u}{2\pi m_e c}$$

- Field on axis

$$B = B_0 \cos\left(\frac{2\pi z}{\lambda_u}\right)$$

- is given by the pole tip field B_t:

$$B_0 = \frac{B_t}{\cosh\left(\frac{\pi g}{\lambda_u}\right)}$$

Undulator Emission

Previously we argued that light of the same wavelength was contained in a narrow angular width

(interference effect)

$$\Delta\theta = \sqrt{\frac{2\lambda}{N\lambda_u}}$$

Assuming that the SR is emitted in angle with a Gaussian distribution with standard deviation $\sigma_{r'}$ then we can approximate:

$$\sigma_{r'} = \sqrt{\frac{\lambda}{N\lambda_u}} = \sqrt{\frac{\lambda}{L}}$$

From the diffraction limit $\sigma_r \cdot \sigma_{r'} = \frac{\lambda}{4\pi}$

we get: $\sigma_r = \frac{1}{4\pi}\sqrt{\lambda L}$

For a Gaussian $\frac{d\dot{N}}{d\Omega} = \left.\frac{d\dot{N}}{d\Omega}\right|_{\theta=0} \exp\left(-\frac{\theta^2}{2\sigma_{r'}^2}\right)$

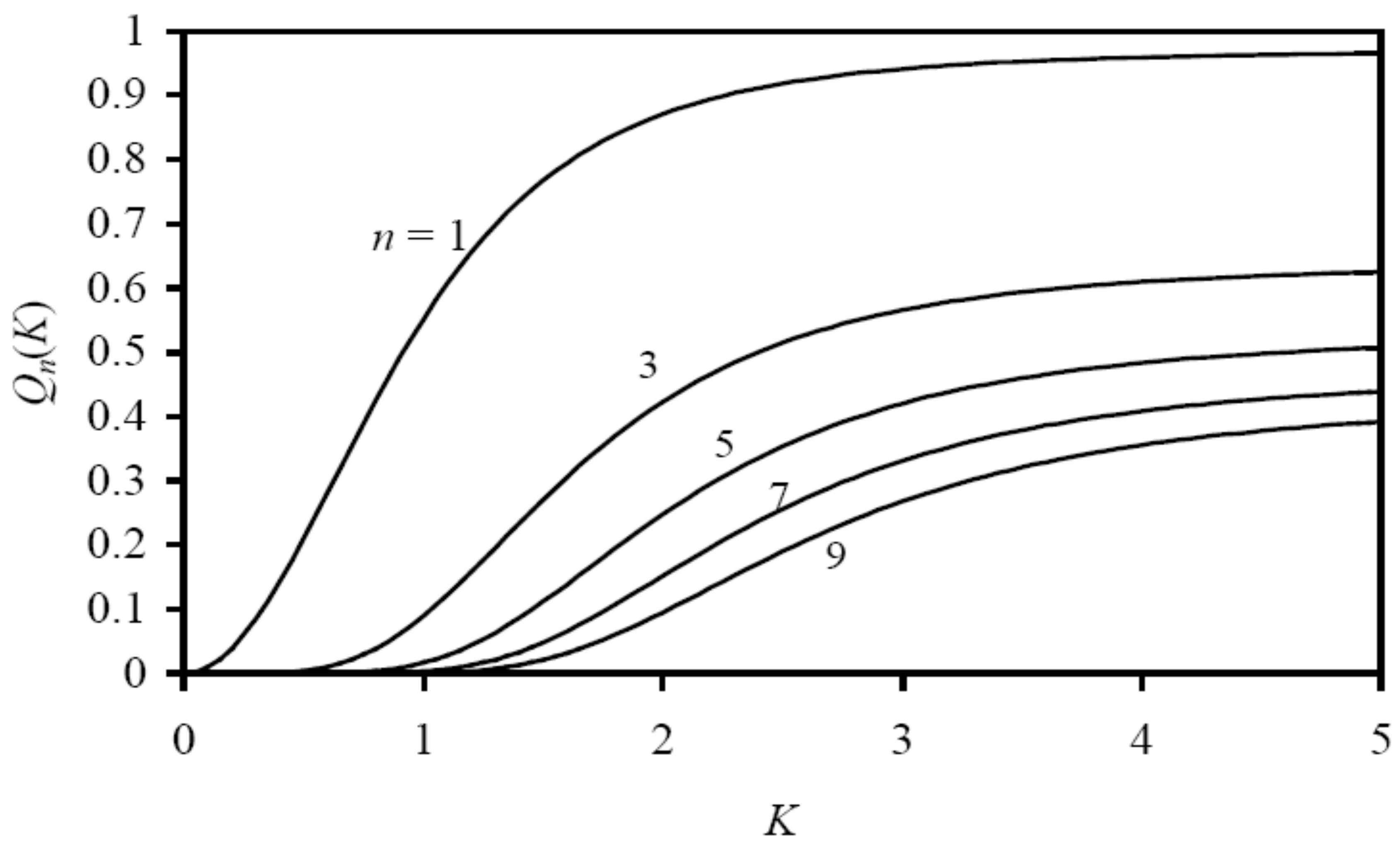
Integrating over all angles gives $\dot{N} = 2\pi\sigma_{r'}^2 \left.\frac{d\dot{N}}{d\Omega}\right|_{\theta=0}$

Undulator Flux

In photons/sec/0.1% bandwidth the flux in this central cone is

$$\dot{N} = 1.43 \times 10^{14}\, N I_b Q_n(K)$$

$$Q_n(K) = \frac{1 + K^2/2}{n} F_n(K)$$



Where: n is the harmonic number

N is the number of periods

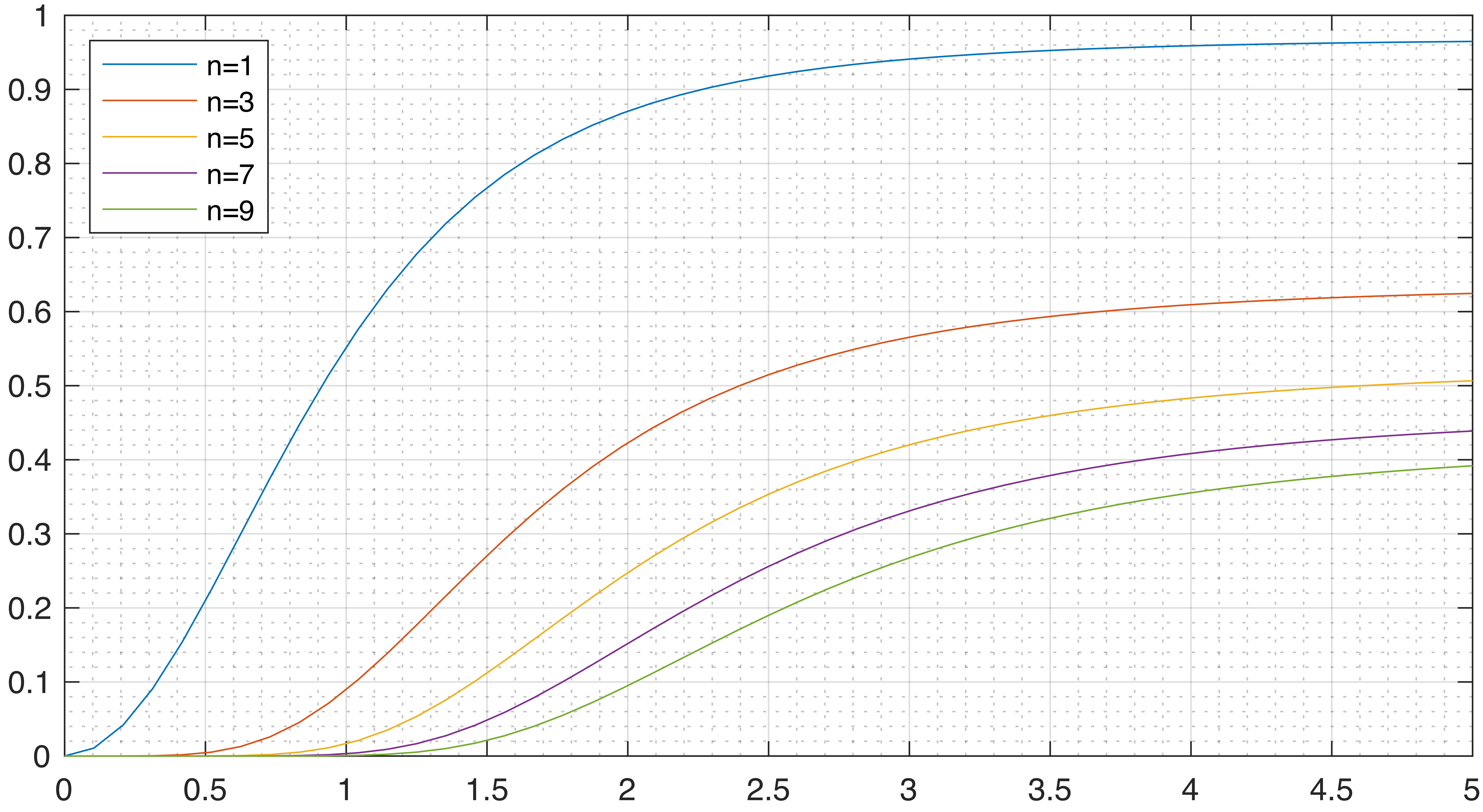
I_b is the beam current in A

$F_n(K)$ is defined below (J are Bessel functions)

$$F_n(K) = \frac{n^2K^2}{(1+K^2/2)^2}\left(J_{(n+1)/2}(Y) - J_{(n-1)/2}(Y)\right)^2$$

$$Y = \frac{nK^2}{4(1+K^2/2)}$$

Plot of $Q_n(K)$



Undulator Brilliance

- To compute the brilliance of an undulator

$$\mathcal{B} = \frac{\dot{N}_\gamma}{4\pi^2 \sigma_x \sigma_y \sigma_{x'} \sigma_{y'} (0.1\%\mathrm{BW})}$$

- … one first has to determine the effective source size σ.
- This is given by the electron beam size σ_x and divergence $\sigma_{x'}$ the diffraction limit for photons σ_r and $\sigma_{r'}$

$$\sigma_{x\mathrm{eff}} = \sqrt{\sigma_x^2 + \sigma_r^2} \qquad \sigma_{x'\mathrm{eff}} = \sqrt{\sigma_{x'}^2 + \sigma_{r'}^2}$$

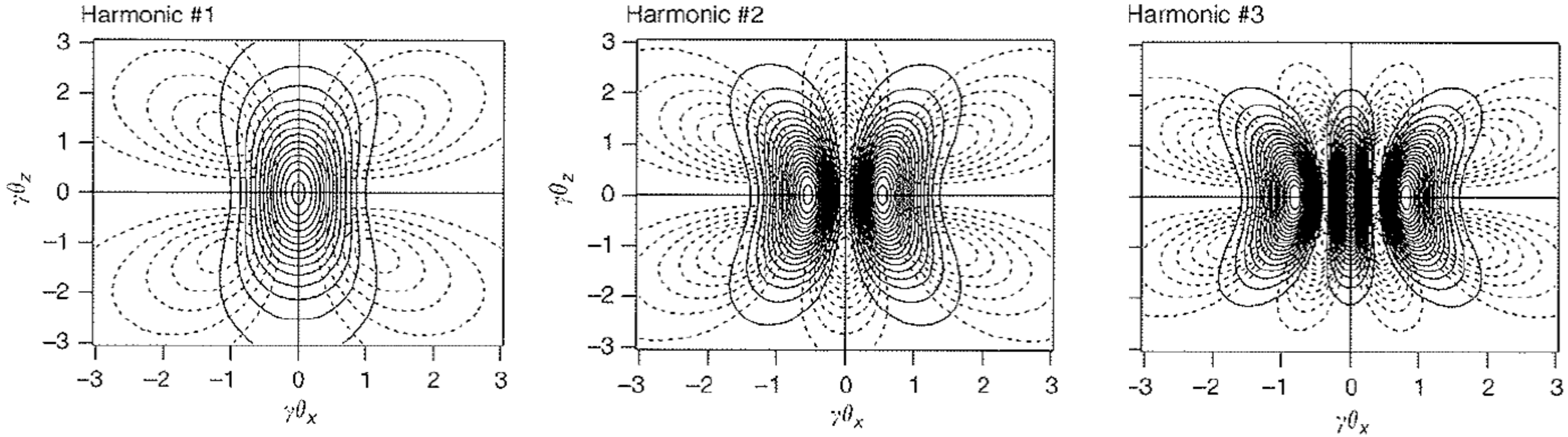
- The diffraction limit for an undulator is:

$$\sigma_r = \frac{1}{4\pi}\sqrt{\lambda L} \qquad \sigma_{r'} = \sqrt{\frac{\lambda}{L}}$$

- (exactly the same for y)
- We will come back to this in the lecture on diffraction-limited storage rings!

Angular patterns of the radiation emitted on harmonics

Angular spectral flux as a function of frequency for a linear undulator; linear polarisation solid, vertical polarisation dashed (K = 2)



$$\lambda_1 = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)$$

$$\lambda_2 = \frac{\lambda_1}{2}$$

$$\lambda_2 = \frac{\lambda_1}{3}$$

Fundamental wavelength emitted by the undulator

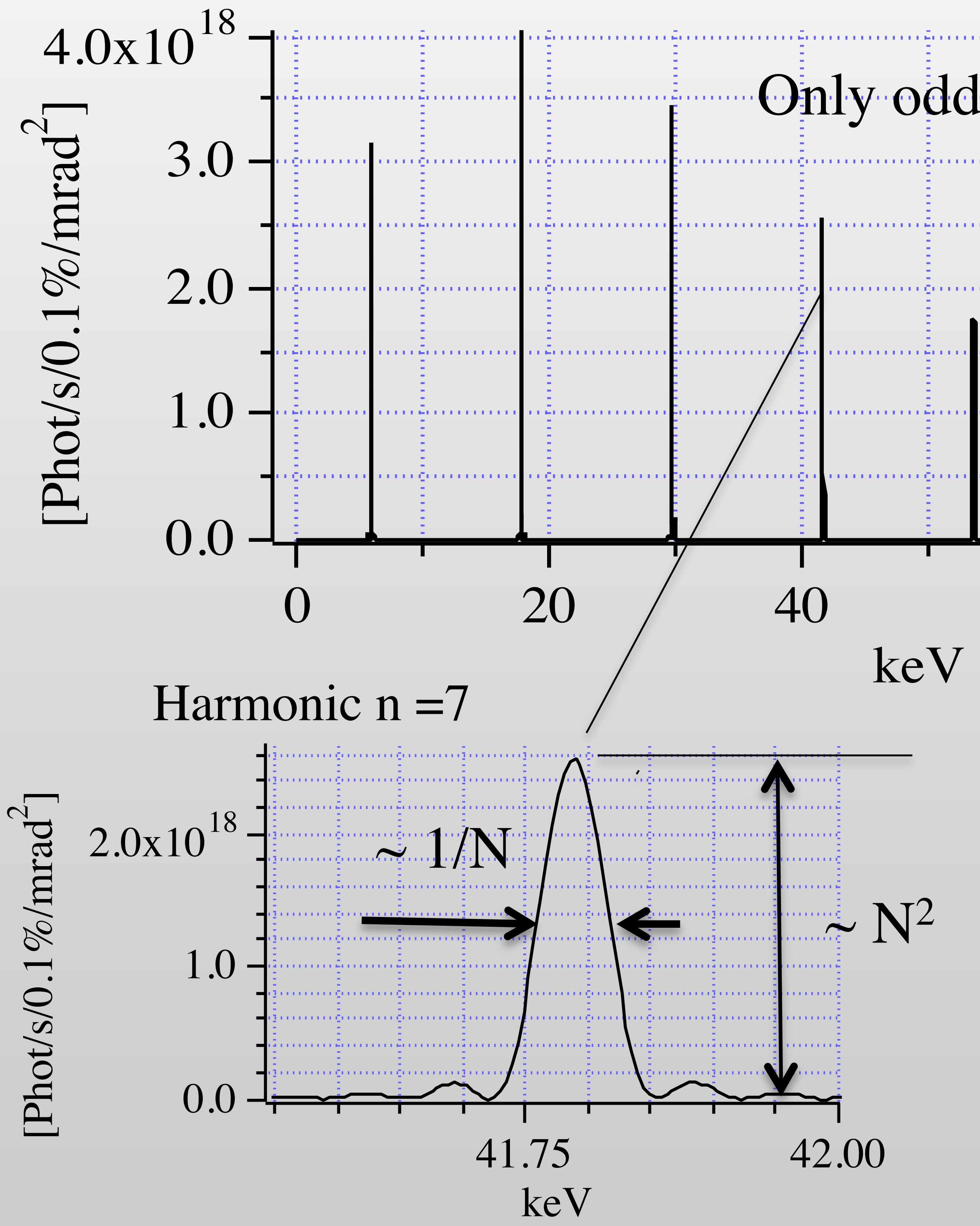
2nd harmonic, not emitted on-axis !

3rd harmonic, emitted on-axis !

Undulators

- Bandwidth of the undulator

Ideal on axis angular spectral flux with filament electron beam (zero emittance)



Undulator:
Period λ_0= 22 mm
Number of period **N**=90
K=1.79

Relative bandwidth at harmonic n:

$\Delta E/E=1/nN$

Radiated power: ~ $N^2/N=N$ proportional to N

Synchrotron Radiation and Storage Ring Dynamics

- Charged particles radiate when accelerated
- Transverse acceleration induces significant radiation (synchrotron radiation) while longitudinal acceleration generates negligible radiation ($1/\gamma^2$).

$$\frac{dU}{dt} = -P_{SR} = -\frac{2\, c\, r_e}{3\left(m_0 c^2\right)^3} \frac{E^4}{\rho^2}$$

$r_e \equiv$ *classical electron radius*

$\rho \equiv$ *trajectory curvature*

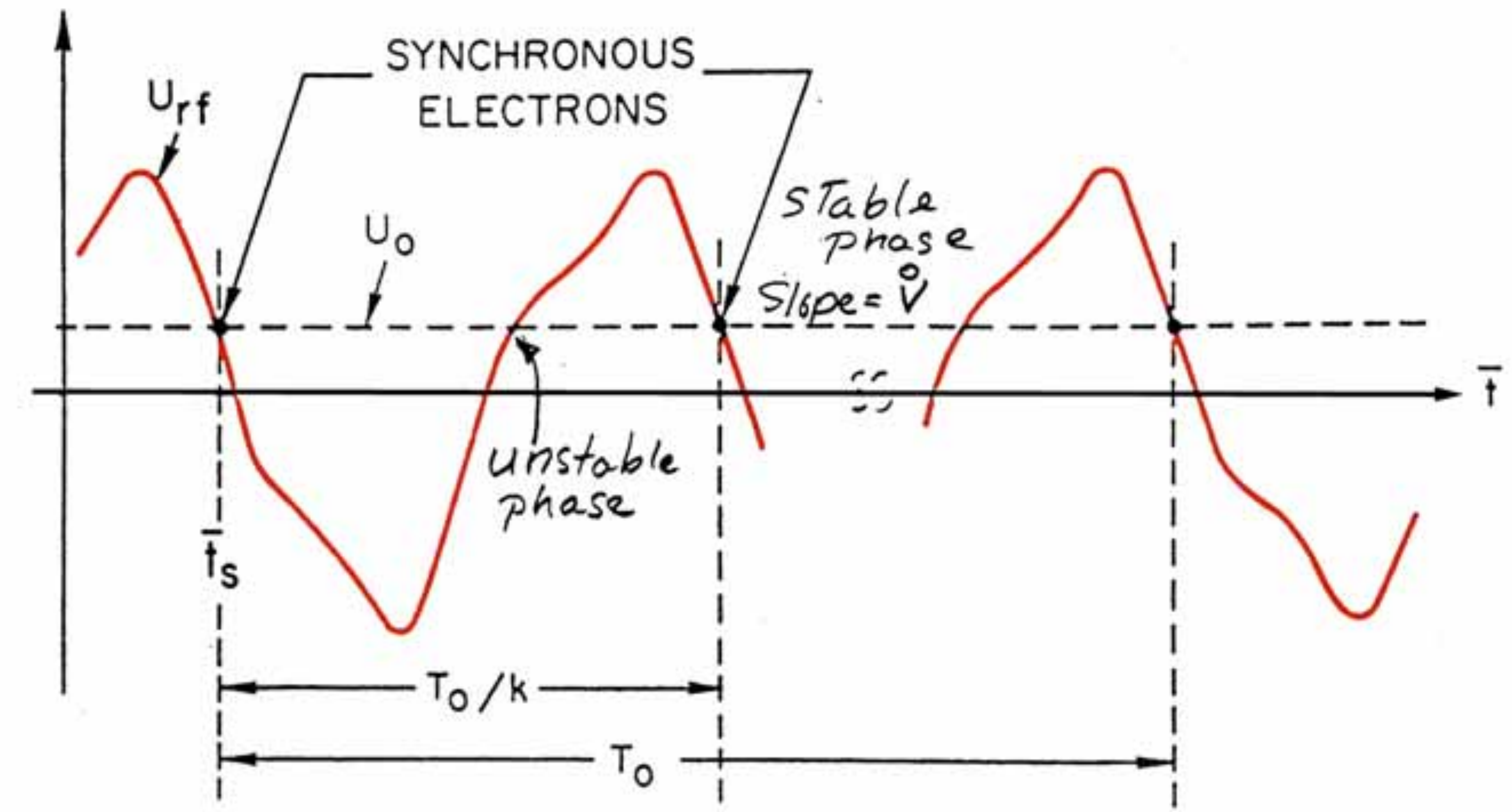
$$U_0 = \int_{finite\ \rho} P_{SR}\, dt \quad \textit{energy lost per turn}$$

$$\alpha_D = -\frac{1}{2T_0} \frac{dU}{dE}\bigg|_{E_0} = \frac{1}{2T_0} \frac{d}{dE}\left[\oint P_{SR}(E_0)\, dt \right]$$

α_{DX}, α_{DY} *damping in all planes*

$\frac{\sigma_p}{p_0}$, $\varepsilon_X, \varepsilon_Y$ *equilibrium momentum spread and emittances*

RF System Restores Energy Loss



Particles change energy according to the phase of the field in the RF cavity

$$\Delta E = eV(t) = eV_o \sin(\omega_{RF} t)$$

For the synchronous particle

$$\Delta E = U_0 = eV_0 \sin(\varphi_s)$$

Radiation Damping of Energy Fluctuations

Say that the energy loss per turn due to synchrotron radiation loss is U_0

The synchronous phase is such that $U_0 = eV_0 \sin(\varphi_s)$

But U_0 depends on energy E ==> Rate of change of the energy will be given

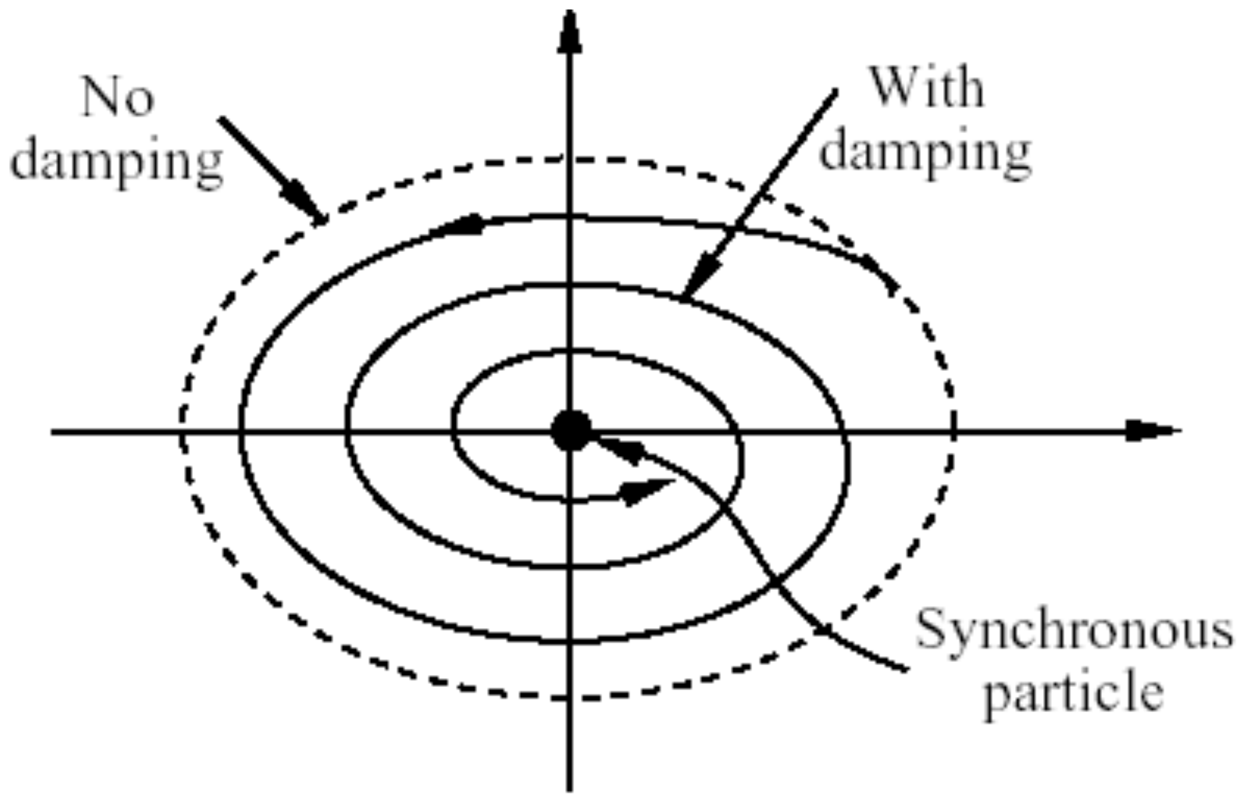
$$\frac{\Delta E}{T_0} = \frac{eV(t) - U_0(E)}{T_0}$$

For ΔE << E and $\boldsymbol{\tau}$ << T_0 we can expand

$$\frac{d\varepsilon}{dt} = \frac{\left(U_0(0) + e\frac{dV}{dt}\boldsymbol{\tau}\right) - \left(U_0(0) + \frac{dU_0}{dE}\varepsilon\right)}{T_0} = \frac{e}{T_o}\frac{dV}{dt}\boldsymbol{\tau} - \frac{1}{T_0}\frac{dU_0}{dE}\varepsilon$$

$$\frac{d\boldsymbol{\tau}}{dt} = -\alpha_c \frac{\varepsilon}{E_s}$$

Energy Damping



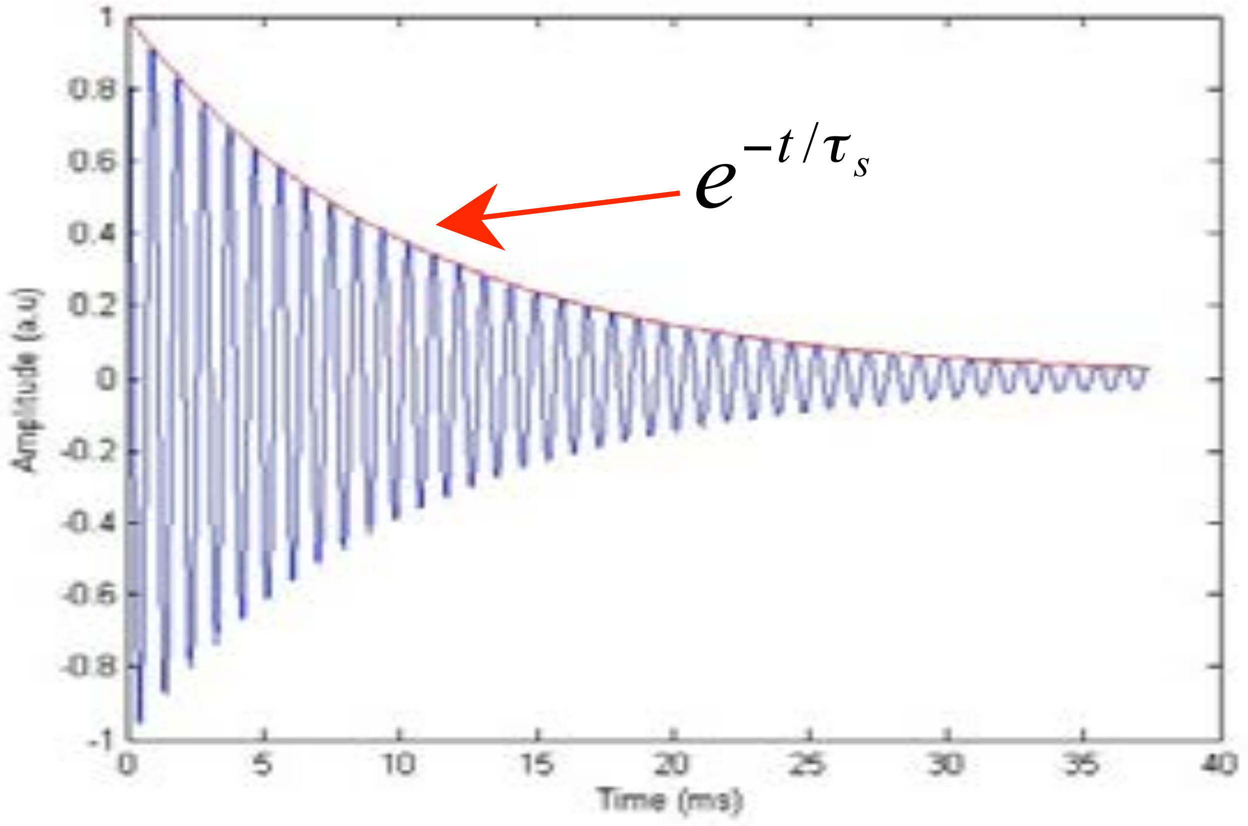
The derivative $\frac{dU_0}{dE}$ (> 0)
is responsible for the damping of the longitudinal oscillations

Combine the two equations for (ε, τ) in a single 2nd order differential equation

$$\frac{d^2\varepsilon}{dt^2} + \frac{2}{\tau_s}\frac{d\varepsilon}{dt} + \omega_s^2\varepsilon = 0 \quad \longrightarrow \quad \varepsilon = Ae^{-t/\tau_s}\sin\left(\sqrt{\omega_s^2 - \frac{4}{\tau_s^2}}\,t + \varphi\right)$$

$$\omega_s^2 = \frac{\alpha e\dot{V}}{T_0 E_0}$$ **angular synchrotron frequency**

$$\frac{1}{\tau_s} = \frac{1}{2T_0}\frac{dU_0}{dE}$$ **longitudinal damping time**



Damping Coefficients

$$\frac{dU}{dt} = -P_{SR} = -\frac{2\,c\,r_e}{3\left(m_0 c^2\right)^3}\frac{E^4}{\rho^2} \qquad \alpha_D = -\frac{1}{2T_0}\left.\frac{dU}{dE}\right|_{E_0} = \frac{1}{2T_0}\frac{d}{dE}\left[\oint P_{SR}\left(E_0\right)dt\right]$$

By performing the calculation one obtains:

$$\alpha_D = \frac{U_0}{2T_0 E_0}\left(2 + D\right)$$

Where D depends on the lattice parameters.
For the *iso-magnetic separate function* case:

$$D = \alpha_C \frac{L}{2\pi\rho} \quad (\ll 1)$$

Damping time ~ time required to replace all the original energy

Analogously, for the transverse plane:

$$\alpha_X = \frac{U_0}{2T_0 E_0}\left(1 - D\right) \quad \text{and} \quad \alpha_Y = \frac{U_0}{2T_0 E_0}$$

Damping Times

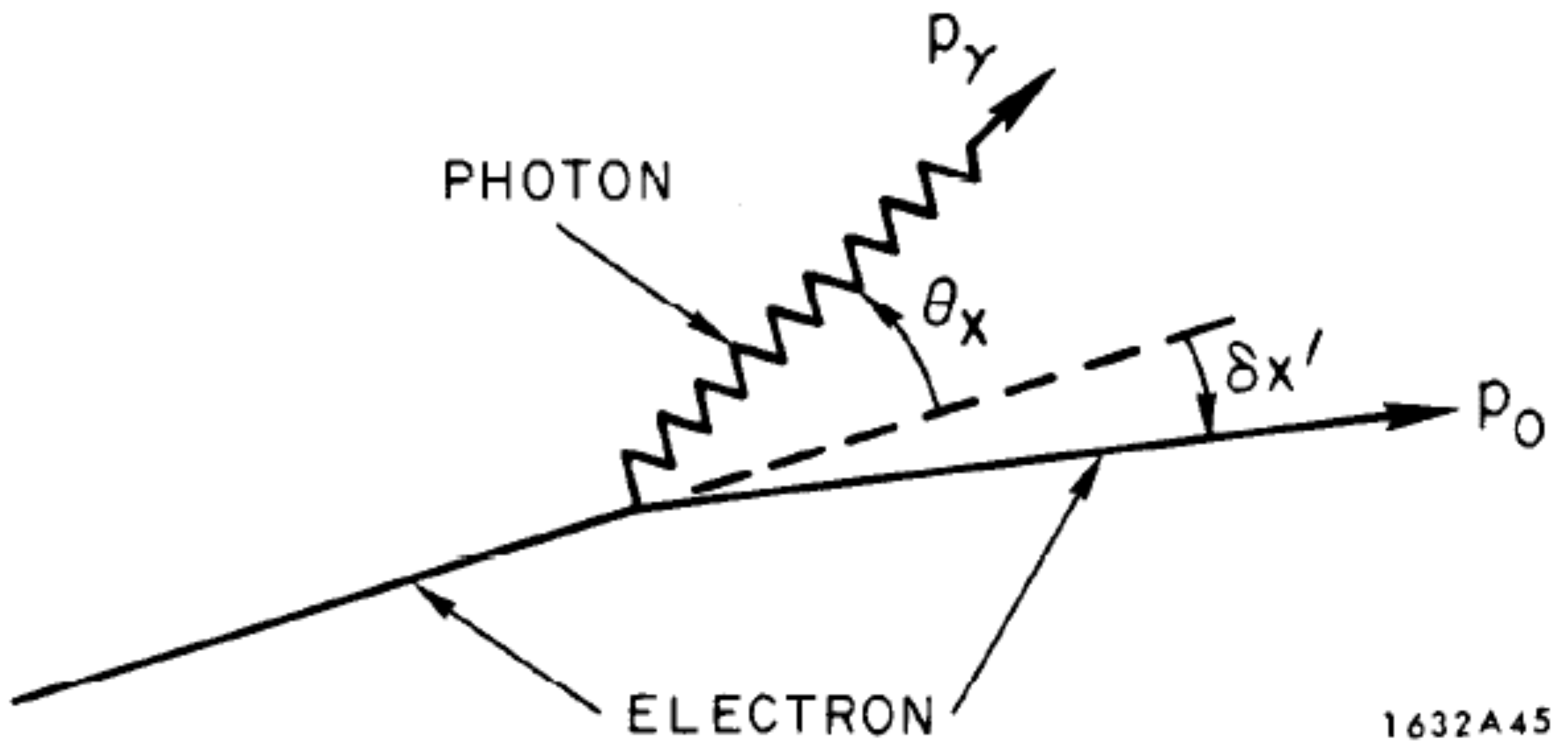
- The energy damping time ~ the time for beam to radiate its original energy
- Typically

$$T_i = \frac{4\pi}{C_\gamma} \frac{R\rho}{J_i E_o^3}$$

- Where $J_e \approx 2$, $J_x \approx 1$, $J_y \approx 1$ and $C_\gamma = 8.9 \times 10^{-5} \, meter - GeV^{-3}$
- Note $\Sigma \, J_i = 4$ (partition theorem)

Quantum Nature of Synchrotron Radiation

- Synchrotron radiation induces damping in all planes.
  - → Collapse of beam to a single point is prevented by the *quantum nature of synchrotron radiation*
- Photons are randomly emitted in quanta of discrete energy
  - → Every time a photon is emitted the parent electron "jumps" in energy and angle
- Radiation perturbs excites oscillations in all the planes.
  - → Oscillations grow until reaching *equilibrium* balanced by radiation damping.



What Other Quantum Nature Could We Consider?

Nick Veasey

Energy Fluctuations

- Expected $\Delta E_{quantum}$ comes from the deviation of $\langle N_\gamma \rangle$ emitted in one damping time, τ_E

- $\langle N_\gamma \rangle = n_\gamma \tau_E$

  $\Longrightarrow \Delta \langle N_\gamma \rangle = (n_\gamma \tau_E)^{1/2}$

- The mean energy of each quantum $\sim \varepsilon_{crit}$

- $\Longrightarrow \sigma_\varepsilon = \varepsilon_{crit}(n_\gamma \tau_E)^{1/2}$

- Note that $n_\gamma = P_\gamma / \varepsilon_{crit}$ and $\tau_E = E_o / P_\gamma$

Therefore, ...

- The quantum nature of synchrotron radiation emission generates energy fluctuations

$$\frac{\Delta E}{E} \approx \frac{\left\langle E_{crit} E_o \right\rangle^{1/2}}{E_o} \approx \frac{C_q \gamma_o^2}{J_\varepsilon \rho_{curv} E_o} \sim \frac{\gamma}{\rho}$$

where C_q is the Compton wavelength of the electron

$$C_q = 3.8 \text{ x } 10^{-13} \text{ m}$$

- Bunch length is set by the momentum compaction & V_{rf}

$$\sigma_z^2 = 2\pi \left(\frac{\Delta E}{E} \right) \frac{\alpha_c R E_o}{e \dot{V}}$$

- Using a harmonic rf-cavity can produce shorter bunches

Emittance and Momentum Spread

- At equilibrium the momentum spread is given by:

$$\left(\frac{\sigma_p}{p_0}\right)^2 = \frac{C_q \gamma_0^2}{J_S} \frac{\oint 1/\rho^3 \, ds}{\oint 1/\rho^2 \, ds} \quad where \;\; C_q = 3.84 \times 10^{-13} \, m$$

$$\left(\frac{\sigma_p}{p_0}\right)^2 = \frac{C_q \gamma_0^2}{J_S \rho}$$

$$iso-magnetic \;\; case$$

- For the horizontal emittance at equilibrium:

$$\varepsilon = C_q \frac{\gamma_0^2}{J_X} \frac{\oint H/\rho^3 ds}{\oint 1/\rho^2 ds}$$

where: $H(s) = \beta_T D'^2 + \gamma_T D^2 + 2\alpha_T DD'$

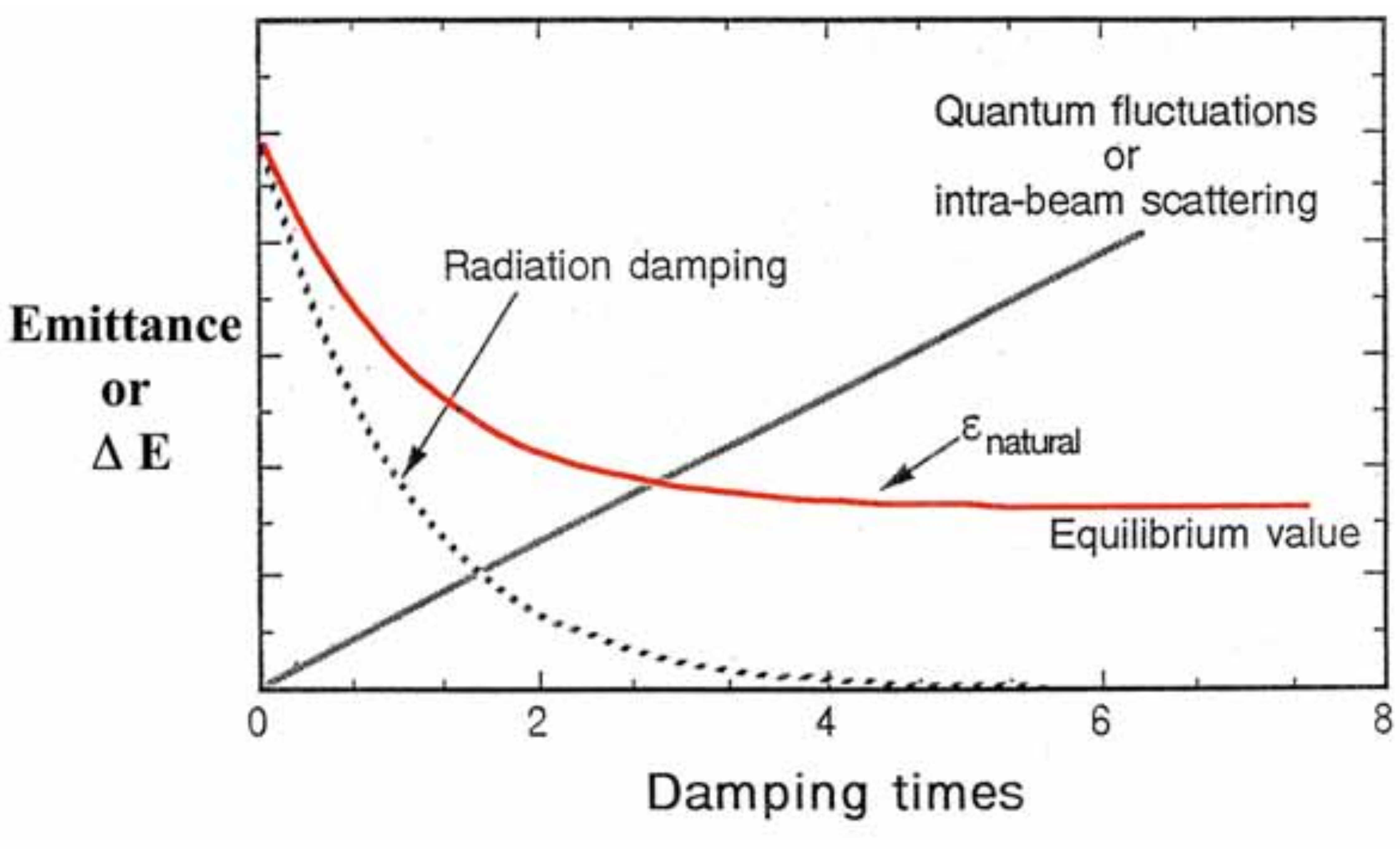
- In the vertical plane, when no vertical bend is present, the synchrotron radiation contribution to the equilibrium emittance is very small

- Vertical emittance is defined by machine imperfections & nonlinearities that couple the horizontal & vertical planes:

$$\varepsilon_Y = \frac{\kappa}{\kappa+1}\varepsilon \quad and \quad \varepsilon_X = \frac{1}{\kappa+1}\varepsilon$$

$$with \;\; \kappa \equiv coupling \;\; factor$$

Equilibrium Emittance and Energy Spread



- Set

Growth rate due to fluctuations (linear) = exponential damping rate due to radiation

==> equilibrium value of emittance or ΔE

$$\varepsilon_{natural} = \varepsilon_1 e^{-2t/\tau_d} + \varepsilon_{eq}\left(1 - e^{-2t/\tau_d}\right)$$

Quantum Lifetime

- At a fixed observation point, transverse particle motion looks sinusoidal

$$x_T = a\sqrt{\beta_n}\sin\left(\omega_{\beta_n} t + \varphi\right) \qquad T = x, y$$

- Tunes are chosen in order to avoid resonances.
  - ➝ At a fixed azimuth, turn-after-turn a particle sweeps all possible positions within the envelope
- Photon emission randomly changes the "invariant" a
  - ➝ Consequently changes the trajectory envelope as well.
- Cumulative photon emission can bring the envelope beyond acceptance at some azimuth
  - ➝ The particle is lost.

This mechanism is called the transverse quantum lifetime

Quantum lifetime was first estimated by Bruck and Sands

$$\tau_{Q_T} \cong \tau_{D_T} \frac{\sigma_T^2}{A_T^2} \exp\left(A_T^2/2\sigma_T^2\right) \quad T = x, y$$

Transverse quantum lifetime

$$where \quad \sigma_T^2 = \beta_T \varepsilon_T + \left(D_T \frac{\sigma_E}{E_0}\right)^2 \quad T = x, y$$

$$\tau_{D_T} \equiv transverse\ damping\ time$$

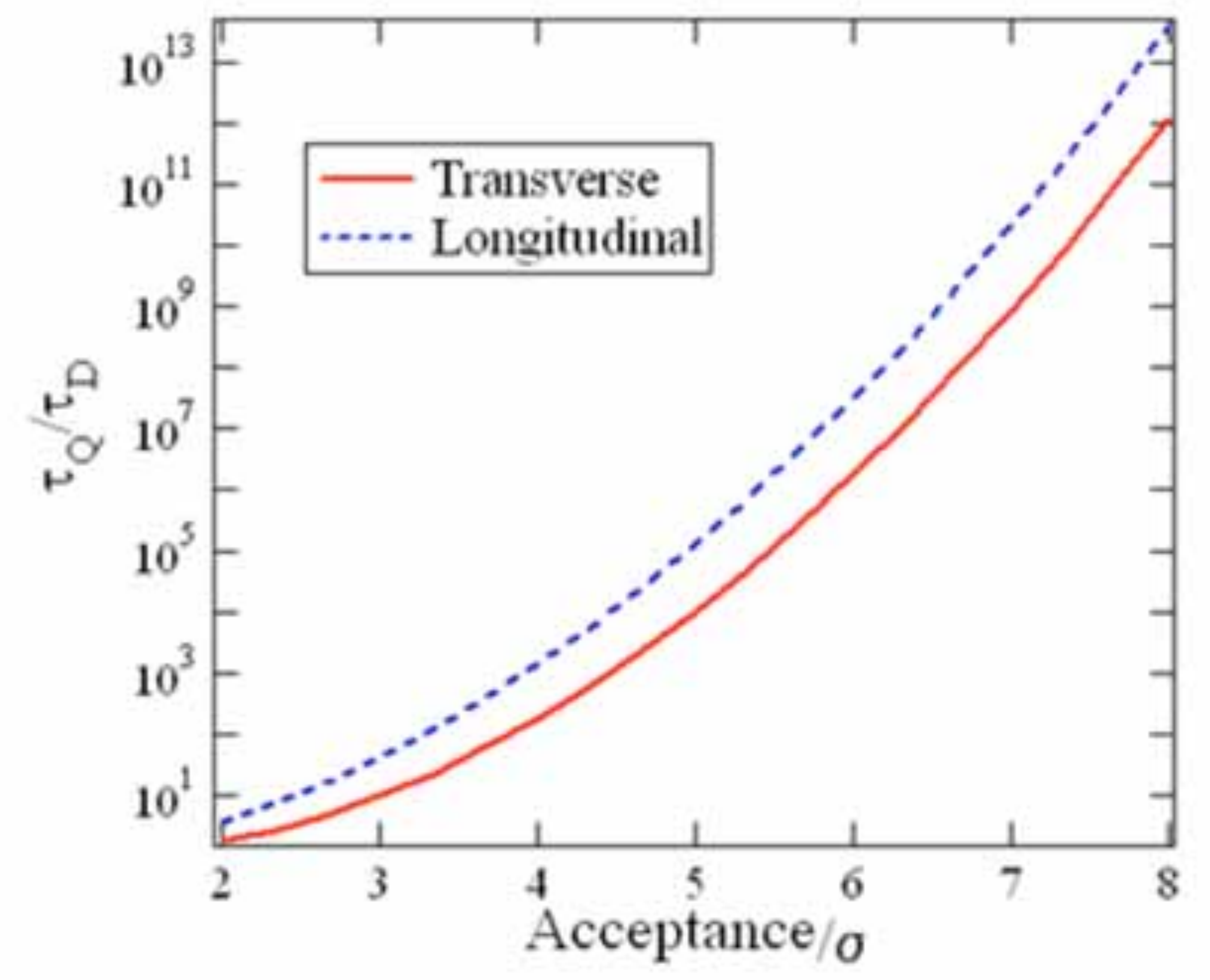
$$\tau_{Q_L} \cong \tau_{D_L} \exp\left(\Delta E_A^2/2\sigma_E^2\right)$$

Longitudinal quantum lifetime

For an iso-magnetic ring:

$$\frac{\Delta E_A^2}{2\sigma_E^2} \approx \frac{J_L E_0}{\alpha_C h E_1}\left(2\frac{e\hat{V}_{RF}}{U_0} - \pi\right)$$

$$E_1 \cong 1.08 \times 10^8 \ eV$$



- τ_Q varies very strongly with the ratio between acceptance & rms size.

Values for this ratio > 6 are usually required.

Time Scales in Storage Rings

- Damping: several ms for electrons, ~ infinity for heavier particles
- Synchrotron oscillations: ~ tens of ms
- Revolution period: ~ hundreds of ns to ms
- Betatron oscillations: ~ tens of ns

Bill Barletta

Questions?

Nick Veasey