Exercises on Space Charge

Exercise 1

Compute the transverse space charge forces and the incoherent tune shifts for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian.

Evaluate also the tune spread (max tune shift – min tune shift) produced by the space charge forces with the same distributions.

parabolic $$\lambda(z) = \frac{3Ne}{2l_o}\left[1 - \left(\frac{2z}{l_o}\right)^2\right]$$

sinusoidal modulation $$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \quad ; \quad k_z = 2\pi / \lambda_w$$

Gaussian $$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\boxed{\begin{aligned} E_r(\boldsymbol{r}) &= \frac{\lambda(z)}{2\pi\ \varepsilon_o}\frac{r}{a^2}; \quad B_\theta(\boldsymbol{r}) = \frac{\lambda(z)\beta}{2\pi\varepsilon_o c}\frac{r}{a^2} \\ F_\perp(r) &= e(E_r - \beta c\ B_\theta) = \frac{e}{\gamma^2}E_r = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\ \varepsilon_o}\frac{r}{a^2} \end{aligned}}$$

$$\boxed{\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right) = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\ \varepsilon_o a^2} = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}}$$

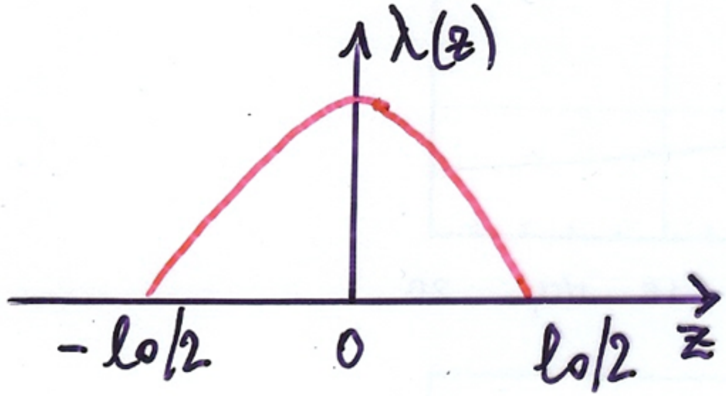
$$r_{e,p} = \frac{e^2}{4\pi\varepsilon_o m_o c^2}\ (electrons : 2.82\ 10^{-15} m,\ protons : 1.53\ 10^{-18} m)$$

$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

a) Parabolic bunch (q_0 = Ne)

$$\lambda(z) = \frac{3Ne}{2l_o}\left[1-\left(\frac{2z}{l_o}\right)^2\right]$$



$$\Delta Q_{\max}(\text{at } z=0) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3 a^2 Q_{x0}}\frac{3N}{2l_0}$$

$$\Delta Q_{\min}\left(\text{at } z=\pm\frac{l_0}{2}\right) = 0$$

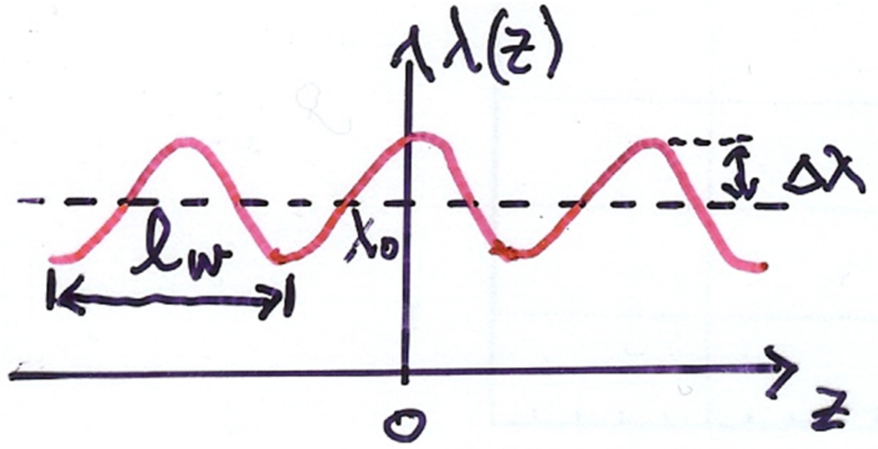
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

$$F_{\perp}(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

b) Sinusoidal modulation (λ_0 =Ne/l_0)

$$\lambda(z) = \lambda_o + \Delta\lambda \cos(k_z z) \;\; ; \;\; k_z = 2\pi / \lambda_w$$



$$\Delta Q_{\max}\left(\text{at } k_z z = 2n\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 + \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$\Delta Q_{\min}\left(\text{at } k_z z = (2n+1)\pi\right) = -\frac{\rho_x^2 r_{e,p}\left(\lambda_0 - \Delta\lambda\right)}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

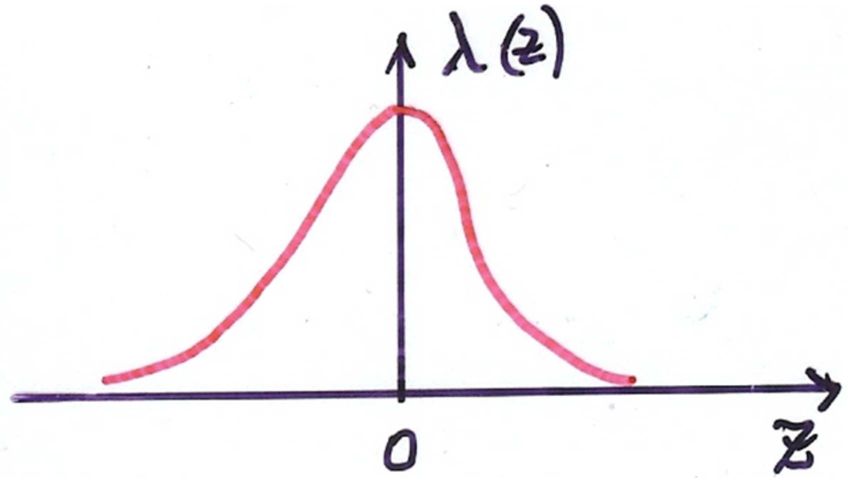
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = -\frac{2\rho_x^2 r_{e,p}\Delta\lambda}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

$$F_\perp(r) = \frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\varepsilon_0}\frac{r}{a^2}$$

$$\Delta Q_x = -\frac{\rho_x^2 \lambda(z) r_{e,p}}{e\beta^2\gamma^3 a^2 Q_{x0}}$$

c) Gaussian bunch (q_0 = Ne)

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

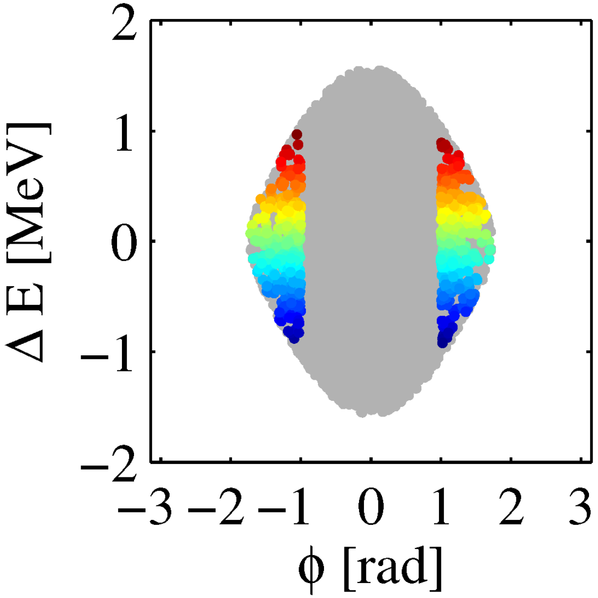


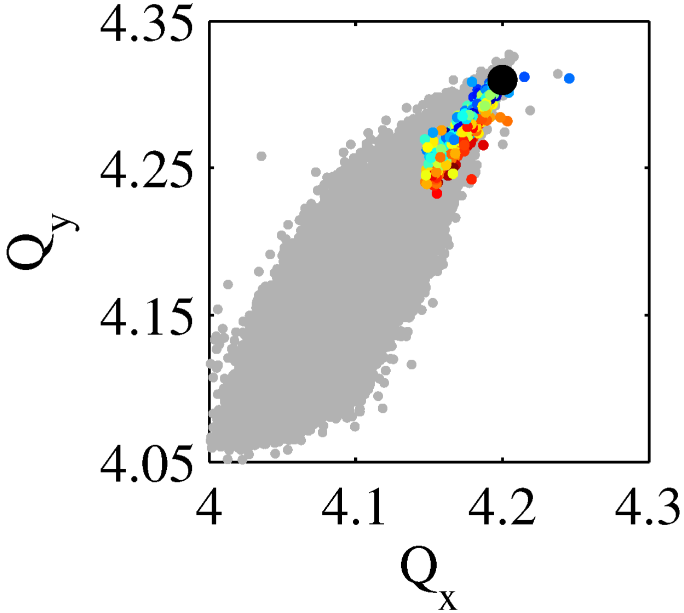
$$\Delta Q_{\max}\left(\text{at } z=0\right) = -\frac{\rho_x^2 r_{e,p}}{\beta^2\gamma^3 a^2 Q_{x0}}\frac{N}{\sqrt{2\pi}\sigma_z}$$

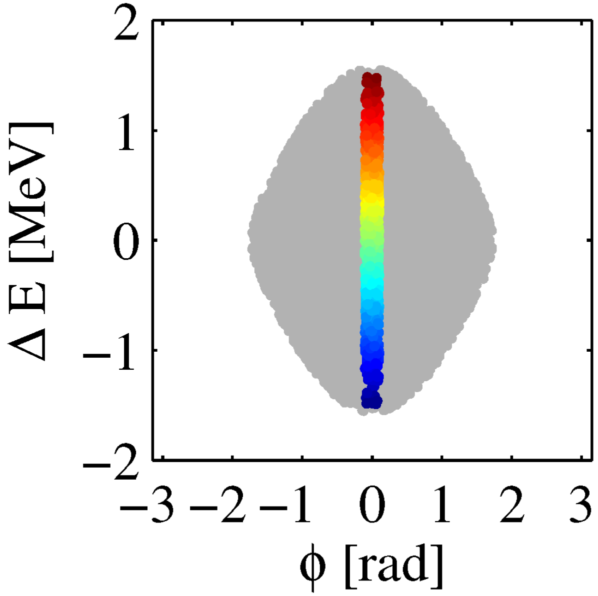
$$\Delta Q_{\min}\left(\text{at } z \to \pm\infty\right) = 0$$

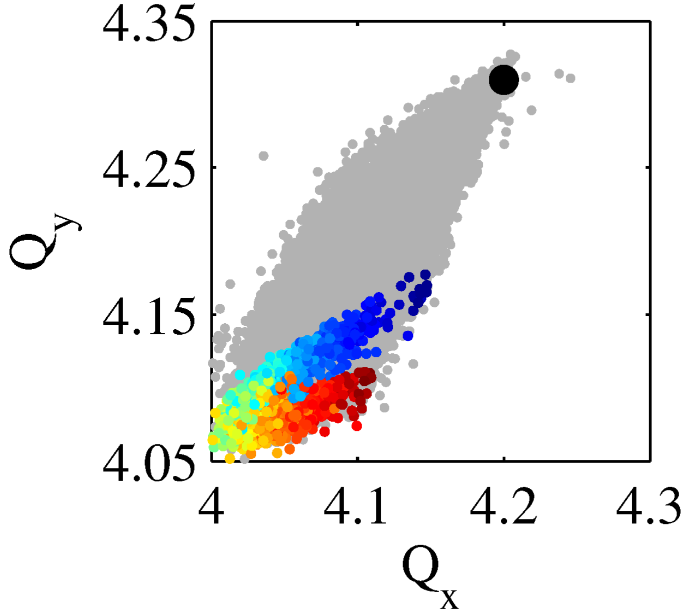
$$\Delta Q_{spread} = \Delta Q_{\max} - \Delta Q_{\min} = \Delta Q_{\max}$$

Effect of longitudinal distribution







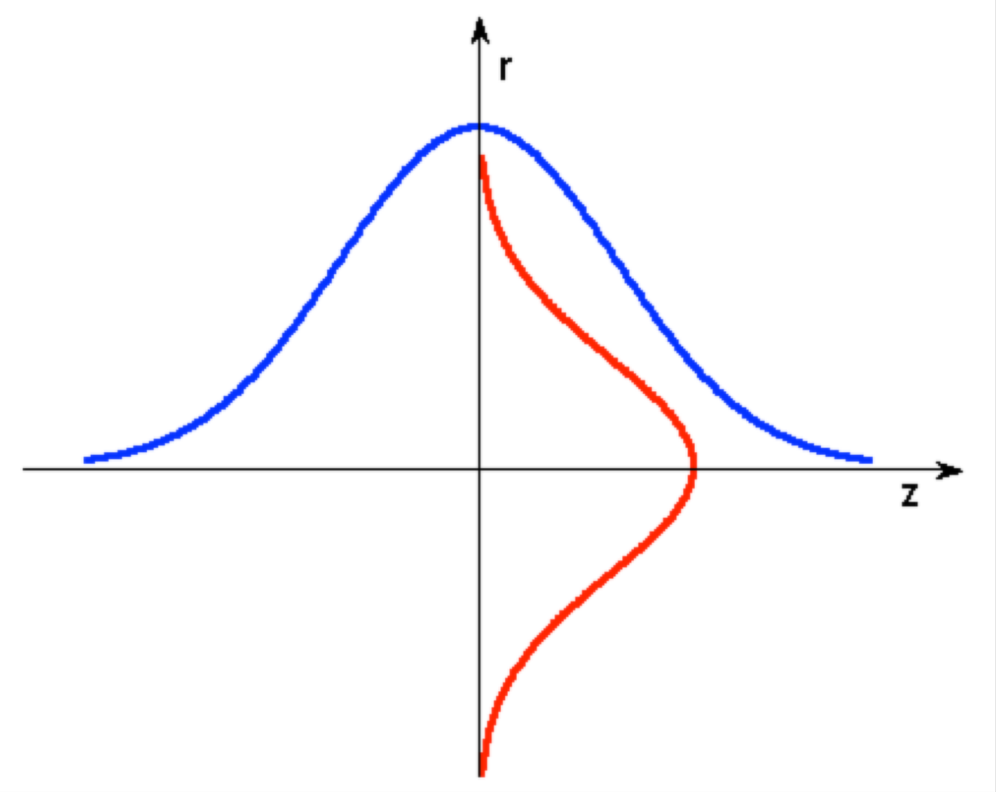


Longitudinal phase-space ($\Delta\phi$, ΔE) scatter plot of the bunch tune footprint. The black dot is the bare tune. Particles at the edges of the bunch have tunes close to the bare tune in the necktie.
Indeed, in this longitudinal region, the beam line density is smaller with respect to the centre of the bunch, therefore also the space charge detuning is small.

(Courtesy of V. Forte, 'Performance of the CERN PSB at 160 MeV with H- charge exchange injection', PhD thesis, Université Blaise Pascal, Clermont-Ferrand, France, 2016)

Exercise 2

Compute the transverse space charge force and the incoherent tune shift for a cylindrical beam in a circular beam pipe, having a bi-Gaussian longitudinal and transverse distribution.



$bi-Gaussian$

$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r,z) = \frac{\lambda(z)}{2\pi\sigma_r^2}\exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

bi-Gaussian

$$\lambda(z)=\frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\rho(r,z)=\frac{\lambda(z)}{2\pi\sigma_r^2}\exp\left(\frac{-r^2}{2\sigma_r^2}\right)$$

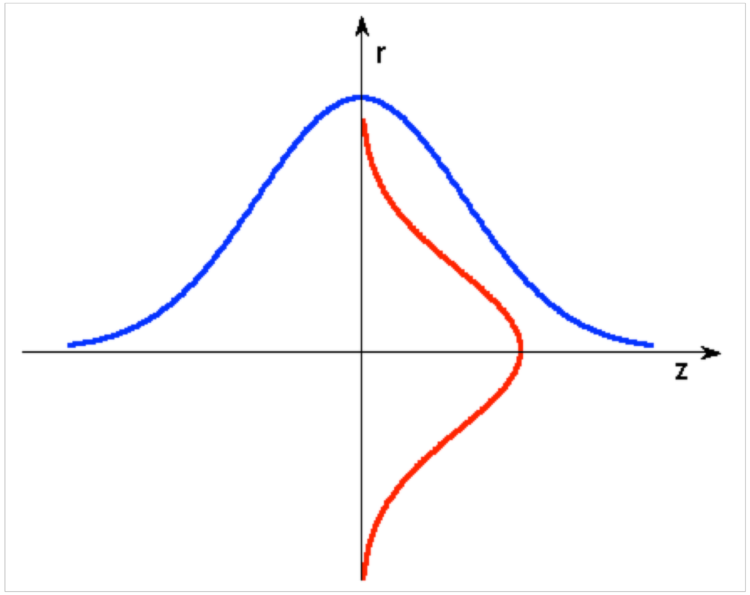
$$\int \vec{E}\cdot\hat{n}\,dS=\frac{q(r)}{\varepsilon_0} \qquad \boxed{\left(\gamma>>1 \qquad E_z\approx 0\right)}$$

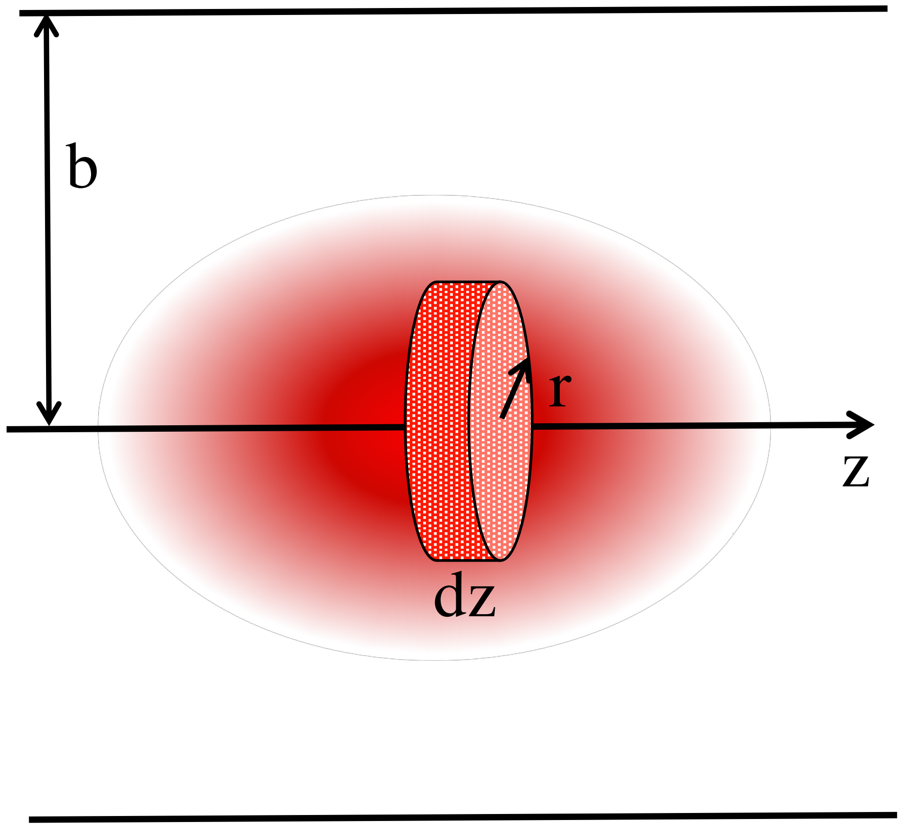
$$E_r(r)2\pi r dz=\frac{dz}{\varepsilon_0}\int_0^r \rho(r',z)2\pi r'\,dr'$$

$$E_r(r)=\frac{\lambda(z)}{2\pi\varepsilon_0\sigma_r^2 r}\int_0^r \exp\left(\frac{-r'^2}{2\sigma_r^2}\right)r'dr'$$

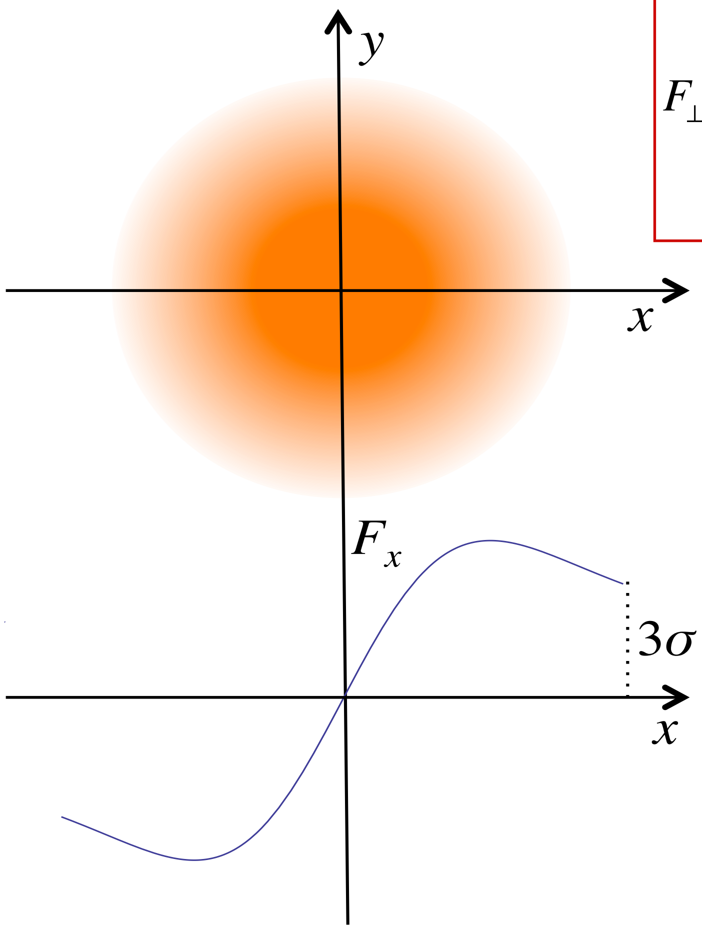
$$E_r(r)=\frac{\lambda(z)}{2\pi\varepsilon_0 r}\left[-\exp\left(\frac{-r'^2}{2\sigma_r^2}\right)\right]_0^r=\frac{\lambda(z)}{2\pi\varepsilon_0}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$

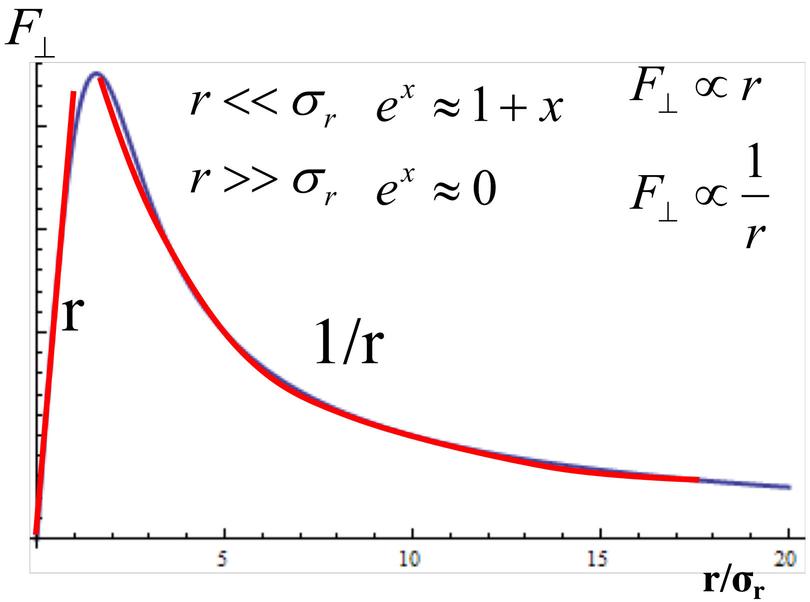
$$F_\perp(r)=\frac{e}{\gamma^2}E_r=\frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]$$





$$F_{\perp}(r)=\frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1-\exp\left(\frac{-r^2}{2\sigma_r^2}\right)}{r}\right]\approx\frac{e\lambda(z)}{2\pi\varepsilon_0\gamma^2}\frac{r}{2\sigma_r^2}\qquad (r<<\sigma_r)$$





$(r<<\sigma_r)$

$$\Delta Q_x=-\frac{\rho_x^2}{2\beta^2E_0Q_{x0}}\left(\frac{\partial F_x^{s.c.}}{\partial x}\right)=-\frac{\rho_x^2}{2\beta^2E_0Q_{x0}}\frac{e}{\gamma^2}\frac{\lambda(z)}{2\pi\,\varepsilon_o}\frac{1}{2\sigma_x^2}$$

If the charge distribution is Gaussian but with different σ_x and σ_y (not cylindrical geometry), it is still possible to obtain the transverse electric field. The expression is known as Bassetti-Erskine formula: M. Bassetti and G.A. Erskine, "Closed expression for the electrical field of a two-dimensional Gaussian charge", CERN-ISR-TH/80-06 (1980).

$$E_x = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Im}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

$$E_y = \frac{Q}{2\varepsilon_o\sqrt{2\pi(\sigma_x^2-\sigma_y^2)}}\,\mathrm{Re}\left[w\left(\frac{x+iy}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right) - e^{\left[-\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2}\right]} w\left(\frac{x\frac{\sigma_y}{\sigma_x}+iy\frac{\sigma_x}{\sigma_y}}{\sqrt{2(\sigma_x^2-\sigma_y^2)}}\right)\right]$$

with the complex error function w(z) given by

$$w(z) = e^{-z^2}\left[1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{\zeta^2}\,d\zeta\right]$$

NB: here Q is the line density.

In the limit $\sigma_x \to \sigma_y$ the above electric field is the one that we have obtained previously

This complicated expression is highly non-linear. It is however possible to obtain a simple expression in the linear approximation which gives

$$E_x \approx \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{x}{\sigma_x(\sigma_x + \sigma_y)}$$

$$E_y \approx \frac{\lambda(z)}{2\pi\varepsilon_0}\frac{y}{\sigma_y(\sigma_x + \sigma_y)}$$

As for the cylindrical symmetry case, there are also magnetic fields associated with the electric fields, so that the transverse force is

$$F_{x,y} \approx \frac{e}{\gamma^2} E_{x,y}$$

and, as in the previous cases, it is possible to obtain the incoherent tune shift (but remember that we are in the linear approximation).

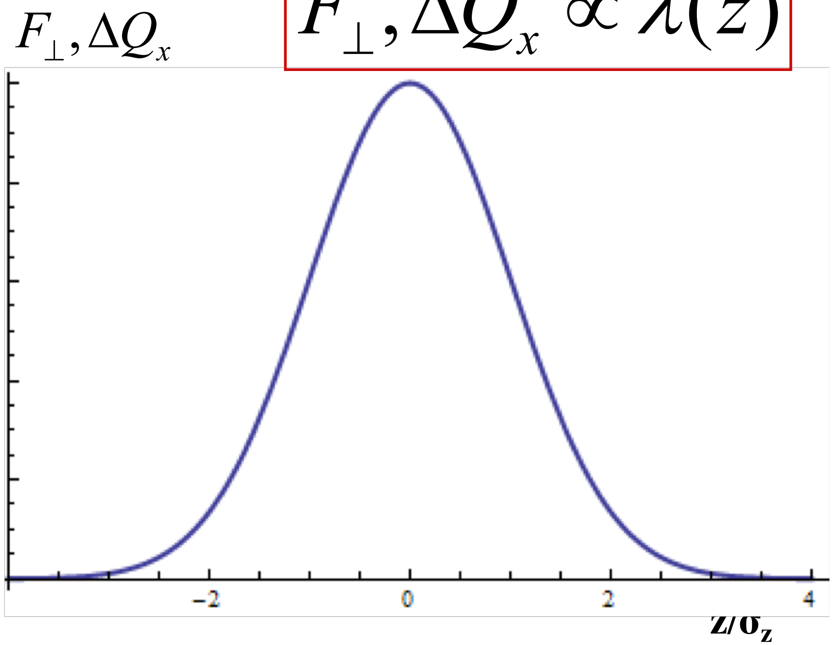
Exercise 3

Evaluate the dependence of the longitudinal and transverse space charge force with z at fixed r (e.g. $<< \sigma_r$) for the bi-Gaussian distribution

Exercise 3

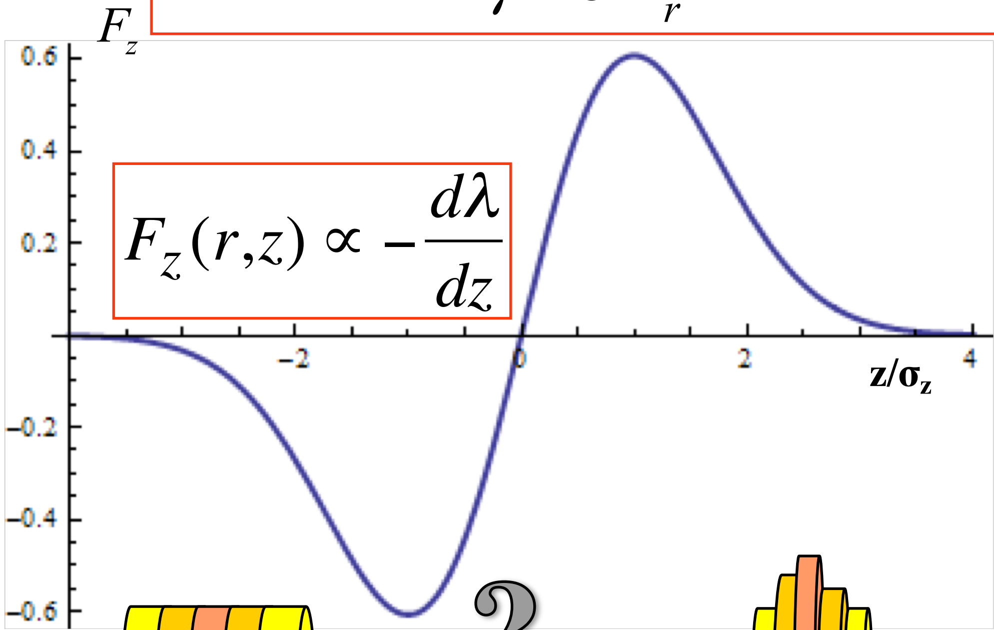
Evaluate the dependence of the longitudinal and transverse space charge force with z at fixed r (e.g. << σ_r) for the bi-Gaussian distribution

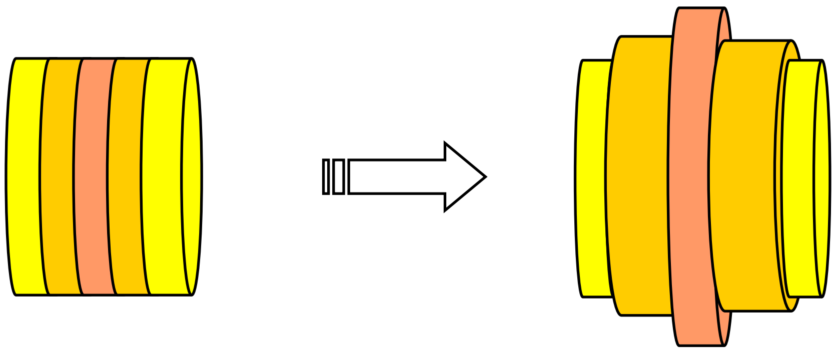
$$F_\perp, \Delta Q_x \propto \lambda(z)$$

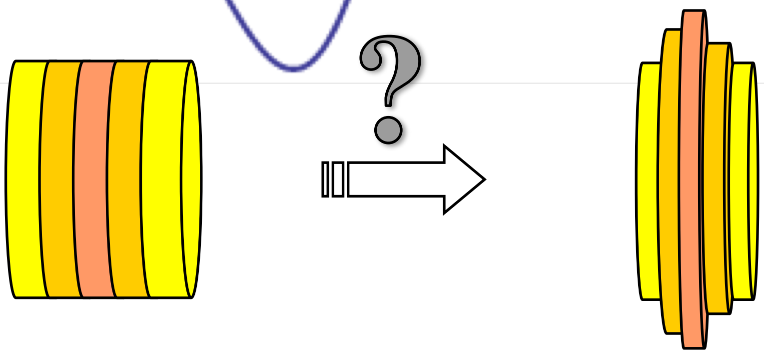


$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)\,dr'$$

$$F_z(r,z) \propto -\frac{d\lambda}{dz}$$







Exercise 4

Compute the longitudinal space charge force of a transverse uniform cylindrical beam in a circular perfectly conducting beam pipe

Exercise 4

Compute the longitudinal space charge force of a transverse uniform cylindrical beam in a circular perfectly conducting beam pipe

$$E_z(r,z) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'$$

$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'$$

$$E_r(r \le a) = \frac{\lambda(z)}{2\pi\,\varepsilon_0}\frac{r}{a^2}$$

$$E_r(r \ge a) = \frac{\lambda(z)}{2\pi\varepsilon_0 r}$$

$$F_z(r,z) = -\frac{e}{2\pi\varepsilon_0\gamma^2}\left[\int_r^a \frac{r'}{a^2}dr' + \int_a^b \frac{1}{r'}dr'\right]\frac{\partial\lambda(z)}{\partial z}$$

$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$

Exercise 5

Compute the longitudinal space charge forces for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian

parabolic
$$\lambda(z) = \frac{3Ne}{2l_o}\left[1-\left(\frac{2z}{l_o}\right)^2\right]$$

sinusoidal modulation
$$\lambda(z) = \lambda_o + \Delta\lambda \cos\left(k_z z\right) \quad ; \quad k_z = 2\pi / l_w$$

Gaussian
$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

Exercise 5

Compute the longitudinal space charge forces for a cylindrical beam in a circular beam pipe, having the following longitudinal distributions: parabolic, sinusoidal modulation, Gaussian

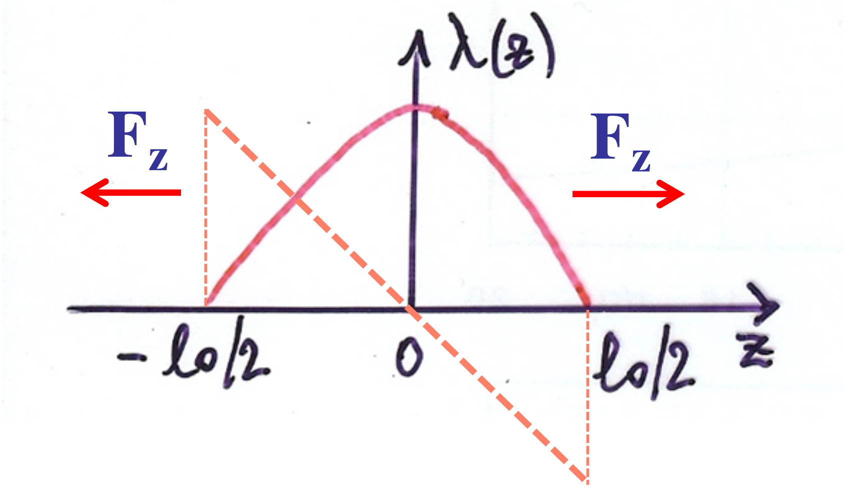
parabolic $$\lambda(z) = \frac{3Ne}{2l_o}\left[1-\left(\frac{2z}{l_o}\right)^2\right]$$

sinusoidal modulation $$\lambda(z) = \lambda_o + \Delta\lambda\cos(k_z z) \quad ; \quad k_z = 2\pi / l_w$$

Gaussian $$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

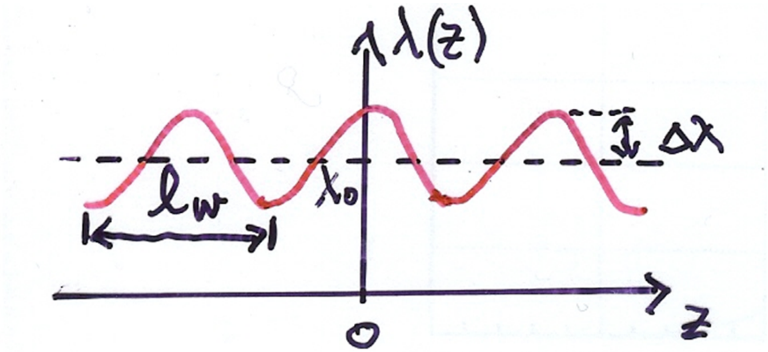
$$F_z(r,z) = -\frac{e}{\gamma^2}\frac{\partial}{\partial z}\int_r^b E_r(r',z)dr'$$

$$F_z(r,z) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\left(1-\frac{r^2}{a^2}+2\ln\frac{b}{a}\right)\frac{\partial\lambda(z)}{\partial z}$$



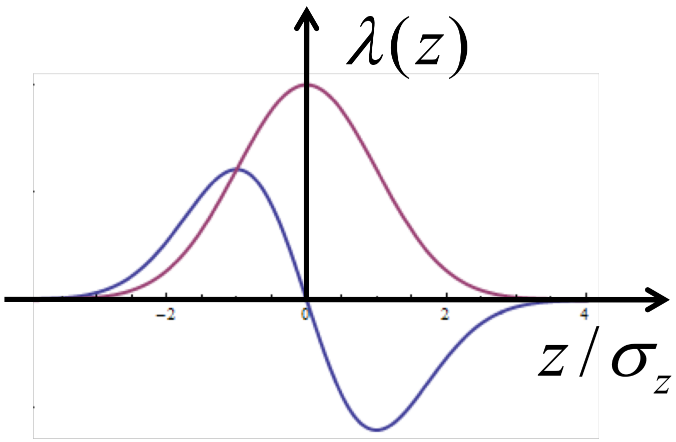
$$\lambda(z) = \frac{3Ne}{2l_o}\left[1-\left(\frac{2z}{l_o}\right)^2\right]$$

$$\frac{d\lambda(z)}{dz} = -\frac{12Ne}{l_0^3}z$$



$$\lambda(z) = \lambda_o + \Delta\lambda\cos(k_z z) \quad ; \; k_z = 2\pi / l_w$$

$$\frac{d\lambda(z)}{dz} = -\Delta\lambda k_z \sin(k_z z)$$



$$\lambda(z) = \frac{Ne}{\sqrt{2\pi}\sigma_z}\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

$$\frac{d\lambda(z)}{dz} = -\frac{Ne}{\sqrt{2\pi}\sigma_z^3}z\exp\left(-\frac{z^2}{2\sigma_z^2}\right)$$

Exercise 6

Compute the incoherent betatron tune shift of a uniform proton beam inside two perfectly conducting parallel plates

Exercise 6

Compute the incoherent betatron tune shift of a uniform proton beam inside two perfectly conducting parallel plates

$$\Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_0 Q_{x0}} \left(\frac{\partial F_x^{s.c.}}{\partial x} \right)$$

$$F_x(z,x) = \frac{e\lambda_0 x}{\pi\ \varepsilon_0} \left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2} \right)$$

$$\Delta Q_x = -\frac{\rho_x^2 e\lambda_0}{2\pi\ \varepsilon_0 \beta^2 E_0 Q_{x0}} \left(\frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2} \right)$$