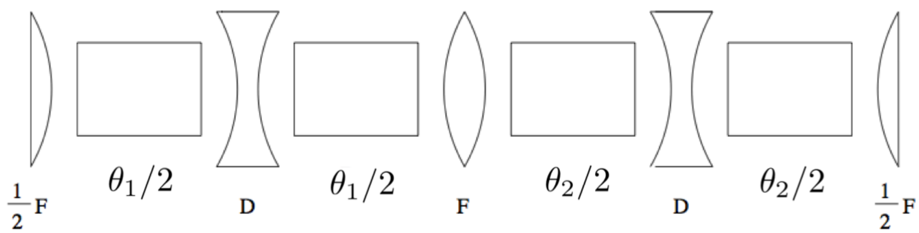
Transverse Beam Dynamics

JUAS 2019 - Tutorial 4 (solutions)

1 Exercise: Dispersion suppressor

In several straight sections of the accelerator, like the ones hosting RF cavities, extraction systems or other devices such as detectors, it is preferable to have no dispersion $\eta(s) = \eta'(s) = 0$. For example, in big colliders, such as the LHC, where small spot sizes are required at the interaction points, the dispersion is reduced to zero at the detector positions. The most common dispersion suppressors consists of two FODO cells of equal length L and equal quadrupole strengths. Bending magnets are placed in the space between the quadrupoles with a different bending field in each FODO. Figure below shows a typical dispersion suppressor.



1. Considering two FODO cells with different total bend angles, $\theta_1 \neq \theta_2$, calculate the relation between the angles θ_1 and θ_2 which must be satisfied to cancel the dispersion at the end of the lattice.

   Hint:

   For each FODO cell, $M_{\text{FODO}} = M_{1/2\text{F}} \cdot M_{\text{dipole}} \cdot M_{\text{D}} \cdot M_{\text{dipole}} \cdot M_{1/2\text{F}}$, in thin-lens approximation we have the following 3×3 matrix:

$$M_{\text{FODO}\ j} = \begin{pmatrix} 1 - \frac{L^2}{8f^2} & L\left(1 + \frac{L}{4f}\right) & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{L}{4f^2}\left(1 - \frac{L}{4f}\right) & 1 - \frac{L^2}{8f^2} & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\mu & \beta\sin\mu & \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_j \\ -\frac{\sin\mu}{\beta} & \cos\mu & \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_j \\ 0 & 0 & 1 \end{pmatrix}$$

   where $j = 1, 2$ (1=first cell, 2=second cell).

   The following condition must be satisfied:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = M_{\text{suppressor}} \begin{pmatrix} \eta_0 \\ 0 \\ 1 \end{pmatrix} \qquad (1)$$

   where η_0 is the initial dispersion (at the middle of the first focusing quadrupole). It can be demonstrated that for a FODO lattice the dispersion has its maximum at the middle of the focusing quadrupole:

$$\eta_0 = \frac{4f^2}{L}\left(1 + \frac{L}{8f}\right)\theta \qquad (2)$$

with $\theta = \theta_1 + \theta_2$ the total bend in the suppressor.

Answer.

Performing the corresponding matrix multiplication yields

$$M_{\text{suppressor}} = \begin{pmatrix} \cos 2\mu & \beta \sin 2\mu & D_x \\ -\frac{\sin 2\mu}{\beta} & \cos 2\mu & D'_x \\ 0 & 0 & 1 \end{pmatrix}$$

where:

$$\begin{aligned}
\cos 2\mu &= 1 - \frac{L^2}{2f^2} + \frac{L^4}{34f^4} \\
\beta \sin 2\mu &= 2L\left(1 - \frac{L^2}{8f^2}\right)\left(1 + \frac{L}{4f}\right) \\
\frac{\sin 2\mu}{\beta} &= \frac{L}{2f^2}\left(1 - \frac{L^2}{8f^2}\right)\left(1 - \frac{L}{4f}\right) \\
D_x &= \cos\mu \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \beta \sin\mu\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_2 \\
D'_x &= -\frac{\sin\mu}{\beta}\frac{L}{2}\left(1 + \frac{L}{8f}\right)\theta_1 + \cos\mu\left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\theta_1 + \left(1 - \frac{L}{8f^2} - \frac{L^2}{32f^2}\right)\theta_2
\end{aligned} \tag{3}$$

Taking into account:

$$\cos\mu = 1 - \tfrac{L^2}{8f^2};\ \beta \sin\mu = L + \tfrac{L^2}{4f} \text{ and } \tfrac{\sin\mu}{\beta} = \tfrac{1}{4f^2}\left(1 - \tfrac{L}{4f}\right)$$

the elements D_x and D'_xmay also be written as

$$\begin{aligned}
D_x &= \frac{L}{2}\left(1 + \frac{L}{8f}\right)\left[\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right] \\
D'_x &= \left(1 - \frac{L}{8f} - \frac{L^2}{32f^2}\right)\left[\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2\right]
\end{aligned} \tag{4}$$

From the condition Eq. (1) we have

$$\begin{aligned}
\eta_0 \cos 2\mu + D_x &= 0 \\
-\eta_0 \frac{\sin 2\mu}{\beta} + D'_x &= 0
\end{aligned} \tag{5}$$

Substituting Eq. (2) in Eq. (5) one obtains:

$$\begin{aligned}
\left(3 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 &= \left(4 - \frac{L^2}{4f^2} - \frac{8f^2}{L^2}\right)\theta \\
\left(1 - \frac{L^2}{4f^2}\right)\theta_1 + \theta_2 &= \left(2 - \frac{L^2}{4f^2}\right)\theta
\end{aligned}$$

In terms of phase advance μ this can be written as:

$$\begin{aligned}
\theta_1 &= \left(1 - \frac{1}{4\sin^2\frac{\mu}{2}}\right)\theta \\
\theta_2 &= \frac{1}{4\sin^2\frac{\mu}{2}}\theta
\end{aligned} \tag{6}$$

where $\theta_1 + \theta_2 = \theta$.

2. Obtain the relation between the angles for the cases of phase-advance per cell $\mu = \pi/3$ and $\pi/2$

   Answer.

   - For $\mu = \pi/3 \longrightarrow 4\sin^2\frac{\mu}{2} = 1$ and therefore (using Eq. (6)) $\theta_1 = 0$ and $\theta_2 = \theta$. This corresponds to a dispersion suppressor with missing magnets.
   - For $\mu = \pi/2 \longrightarrow 4\sin^2\frac{\mu}{2} = 2$ and therefore $\theta_1 = \theta_2 = \theta/2$.

2 Exercise: Double-Bend Achromat (DBA) lattice

A Double-Bend Achromat (DBA) can be made from two dipoles with a horizontally focusing quadrupole between them. The transfer matrix through the achromat is:

$$M_{\text{DBA}} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}}$$

Note that this magnet configuration does not produce vertical focusing, therefore it will not be enough to create a stable lattice. A full DBA typically comprises additional quadrupole doublets before and after the bending section. For sake of simplicity, we will neglect them.

1. Use the thin-lens approximation for quadrupoles and small-angle approximation for bends to find the dispersion in the middle of the quadrupole. Write the focal length in terms of the drift and bend parameters.

   Answer. Let us consider the 3×3 transfer matrices of each element of the lattice (using the thin lens approximation and small angle approximation for the bending magnets) for the beam coordinates x, $x'$and $\Delta p/p$:

$$M_{\text{bend}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{\text{drift}} = \begin{pmatrix} 1 & L & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_{1/2\text{F}} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Assuming the initial dispersion vector $(\eta_0, \eta_0', 1) = (0,0,1)$ and propagating it to the centre of the quadrupole:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = M_{1/2\text{F}} M_{\text{drift}} M_{\text{bend}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Here we take into account that η'=0 at the centre of a quadrupole. After matrix multiplication we obtain:

$$\begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & L & L\theta \\ -\frac{1}{2f} & 1-\frac{L}{2f} & \theta\left(1-\frac{L}{2f}\right) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   Therefore, one obtains:

$$\eta_c = L\theta$$

$$1 - \frac{L}{2f} = 0 \Rightarrow L = 2f$$

2. Show that the dispersion vanishes after the bend.

   Answer. Propagate the dispersion vector from the centre of the quadrupole to the end of the lattice:

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = M_{\text{bend}} M_{\text{drift}} M_{1/2\text{F}} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} \eta_{end} \\ \eta'_{end} \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{2f} & L & 0 \\ -\frac{1}{2f} & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_c \\ 0 \\ 1 \end{pmatrix}.$$

Taking into account $L = 2f$, we obtain:

$$\eta_{end} = (2f - L)\theta = 0,$$

$$\eta'_{end} = \theta - \frac{1}{2f}\eta_c = \theta - \frac{1}{2f}(2f\theta) = 0$$

3. Compute the parameters L, f for a 10-meter long DBA which bends the beam by an angle of 1 radians. What is the dispersion in the centre? Given a particle with 1% energy deviation, compute the displacement at the centre of the cell.

   Answer.

$$L = 5 \text{ m}$$

$$f = 2.5 \text{ m}$$

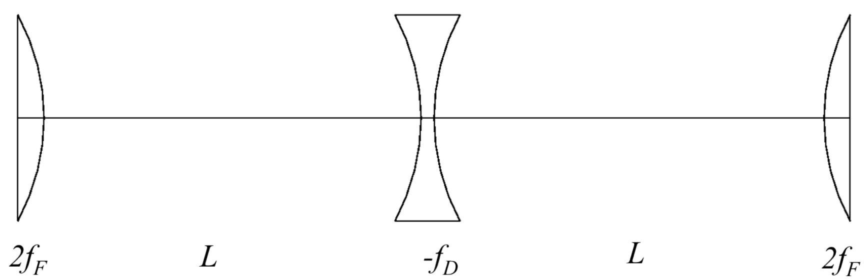
$$D = L \cdot \theta = 5 \text{ m}$$

$$x = \delta D = 0.01 * 5 \text{ m} = 5 \text{ cm}$$

3 Exercise: Chromaticity in a FODO cell

Consider a ring made of N_{cell} identical FODO cells with equally spaced quadrupoles. Assume that the two quadrupoles are both of length l_q, but their strengths may differ.

1. Calculate the maximum and the minimum betatron function in the FODO cell. (*Use the thin-lens approximations*)



Answer. First we calculate the transfer matrix for a FODO cell (see figure). We start from the centre of the focusing quadrupole where the betatron function is maximum. This exercise considers a general case where f_F is not necessarily equal to f_D. Using the thin lens approximation for the FODO cell with drifts of length L we get the following matrix:

$$\begin{aligned} M_{cell} &= \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f_F} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) & 2L + \frac{L^2}{f_D} \\ \frac{1}{f_D} - \frac{1}{f_F}(1 - \frac{L}{2f_F} + \frac{L}{f_D} - \frac{L^2}{4f_F f_D}) & 1 - L(\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{2f_F f_D}) \end{pmatrix} \end{aligned} \tag{7}$$

Remember that, in terms of betatron functions and phase advance, the matrix of a FODO cell is given by:

$$M_{cell} = \begin{pmatrix} cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix} \tag{8}$$

Since β has a maximum at the centre of the focusing quadrupole, then $\alpha = -\beta'/2 = 0$, and we can also write:

$$M_{cell} = \begin{pmatrix} cos\mu & \beta \sin\mu \\ -\frac{\sin\mu}{\beta} & \cos\mu \end{pmatrix}$$

Equating Eq. (7) to Eq. (9) we obtain:

$$\cos\mu = \frac{1}{2}\mathrm{tr}(M_{cell}) = 1 + \frac{L}{f_D} - \frac{L}{f_F} - \frac{L^2}{2f_D f_F} = 1 - 2\sin^2\frac{\mu}{2}$$

or

$$2\sin^2\frac{\mu}{2} = \frac{L}{f_F} - \frac{L}{f_D} + \frac{L^2}{2f_D f_F} \tag{9}$$

Where we have applied the following trigonometric identity: $\cos\mu = 1 - 2\sin^2\frac{\mu}{2}$.

The maximum for the betatron function β_{max} occurs at the focusing quadrupole. Since Eq. (7) is for a periodic cell starting at the centre of the focusing quadrupole, the m_{12} component of the matrix gives us

$$\beta_{max}\sin\mu = 2L + \frac{L^2}{f_D}$$

Rearranging:

$$\beta_{max} = \frac{2L + \frac{L^2}{f_D}}{\sin\mu} \tag{10}$$

On the other hand, the minimum for the betatron function occurs at the defocusing quadrupole position. Therefore, interchanging f_F with $-f_D$ for a FODO cell gives:

$$\beta_{min} = \frac{2L - \frac{L^2}{f_F}}{\sin\mu} \tag{11}$$

2. Calculate the natural chromaticities for this ring.

   Answer. Let us remember the definition of natural chromaticity. The so-called “natural” chromaticity is the chromaticity that derives from the energy dependence of the quadrupole focusing, i.e. the chromaticity arising only from quadrupoles. The chromaticity is defined in the following way:

$$\xi = \frac{\Delta Q}{\Delta P/P_0} \tag{12}$$

   where ΔQ is the tune shift due to the chromaticity effects and $\Delta P/P_0$ is the momentum offset of the beam or the particle with respect to the nominal momentum p_0.

   The natural chromaticity is defined as (remember from Lecture 4):

$$\xi_N = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \tag{13}$$

   Sometimes, especially for small accelerators, the chromaticity is normalised to the machine tune Q and defined also as:

$$\xi' = \frac{\Delta Q/Q}{\Delta P/P_0} \tag{14}$$

$$\xi'_N = -\frac{1}{4\pi Q}\oint \beta(s)k(s)ds \tag{15}$$

For this exercise, either you decide to use Eq. (13) or Eq. (15) it is fine! From now on let us use Eq. (13):

$$\begin{aligned}
\xi_N &= -\frac{1}{4\pi}\oint \beta(s)k(s)ds \\
&= -\frac{1}{4\pi} \times N_{cell}\int_{cell} \beta(s)k(s)ds \\
&= -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i (kl_q)_i
\end{aligned}$$

Here we have used the following approximation valid for thin lens:

$$\int_{cell} \beta(s)k(s)ds \simeq \sum_{i\in\{quads\}} \beta_i (kl_q)_i$$

where we sum over each quadrupole i in the cell. In the case of the FODO cell we have two half focusing quadrupoles and one defocusing quadrupole. Taking into account that $(kl_q)_i = 1/f_i$, we have:

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{4\pi}\sum_{i\in\{quads\}} \beta_i (kl_q)_i \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{2f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right) + \beta_{max}\left(\frac{1}{2f_F}\right)\right] \\
&= -\frac{N_{cell}}{4\pi}\left[\beta_{max}\left(\frac{1}{f_F}\right) + \beta_{min}\left(-\frac{1}{f_D}\right)\right] \\
&= -\frac{N_{cell}}{4\pi\sin\mu}\left[\left(2L + \frac{L^2}{f_D}\right)\frac{1}{f_F} - \left(2L - \frac{L^2}{f_F}\right)\frac{1}{f_D}\right] \\
&= -\frac{N_{cell}L}{2\pi\sin\mu}\left[\frac{1}{f_F} - \frac{1}{f_D} + \frac{L}{f_F f_D}\right]
\end{aligned}$$

Here we have used the expressions (10) and (11) for β_{max} and β_{min}.

3. Show that for short quadrupoles, if $f_F \simeq f_D$,

$$\xi_N \simeq -\frac{N_{cell}}{\pi}\tan\frac{\mu}{2}.$$

Answer. If $f_F \simeq f_D$, we have

$$\begin{aligned}
\xi_N &\simeq -\frac{N_{cell}}{2\pi\sin\mu}\frac{L^2}{f_F f_D} \\
&= -\frac{N_{cell}}{4\pi\sin\frac{\mu}{2}\cos\frac{\mu}{2}} 4\sin^2\frac{\mu}{2}
\end{aligned}$$

where we have used the trigonometric identity: $\sin\mu = 2\sin\frac{\mu}{2}\cos\frac{\mu}{2}$

Considering Eq. (9), we have

$$4\sin^2\frac{\mu}{2} = \frac{L^2}{f_F f_D}$$

which finally gives:

$$\xi_N \simeq -\frac{N_{cell}}{\pi} \tan \frac{\mu}{2}$$

Q.E.D.!

4. Design the FODO cell such that it has: phase advance $\mu = 90$ degrees, a total length of 10 m, and a total bending angle of 5 degrees. What are β_{max}, β_{min}, D_{max}, D_{min}?

   Answer. Lattice parameters: $L = 10$ m, $\theta = 5$ degrees$= 0.087266$ rad, $f = \frac{1}{\sqrt{2}} \frac{L}{2} = 3.535$ m

   Maximum and minimum betatron functions:

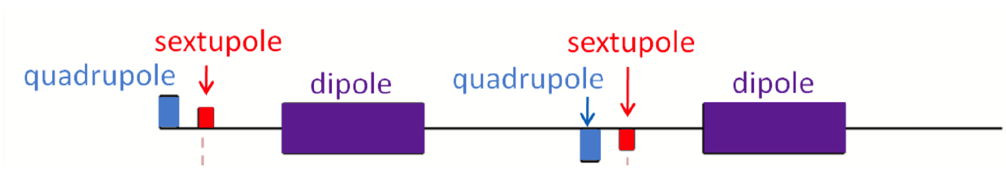
$$\beta_{max} = \frac{L + \frac{L^2}{4f}}{\sin \mu} = L + \frac{L^2}{4f} = 17.07 \text{ m}, \quad \beta_{min} = \frac{L - \frac{L^2}{4f}}{\sin \mu} = L - \frac{L^2}{4f} = 2.93 \text{ m}$$

   Maximum and minimum dispersion:

$$D_{max} = \frac{L\theta \left(1 + \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f + \frac{L}{2}\right) \theta = 0.59060 \text{ m}, \quad D_{min} = \frac{L\theta \left(1 - \frac{1}{2} \sin \frac{\mu}{2}\right)}{4 \sin^2 \frac{\mu}{2}} = \frac{f}{L} \left(4f - \frac{L}{2}\right) \theta = 0.28207 \text{ m}$$

5. Add two sextupoles at appropriate locations to correct horizontal and vertical chromaticities. (hints: use 1 sextupole for the horizontal plane and 1 for the vertical plane; do not consider geometric aberrations).

   Answer. By locating sextupoles with strength $K_s > 0$ where β_x is large and β_y is small, we can correct the horizontal chromaticity with relatively little impact on the vertical chromaticity. Similarly, by locating sextupoles with $K_s < 0$ where β_y is large and β_x is small, we can correct the vertical chromaticity with relatively little impact on the horizontal chromaticity. See figure below.



   Let us assume the case of a FODO lattice where $f_F = f_D = f$. Then the natural chromaticity of this FODO cell is given by the expression (exercise 1.3):

$$\xi_N \simeq -\frac{1}{\pi} \tan \frac{\mu}{2}$$

   For $\mu = 90$ it is $\xi_N \simeq -1/\pi$ in both horizontal and vertical plane. Therefore, we need to adjust the strength of the sextupoles to cancel this chromaticity:

$$-\frac{1}{4\pi} \left[K_{2F} D_{max} \beta_{max} + K_{2D} D_{min} \beta_{min}\right] \simeq -\frac{1}{\pi}$$

   where $K_{2F} = k_{2F} l_s$ is the normalised integrated strength of the sextupole located near the focusing quadrupole, and $K_{2D} = k_{2D} l_s$ the normalised integrated strength of the sextupole near the defocusing quadrupole (with l_s the effective length of the sextupole). For an effective cancellation of the chromaticity in both planes, usually $K_{2F} > 0$ and $K_{2D} < 0$. Substituting the values for the maximum and minimum dispersion and betatron function in terms of the total length of the lattice L and the focal length of the quadrupoles f, one obtains the following expression:

$$-\frac{1}{4\pi} \frac{f}{L} \theta \left[K_{2F} \left(4f + \frac{L}{2}\right) \left(L + \frac{L^2}{4f}\right) + K_{2D} \left(4f - \frac{L}{2}\right) \left(L - \frac{L^2}{4f}\right)\right] \simeq -\frac{1}{\pi}$$

Considering the same absolute value for the strength of the sextupoles, $K_{2F} = -K_{2D} = K_s$, we can write then:

$$\frac{3}{4\pi} K_s L f \theta = \frac{1}{\pi}$$

The strength of the sextupole is given then by:

$$K_s = \frac{4}{3 L f \theta}$$

Then, substituting all the numerical values for the lattice parameters:
$K_{2F} = 0.865$ m^{-2}
$K_{2D} = -0.865$ m^{-2}

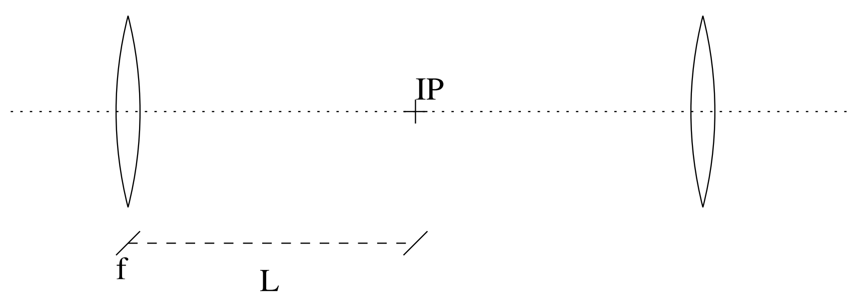
6. If the gradient of all focusing quadrupoles in the ring is wrong by +10%, how much is the tune-shift with and without sextupoles?

Answer.

If the gradient of the focusing quadrupole has and error of 10%, then the corresponding quad. strength error is also 10%. We calculate the number of cells of a ring made of these FODO cells, $N_{cell} = 72$ cells, and then we calculate the total tune-shift in both planes:
$\Delta Q_x = N_{cell} \frac{\Delta K_F \beta_{max}}{4\pi} = 9.78$
$\Delta Q_y = N_{cell} \frac{\Delta K_F \beta_{min}}{4\pi} = 1.68$
When the sextupoles correct for the chromaticity, the particles have, in principle, no tune-shift with energy. In real machines, one wants to have a non-zero residual chromaticity to stabilise the beam against resonant imperfections.

4 Exercise: Low-Beta Insertion

Consider the following low-beta insertion around an interaction point (IP). The quadrupoles are placed with mirror-symmetry with respect to the IP:



The beam enters the quadrupole with Twiss parameters $\beta_0 = 20$ m and $\alpha_0 = 0$. The drift space has length $L = 10$ m.

(i) Determine the focal length of the quadrupole in order to locate the waist at the IP.

(ii) What is the value of $\beta^\star$?

(iii) What is the phase advance between the quadrupole and the IP?

Solution.

$$M = \begin{pmatrix} 1-\frac{L}{f} & L \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_{\mathrm{IP}} = M \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}_0 \cdot M^T$$

$$\begin{pmatrix} \beta_{\mathrm{IP}} & 0 \\ 0 & 1/\beta_{\mathrm{IP}} \end{pmatrix} = M \cdot \begin{pmatrix} \beta_0 & 0 \\ 0 & 1/\beta_0 \end{pmatrix} \cdot M^T$$

We get a system of equations:

$$\begin{cases} \beta_{\mathrm{IP}} = & \beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0} \\ \frac{1}{\beta_{\mathrm{IP}}} = & \frac{\beta_0}{f^2} + \frac{1}{\beta_0} \end{cases}$$

multiplying them:

$$1 = \left(\beta_0 \left(1-\frac{L}{f}\right)^2 + \frac{L^2}{\beta_0}\right)\left(\frac{\beta_0}{f^2} + \frac{1}{\beta_0}\right)$$

and solving for f:

$$f = \frac{\beta_0\sqrt{(\beta_0^2 - 4L^2)} + \beta_0^2}{2L}$$

from which one finds:

$$f = 20 \text{ m}$$

and substituting back into one of the equations in the system:

$$\beta_{\mathrm{IP}} = 10 \text{ m}.$$

The phase advance can be computed remembering that

$$M_{0\to s} = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}}\left(\cos\psi_s + \alpha_0 \sin\psi_s\right) & \sqrt{\beta_s\beta_0}\sin\psi_s \\ \frac{(\alpha_0-\alpha_s)\cos\psi_s - (1+\alpha_0\alpha_s)\sin\psi_s}{\sqrt{\beta_s\beta_0}} & \sqrt{\frac{\beta_0}{\beta_s}}\left(\cos\psi_s - \alpha_s\sin\psi_s\right) \end{pmatrix}$$

In this case, $\alpha_0 = \alpha_{\mathrm{IP}} = 0$,

$$\mathrm{Trace}\,(M) = \frac{3}{2} = \left(\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}\right)\cos\Delta\mu$$

$$\Delta\mu = \arccos\left(\frac{3}{2}\cdot\frac{1}{\sqrt{\frac{\beta^\star}{\beta_0}} + \sqrt{\frac{\beta^0}{\beta^\star}}}\right) = \arccos\left(\frac{3}{2}\cdot\frac{1}{2.1213}\right) = 45 \text{ degrees}$$

Alternatively, given that the system:

$$M = Q\cdot D\cdot D\cdot Q$$

is indeed periodic, one can say:

$$M = \begin{pmatrix} 1-\frac{2L}{f} & 2L \\ \frac{2L}{f^2} - \frac{2}{f} & 1-\frac{2L}{f} \end{pmatrix}$$

$$\cos\Delta\mu_{\mathrm{twice}} = \frac{1}{2}\mathrm{Trace}\,(M) = \frac{1}{2}\mathrm{Trace}\left(2 - \frac{4L}{f}\right) = 0$$

$$\Delta\mu_{\mathrm{twice}} = 90 \text{ degrees} \qquad \Rightarrow \Delta\mu = 45 \text{ degrees}$$