Transverse Beam Dynamics

JUAS 2019 - Tutorial 1 (solutions)

1 Exercise: Wien Filter

A Wien Filter is a device that allows to select particles in a beam according to their velocity.

1. Write down the expression of the Lorentz force.

   Answer.

   $$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

2. How should we orient an electric field $\vec{E}$ if we want to compensate the force of an uniform magnetic field $\vec{B}$?

   Answer. The electric field must be oriented perpendicularly to both $\vec{v}$ and $\vec{B}$.

3. Assuming the magnetic field is 2 mT, what would be the required electric field (V/m) to select protons travelling with a velocity of $0.15c$?

   Answer.

   $$E = vB = 0.15c \ 2 \ \text{mT} \approx 90'000 \ \text{V/m}$$

4. Assuming that the particles move along the z axis and $\vec{B} = \begin{pmatrix} 0, & B_y, & 0 \end{pmatrix}$, write the equations of motion.

   Answer. If $\vec{B} = \begin{pmatrix} 0, & B_y, & 0 \end{pmatrix}$ and the particles move along the z axis, then $\vec{E} = \begin{pmatrix} E_x, & 0, & 0 \end{pmatrix}$, and the equations of motion read:

   $$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = q\left(E_x - B_y\frac{\mathrm{d}z}{\mathrm{d}t}\right)$$
   $$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 0$$
   $$m\frac{\mathrm{d}^2z}{\mathrm{d}t^2} = qB_y\frac{\mathrm{d}x}{\mathrm{d}t}$$

   with $\frac{\mathrm{d}z}{\mathrm{d}t} = v_z$.

   - Particles with $v = v_z$ are not deflected if

   $$\left(E_x - B_y\frac{\mathrm{d}z}{\mathrm{d}t}\right) = 0 \quad \Rightarrow (E_x - B_yv_z) \quad \Rightarrow E_x = B_yv_z$$

   See: "CONSTRUCTION OF A WIEN FILTER HEAVY ION ACCELERATOR", K. Jensen and E. Veje, Nuclear Instruments and Methods 122 (1974)

5. [Optional] Could we use a Wien filter with a neutral beam (eg. neutron)? What other techniques could be employed to create a velocity filter?

   Answer. A Wien filter will not work with a neutral beam, but still it is possible to filter it by velocity. A possibility consists in having two rotating disks made of an absorbing material, presenting a radial slit. The velocity can be selected tuning the distance, the rotation frequency and the phase between the the discs.

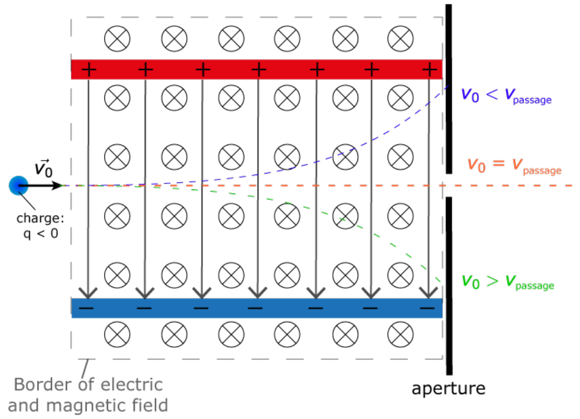
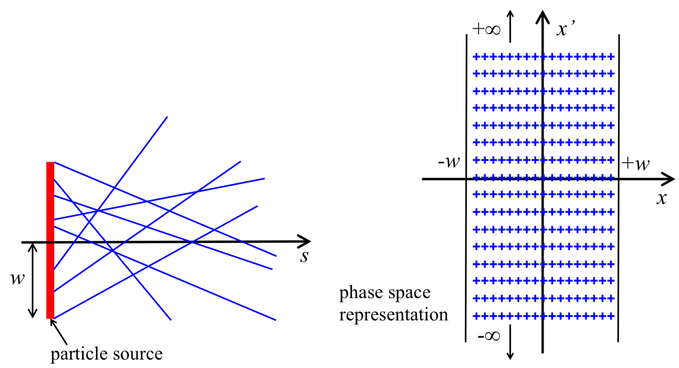


Figure 1: Sketch of a Wien filter

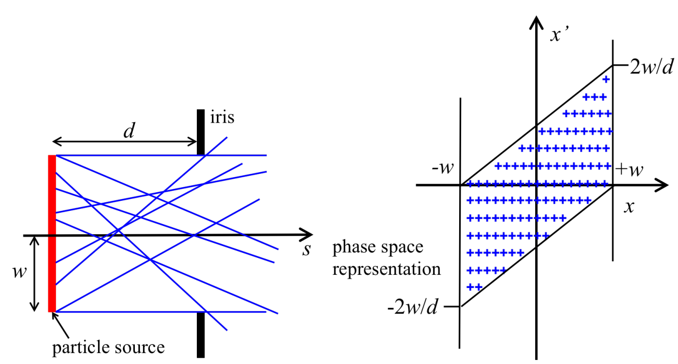
2 Exercise: Understanding the phase space concept

1. Phase Space Representation of a Particle Source:

   - Consider a source at position s_0 with radius w emitting particles. Make a drawing of this setup in the configuration space and in the phase space. Which part of the phase space can be occupied by the emitted particles?
     Answer. Particles are emitted from the entire source surface $x \in [-w, +w]$ with a trajectory slope $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, i.e. the particles can have any $x' \in \mathbb{R}$. The occupied phase space area is infinite.



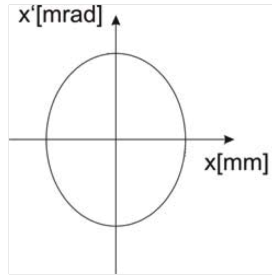
   - Any real beam emerging from a source like the one above will be collimated. This can be modelled by assuming that a distance d away from the source there is an iris with opening radius $R = w$. Draw this setup in the configuration space and in the phase space. Which part of the phase space is occupied by the beam, right after the collimator?
     Answer. Particles with angle $x' = 0$ are emitted from the entire source surface $x \in [-w, +w]$ and arrive behind the iris opening. For $x = \pm w$ there is a maximum angle $x' = \pm 2w/d$ that will still be accepted by the iris. This leads to a parallelogram in phase space. Such a beam has a specific emittance given by the occupied phase space area.



2. Sketch the emittance ellipse of a particle beam in:

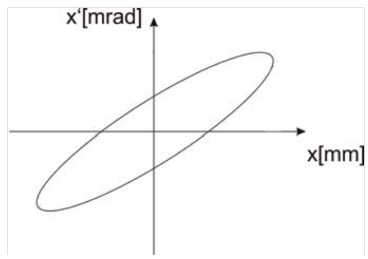
(I) horizontal x-x' phase space at the position of a transverse waist,

Answer. Beam at the position of a transverse (x) waist



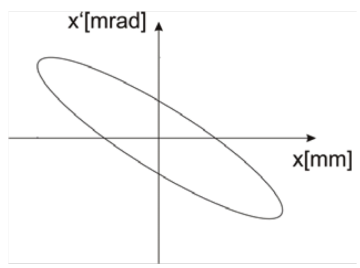
(II) when the beam is divergent, and

Answer. Divergent beam (positive slope):



(III) when the beam is convergent.

Answer. Convergent beam (negative slope):



3 Exercise: Local radius, rigidity

We wish to design a proton ring with a radius of $R = 200$ m. Let us assume that only 50% of the circumference is occupied by bending magnets:

- What will be the local radius of bend ρ in these magnets if they all have the same strength?

  Answer.
  $$2\pi\rho = 50\% \cdot 2\pi R \longrightarrow \rho = 100 \text{ m}.$$

- If the kinetic energy of the protons is 2 GeV, calculate the beam rigidity $B\rho$ and the field in the dipoles.

  Answer. The total energy is given by
  $$E = \sqrt{m_0^2 c^4 + p^2 c^2},$$
  where $m_0 = 938$ MeV$/c^2$ is the rest mass of the proton. Knowing that the kinetic energy is
  $$E_k = E - m_0 c^2 = \sqrt{m_0^2 c^4 + p^2 c^2} - m_0 c^2$$
  then

$$p = 2.78 \text{ GeV}/c.$$

The magnetic rigidity is:

$$B\rho \approx \frac{1}{0.3} p \text{ [GeV/c]=}9.27 \text{ T} \cdot \text{m}$$

and therefore $B \approx 0.09$ T.

4 Exercise: Thin-lens approximation

1. Compute and compare the matrices for the quadrupole of the previous exercise for the thin and thick cases.

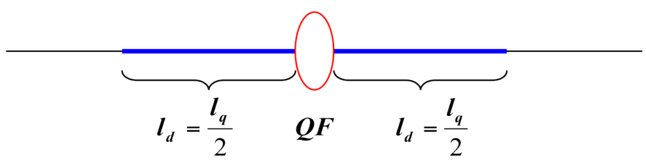
   Answer. The matrix of a focusing quadrupole is given by

$$M_{QF} = \begin{pmatrix} \cos\left(\sqrt{|k|}l_q\right) & \frac{1}{\sqrt{|k|}} \sin\left(\sqrt{|k|}l_q\right) \\ -\sqrt{|k|} \sin\left(\sqrt{|k|}l_q\right) & \cos\left(\sqrt{|k|}l_q\right) \end{pmatrix} = \begin{pmatrix} 0.8525 & 5.22 \\ -0.0522 & 0.8525 \end{pmatrix}$$

   In thin lens approximation we replace the matrix above by the expression

$$M_{QF} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{|f|} & 1 \end{pmatrix} \text{ with focal length } f = \frac{1}{|kl_q|} = 18.2 \text{ m}$$

   The thin lens description has to be completed by the matrix of a drift space of half the quadrupole length in front and after the thin lens quadrupole. The appropriate description is therefore



   So we write

$$M_{thinlens} = \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{l_q}{2} \\ 0 & 1 \end{pmatrix}$$

   Multiplying out we get

$$M_{thinlens} = \begin{pmatrix} 1 + \frac{l_q}{2} kl_q & \frac{l_q}{2}\left(2 + kl_q \frac{l_q}{2}\right) \\ kl_q & 1 + kl_q \end{pmatrix}$$

   With the parameters in the example we get finally

$$M_{thinlens} = \begin{pmatrix} 0.848 & 5.084 \\ -0.055 & 0.848 \end{pmatrix}$$

   which is still quite close to the result of the exact calculation above.

2. Verify if the stability condition is valid

3. Would the answer be the same, if the quadrupole was defocusing?

- **Answer.** The stability condition is:

$$|\mathrm{Tr}\,(M)| \leq 2$$

Indeed,

$$0.848 + 0.848 \leq 2.$$

If the quadrupole was defocusing, then

$$\mathrm{Tr}\,(M) = \cosh\left(\sqrt{|k|}l_q\right) + \cosh\left(\sqrt{|k|}l_q\right)$$

which is always > 2.

5 Exercise: Hill's equation

Solve the Hill's equation:

$$y'' + k(s)y = 0$$

by substituting:

$$y = A\sqrt{\beta(s)}\cos[\phi(s) + \phi_0] \text{ with } \phi' = \tfrac{1}{\beta(s)}, \text{ and where } A \text{ and } \phi_0 \text{ are constants,}$$

demonstrating that a necessary condition is:

$$\tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

The first and second derivative of y:

$$\begin{aligned} y' &= \tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta'}{2}\cos[\phi(s) + \phi_0] - \sin[\phi(s) + \phi_0]) \\ y'' &= \tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta''}{2} - \tfrac{\beta'^2}{4\beta} - \tfrac{1}{\beta})\cos[\phi(s) + \phi_0] \end{aligned}$$

Substituting in the Hill's equation

$$\tfrac{A}{\sqrt{\beta(s)}}(\tfrac{\beta''}{2} - \tfrac{\beta'^2}{4\beta} + k(s)\beta - \tfrac{1}{\beta})\cos[\phi(s) + \phi_0] = 0$$

Since the phase $\phi(s)$ has a different value at every point around the orbit and the amplitude $A \neq 0$, the previous equation can only be satisfied if

$$\tfrac{\beta\beta''}{2} - \tfrac{\beta'^2}{4} + k(s)\beta^2 - 1 \quad = \quad 0$$

and therefore

$$\tfrac{1}{2}\beta\beta'' - \tfrac{1}{4}\beta'^2 + k(s)\beta^2 = 1$$

Q.E.D.!