Transverse Beam Dynamics

JUAS 2019 - Tutorial 2 (solutions)

1 Exercise: Particle momentum, geometry of a storage ring and thin lenses

The LHC storage ring at CERN collides proton beams with a maximum momentum of $p = 7$ TeV/c per beam. The main parameters of this machine are:

Circumference	$C_0 = 26658.9$ m	
Particle momentum	$p = 7$ TeV/c	
Main dipoles	$B = 8.392$ T	$l_B = 14.2$ m
Main quadrupoles	$G = 235$ T/m	$l_q = 5.5$ m

1. Calculate the magnetic rigidity of the design beam, the bending radius of the main dipole magnets in the arc and determine the number of dipoles that is needed in the machine.

   Answer. The beam rigidity is obtained in the usual way by the golden rule:

   $$B\rho = \frac{p}{e} = \frac{1}{0.299792} \cdot p[\mathrm{GeV}/c] = 3.3356 \cdot p[\mathrm{GeV}/c] = 3.3356 \cdot 7000 \text{ Tm} = 23349 \text{ T}\cdot\text{m}$$

   and knowing the magnetic dipole field we get

   $$\rho = \frac{3.3356 \cdot 7000\text{Tm}}{8.392\text{T}} = 2782\text{m}$$

   The bending angle for one LHC dipole magnet:

   $$\theta = \frac{l_B}{\rho} = \frac{14.2\text{m}}{2782\text{m}} = 5.104 \text{ mrad}$$

   and as we want to have a closed storage ring we require an overall bending angle of 2π:

   $$N = \frac{2\pi}{\theta} = 1231 \text{ Magnets}$$

2. Calculate the k-strength of the quadrupole magnets and compare its focal length to the length of the magnet. Can this magnet be treated as a thin lens?

   Answer. We can use the beam rigidity (or the particle momentum) to calculate the normalised quadrupole strength:

   $$k = \frac{G}{B\rho} = \frac{G}{p/e} = 0.299792 \cdot \frac{G}{p[\mathrm{GeV/c}]} = 0.299792 \cdot \frac{235\mathrm{T/m}}{7000\mathrm{GeV/c}} = 0.01 \text{ m}^{-2}$$

   an the focal length:

   $$f = \frac{1}{k \cdot l_q} = 18.2 \text{ m} > l_q$$

   The focal length of this magnet is still quite bigger than the magnetic length l_q. So it is valid to treat that quadrupole in thin lens approximation.

2 Exercise: Stability condition

Consider a lattice composed by a single 2 meters long quadrupole, with $f = 1$ m

- Prove that if the quadrupole is defocusing, then a lattice is not stable
- Prove that if the quadrupole is focusing, then the lattice is stable

Solution:

- Let's work in thin-lens approximation

$$M_{\mathrm{QD}} = \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix}$$

  which can be computed to be

$$M_{\mathrm{QD}} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

  has trace $\mathrm{Tr}\left(M_{\mathrm{QD}}\right) = 4$, which does not fulfill the stability requirement:

$$\left|\mathrm{Tr}\left(M_{\mathrm{QD}}\right)\right| \leq 2$$

- In the case of a focusing quadrupole:

$$M_{\mathrm{QF}} = \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_{\mathrm{quad}}/2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

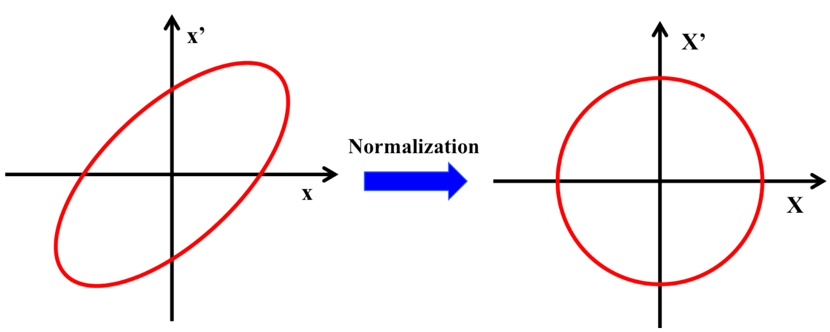
  which clearly satisfies the stability criterion.

3 Exercise: Normalised phase space

Let us consider the following phase space vector: (x, x'). The transformation to a *normalised phase space* (X, X') is given by:

$$\begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_x} & 0 \\ \alpha_x/\sqrt{\beta_x} & \sqrt{\beta_x} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}$$

The normalisation process of the phase space is illustrated in the figure below:



If we know that the transfer matrix between two points 1 and 2 (with phase advance ϕ_x between them) in the phase space (x, x') is given by:

$$M_{1\to 2} = \begin{pmatrix} \sqrt{\frac{\beta_{x2}}{\beta_{x1}}}(\cos\phi_x + \alpha_{x1}\sin\phi_x) & \sqrt{\beta_{x1}\beta_{x2}}\sin\phi_x \\ \frac{(\alpha_{x1}-\alpha_{x2})\cos\phi_x - (1+\alpha_{x1}\alpha_{x2})\sin\phi_x}{\sqrt{\beta_{x2}\beta_{x1}}} & \sqrt{\frac{\beta_{x1}}{\beta_{x2}}}(\cos\phi_x - \alpha_{x2}\sin\phi_x) \end{pmatrix}$$

Obtain the transfer matrix between two points 1 and 2 in the normalised phase space.

Answer. If one writes

$$M_{1\to 2} = U_2^{-1} \cdot R \cdot U_1$$

with U_1 the transformation into normalised coordinates for the Twiss parameters at 1, and U_2 its inverse for the Twiss parameters at 2: i.e.,

$$U_1 = \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}; \qquad U_2^{-1} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix}$$

It can be shown that the matrix M_{12} can be written as:

$$M_{12} = \begin{pmatrix} \sqrt{\beta_2} & 0 \\ -\frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ \frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix}$$

with

$$R = \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix}.$$

4 Exercise: beam size and luminosity

An e^+e^- collider has an interaction Point (IP) with $\beta_x^* = 0.5$ m and $\beta_y^* = 0.1$ cm. The peak luminosity available by a e^+e^- collider can be written as:

$$L = \frac{N_\mathrm{b} N_{e^-} N_{e^+} f_\mathrm{rev}}{4\pi\sigma_x^*\sigma_y^*} \ [\mathrm{cm}^{-2}\mathrm{s}^{-1}]$$

where $N_\mathrm{b} = 80$ is the number of bunches per beam (we assume the same number of bunches for both the e^- and the e^+ beams), $N_{e^-} = N_{e^+} = 5 \times 10^{11}$ is the number of particles per bunch (we assume the same number for both e^- and e^+ bunches), and f_rev is the revolution frequency. The horizontal and vertical normalised beam emittances are respectively: $\epsilon_{x,N} = 2.2$ mm and $\epsilon_{y,N} = 4.7$ μm.

- Compute the revolution frequency f_rev, knowing that the circumference is 80 km and that the beam moves nearly at the speed of light

Solution. The revolution period is given by T_{rev} = circumference$/c$ = 80km/c, and therefore the revolution frequency is:

$$f_{rev} = 1/T_{rev} = c/80 \text{ km} \simeq 3.75 \text{ kHz}$$

- Calculate the beam transverse beam sizes σ_x^* and σ_y^* at the IP, and the luminosity L for two different beam energies: 45 GeV and 120 GeV

Solution. For 45 GeV beam energy: in this case the Lorentz factor is $\gamma = 88062.622$, and $\sigma_x^* = \sqrt{\beta_x^*\epsilon_{x,N}/\gamma} \simeq 111.76$ μm and $\sigma_y^* = \sqrt{\beta_y^*\epsilon_{y,N}/\gamma} \simeq 0.23$ μm, and the luminosity is $L \simeq 2.32 \times 10^{34}$ cm^{-2}s^{-1}

For 120 GeV beam energy: in this case the Lorentz factor is $\gamma = 234833.66$, and $\sigma_x^* = \sqrt{\beta_x^*\epsilon_{x,N}/\gamma} \simeq 68.56$ μm and $\sigma_y^* = \sqrt{\beta_y^*\epsilon_{y,N}/\gamma} \simeq 0.14$ μm, and the luminosity is $L \simeq 6.22 \times 10^{34}$ cm^{-2}s^{-1}

- What are the beam divergences (horizontal and vertical) at the IP for the 45 GeV case?

Solution. Where $\alpha = 0$ we have

$$\sigma_{x'}^* = \sqrt{\frac{\epsilon_{x,N}}{\gamma\beta_x^*}} = ...$$

- What is the value of the betatron function at position $s = 0.5$ m from the IP?

Solution. We know that the betatron function in the drift space of a low beta region (where we have the interaction point) depends on the longitudinal coordinate as follows:

$$\beta(s) = \beta^* + \frac{s^2}{\beta^*}$$

Therefore, $\beta_x(0.5\mathrm{m}) = 1$ m, and $\beta_y(0.5\mathrm{m}) = 250$ m