- Exercice 1 -

TUTORIAL PARTICLE SOURCE

F.1

4/3/19

SOLUTIONS

(1) $\vec{E} = -\vec{\text{grad}}\, V$ (2) $\text{div}\, \vec{E} = \frac{\rho}{\varepsilon_0}$

(1) + (2) $\Rightarrow$ $\text{div}(-\vec{\text{grad}}\, V) = -\frac{\rho}{\varepsilon_0} = \Delta V$

1 Dim, $f(z)$ only $\Rightarrow$ $\frac{d^2 V}{dz^2} = 0$ (no charge $\rho = 0$)

and $\vec{E} = E(z)\vec{z} = -\frac{dV}{dz}\vec{z}$

$\Rightarrow$ $\frac{dV}{dz} = \text{const.} = a$

$\Rightarrow$ $V(z) = a \cdot z + b$

$V(0) = V_0$; $V(d) = 0$ $\Rightarrow$ $\begin{cases} b = V_0 \\ a d + V_0 = 0 \end{cases}$

$\Rightarrow$ $\begin{cases} a = -\frac{V_0}{d} \\ b = V_0 \end{cases}$ so $V(z) = V_0 - \frac{V_0}{d} \cdot z$

$$\boxed{V(z) = \frac{V_0}{d}(d - z) \quad \forall\ z \in [0, d]}$$

$\vec{E} = -\frac{dV}{dz}\vec{z}$ (1 Dim)

$$\Rightarrow \boxed{\vec{E} = \frac{V_0}{d}\vec{z}}$$

- Exercice 2 -

$$m \frac{d\vec{v}}{dt} = q\vec{E} \qquad \text{ioin} \quad m\frac{dv_z}{dt} = q\frac{V_0}{d} \quad \text{along } z$$

$$v_z = v = \frac{dz}{dt}$$

$$m\frac{dv}{dt} = m\frac{dv}{dz}\cdot\frac{dz}{dt} = q\frac{V_0}{d}$$

$$\Rightarrow \quad m\, v\, dv = q\frac{V_0}{d}.dz$$

$$\frac{1}{2}dv^2 = \frac{qV_0}{md}dz \Rightarrow \frac{1}{2}\int_0^z dv^2 = \frac{qV_0}{md}\int_0^z dz$$

$$\Rightarrow \quad v^2(z) - \underset{0}{\underset{\shortparallel}{v^2(0)}} = \frac{2qV_0}{md}.(z-0)$$

$$\Rightarrow \quad v(z) = \sqrt{\frac{2qV_0 z}{md}} \quad , \quad z \leq d$$

$$v(z) = v(d) \quad \text{if } z > d$$

Exercice 3

$\vec{v_y} = v_y \cdot \vec{y}$ at $t=0$ particle at $\begin{cases} z=0 \\ y=0 \\ v_z(0)=0 \\ v_y(0)=v_y = \text{const.} \end{cases}$

$$m\frac{d\vec{v}}{dt} = q\vec{E} \Rightarrow \begin{cases} \dfrac{dv_y}{dt} = 0 \quad (1) \\ \dfrac{dv_z}{dt} = \dfrac{q}{m}\dfrac{V_0}{d} \quad (2) \end{cases} \qquad (v_x = 0 \ \forall t)$$

$(1) \Rightarrow \quad v_y(t) = \text{const} = v_y$

$$\Rightarrow \frac{dy}{dt} = v_y \Rightarrow \overset{(3)}{\boxed{y(t) = v_y \cdot t}} + \underset{0}{\underset{\parallel}{y(0)}}$$

$$(2) \Rightarrow \overset{\text{integration from 0 to } t}{v_z(t)} = \frac{qV_0}{md}(t-0) + \underset{0}{\underset{\parallel}{v_z(0)}} \quad (5)$$

$$\Rightarrow \frac{dz}{dt} = \frac{qV_0 t}{md} \overset{\text{integration}}{\Rightarrow} z(t) - \underset{0}{\underset{\parallel}{z(0)}} = \frac{qV_0}{2md}(t^2 - 0^2) \quad (4)$$

$$(3) + (4) \quad y(t) = v_y \cdot t = v_y \cdot \sqrt{\frac{2md\,z(t)}{qV_0}} \quad (7)$$

From exercice 2: $v_z(z) = \sqrt{\dfrac{2qV_0 z}{md}}$ and $v_z(d) = v = \sqrt{\dfrac{2qV_0}{m}}$ (6)

$$(6) + (7) \Rightarrow \boxed{y(z) = v_y \frac{\sqrt{2}}{v} \cdot \sqrt{2d \cdot z}} = \boxed{\frac{v_y}{v} \cdot 2\sqrt{d \cdot z} \quad z \leq d}$$

$$\Rightarrow y(d) = \frac{2v_y}{v} d$$

at $z = d$, $\begin{cases} y(d) = \dfrac{2 v_y}{v} . d \\ z = d \end{cases}$

for $z \geq d$ $\quad v_z(t) = \text{cnst} = \sqrt{\dfrac{2 q V_0}{m}}$

$\Rightarrow z(t') = v . t' + d$ (for t' such that $z \geq d$)

$t' = 0 \Rightarrow z(t' = 0) = d$

$v_y(t') = \text{cnst} = v_y \Leftrightarrow \dfrac{dy}{dt} = v_y$

$\Rightarrow y(t') = v_y . t' + \underset{\substack{\parallel \\ 2 \frac{v_y}{v} d}}{y(0)}$

$$\Rightarrow y(z) = v_y . \frac{(z - d)}{v} + 2 \frac{v_y}{v} d = \frac{v_y}{v}(z + d), \quad z \geq d$$

Exercice 4 - Tutorial particle sources - Child Langmuir

4.1 Energy = const $= \frac{1}{2} m v^2(z) + qV(z) = \frac{1}{2} m \underset{0}{\underset{\shortparallel}{v^2(0)}} + \underset{qV_0}{\underset{\shortparallel}{qV(0)}}$

$\Rightarrow \frac{1}{2} m v^2(z) + qV(z) = qV_0$

$$\Rightarrow \boxed{v(z) = \sqrt{\frac{2q(V_0 - V(z))}{m}} \quad , \; 0 \leq z \leq d} \quad (1)$$

4.2 $\operatorname{div} \vec{J} = \frac{\partial \rho}{\partial r}$ stationary $\Rightarrow \frac{\partial \rho}{\partial r} = 0$

$\Rightarrow \operatorname{div} \vec{J} = 0$ 1 Dim. problem $J = f(z)$ only

$\Rightarrow \frac{dJ}{dz} = 0 \Rightarrow J(z) = J_0 =$ const.

$\forall z, \; \underset{\text{definition}}{\vec{J} = \rho \vec{v}} \Rightarrow \boxed{J(z) = J_0 = \rho(z) \cdot v(z) = \text{const}} \; (2)$

4.3

$$\Delta V = -\frac{\rho}{\varepsilon_0}$$

here, 1 DIM, $\rho(z)$ and $V(z)$ only

$$\Rightarrow \quad \frac{d^2 V(z)}{dz^2} = -\frac{\rho(z)}{\varepsilon_0} \quad (3)$$

$$(1)+(2) \Rightarrow \rho(z) = \frac{J_0}{v(z)} = J_0 \sqrt{\frac{m}{2q}} \cdot \frac{1}{(V_0 - V(z))^{1/2}} \quad (4)$$

$$(4)+(3) \Rightarrow \boxed{\frac{d^2 V(z)}{dz^2} = -\frac{J_0}{\varepsilon_0}\sqrt{\frac{m}{2q}} \cdot \frac{1}{(V_0 - V(z))^{1/2}}} \quad (5)$$

$$K = \frac{J_0}{\varepsilon_0}\sqrt{\frac{m}{2q}}$$

$$\boxed{U(z) = V_0 - V(z)} \geq 0 \quad \forall z \in [0, d] \quad (6)$$

$$(7) \quad \boxed{\frac{dU}{dz} = -\frac{dV}{dz}}$$

$$(5)+(7) \Rightarrow \frac{d^2 U}{dz^2} = \frac{J_0}{\varepsilon_0}\sqrt{\frac{m}{2q}} \frac{1}{U^{1/2}}$$

$$\boxed{\frac{d^2 U}{dz^2} = \frac{K}{U^{1/2}}} \quad (8)$$

4.4 $\vec{E} = -\vec{\text{grad}}\, V$

1 Dim $\Rightarrow E(z) \cdot \vec{z} = -\frac{dV}{dz} \vec{z}$

$$\Rightarrow E(z) = \frac{dU}{dz}(z)$$

$$\frac{d^2U}{dz^2} = \frac{d}{dz}(E) = \frac{dE}{dz} = \frac{dE}{dU} \cdot \frac{dU}{dz} = E \frac{dE}{dU} \quad (9)$$

$$(8) + (9) \Rightarrow E \frac{dE}{dU} = \frac{K}{U^{1/2}} \Leftrightarrow E\, dE = K \frac{dU}{U^{1/2}} \quad (10)$$

4.5

$$(10) \Rightarrow \int_{E(0)}^{E(z)} E\, dE = K \int_{U(0)}^{U(z)} \frac{dU}{\sqrt{U}}$$

$$\Rightarrow \frac{1}{2} E^2(z) - \frac{1}{2} E^2(0) = 2K \left(\sqrt{U(z)} - \sqrt{\underset{\overset{\shortparallel}{0}}{U(0)}} \right)$$

$$\Rightarrow E^2(z) - E^2(0) = 4K \sqrt{U(z)} \quad (11)$$

4.6 space charge upper limit when $E(0) \to 0$

$$(11) \Rightarrow E^2(z) = \left(\frac{dU}{dz}\right)^2 = 4K\, U^{1/2}$$

$$\Rightarrow \frac{dU}{dz} = \sqrt{4K}\, U^{1/4}$$

$$\Rightarrow \int_{U(0)}^{U(d)} \frac{dU}{U^{1/4}} = \sqrt{4K} \int_0^d dz$$

$$\Rightarrow \left[\frac{4}{3} U^{3/4} \right]_{U(0)}^{U(d)} = \sqrt{4K}(d-0)$$

$$U(0) = V_0 - V(0) = 0$$
$$U(d) = V_0 - V(d) = V_0$$

4.7

$$\Rightarrow \frac{4}{3} V_0^{3/4} = \sqrt{4K}\, d$$

$$K = \frac{J_0}{\varepsilon_0} \sqrt{\frac{m}{2q}} \quad \text{so} \quad \frac{16}{9} \frac{V_0^{3/2}}{d^2} = \frac{4 J_0}{\varepsilon_0} \sqrt{\frac{m}{2q}}$$

$$\Rightarrow \boxed{J_0 = \frac{4}{9} \epsilon_0 \cdot \sqrt{\frac{2q}{m}} \cdot \frac{V_0^{3/2}}{d^2}}$$