RF Measurement Techniques II

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- Introduction to Scattering-parameters (S-parameters)
- The vector network analyzer (VNA)
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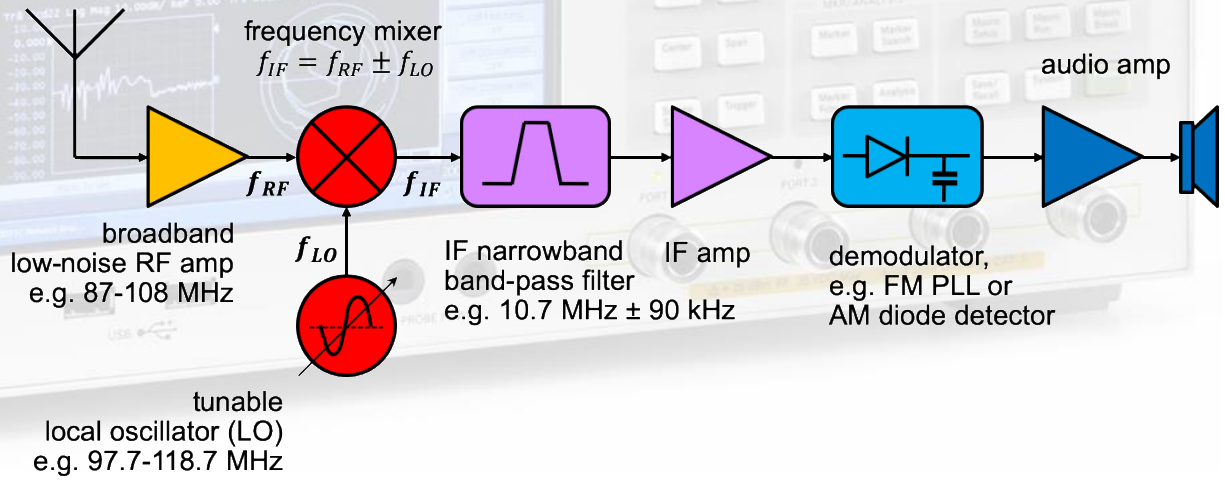
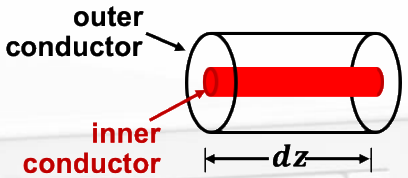
RF Measurement Instruments

- **Oscilloscope**
  - Time domain, arbitrary waveforms, direct digitalization
- **Spectrum analyzer (VSA, FFT)**
  - Frequency domain, down-convert to IF -> digitalization
- **Vector network analyzer (VNA)**
  - Device (DUT) characterizing, frequency and time domain (via iDFT)
- **VSA & VNA are based on the super-heterodyne concept**
  - Down-conversion to an intermediate frequency (IF)
  - Utilizing a mixer, e.g. based on *Schottky* diodes



Transmission-lines & VSWR

TEM transmission-lines, e.g. coaxial cable



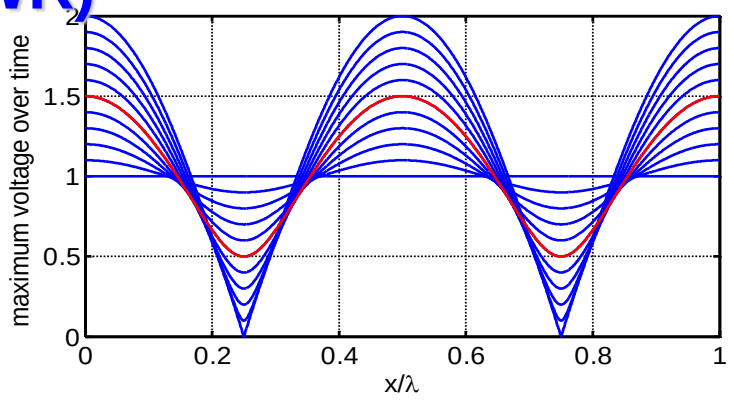
- Characteristic impedance, typically $Z_0 = 50\ \Omega$, is defined by the cross-section geometry.
- Needs to be terminated with $Z_L = Z_0$ to operate without reflections

reflection coefficient: $$\Gamma = \frac{E^{refl}}{E^{inc}} = \frac{b}{a} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- For steady-state sinusoidal signals, and $Z_L \neq Z_0$, the superposition of forward (incident) waves (a) and backward (reflected) wave (b) result in standing waves along the transmission line with voltage maxima V_{max} and minima V_{min}.

Voltage Standing Wave Ratio (VSWR)

$$VSWR = \frac{|V_{max}|}{|V_{min}|} = \frac{|a| + |b|}{|a| - |b|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$



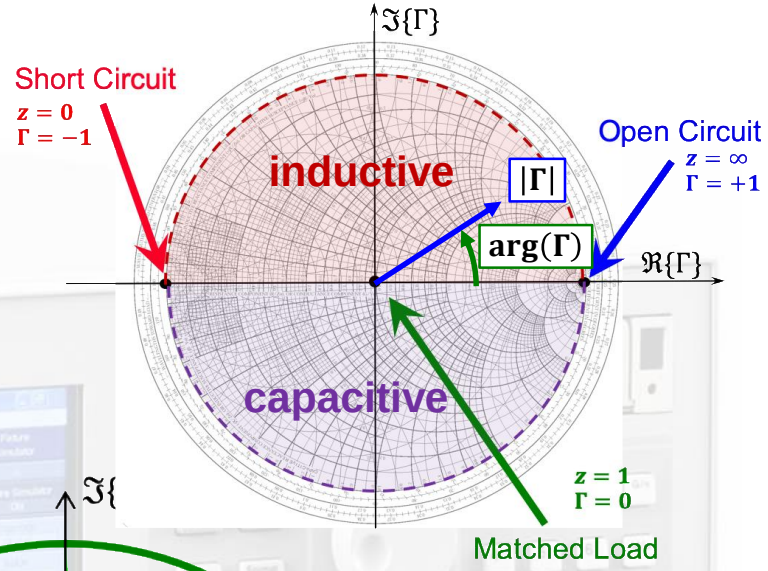
The *Smith* Chart

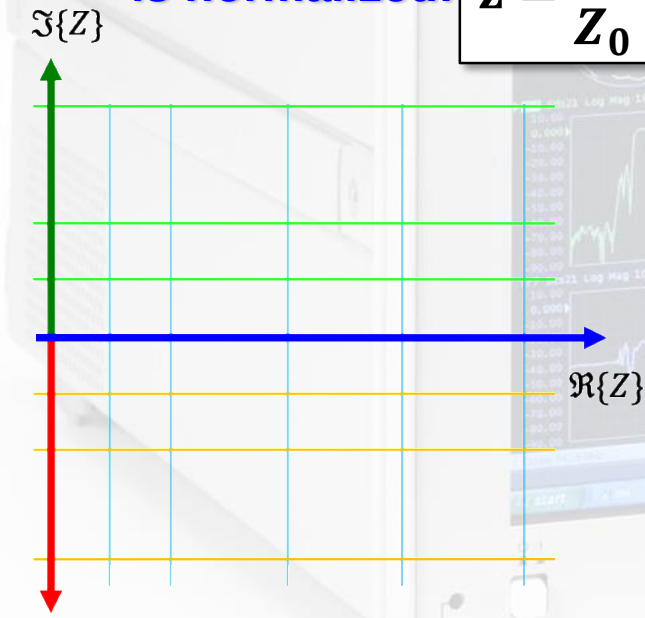
- gives a lot of confusion…
- The *Smith* chart is a conformal mapping of the complex Z-plane on the Γ-plane by applying the transformation:

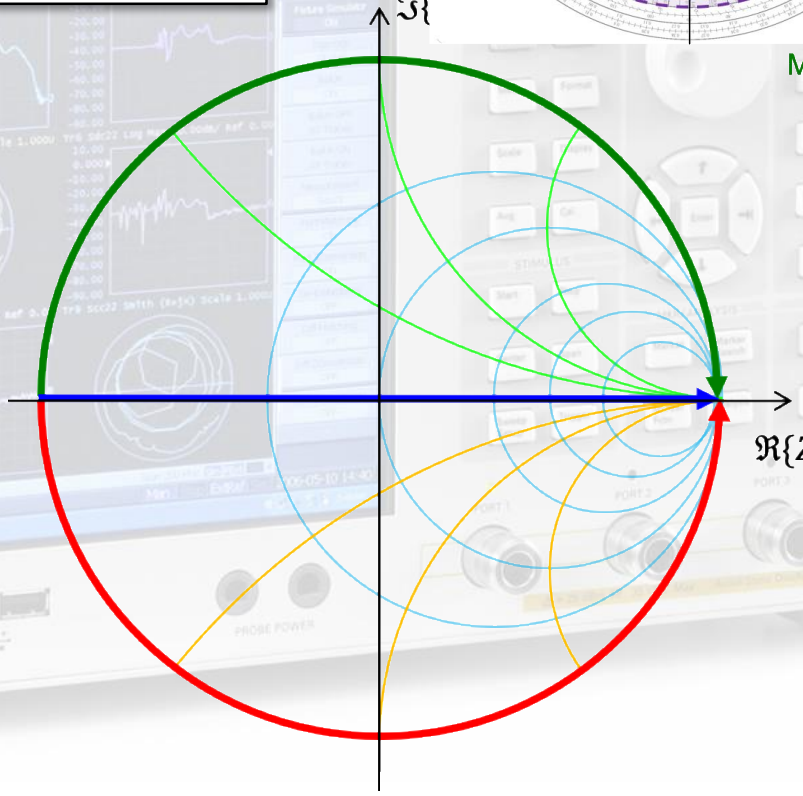
$$\Gamma = \frac{Z - Z_0}{Z + Z_0}$$

- The Z-plane is normalized:

$$z = \frac{Z}{Z_0} = \frac{1 + \Gamma}{1 - \Gamma}$$







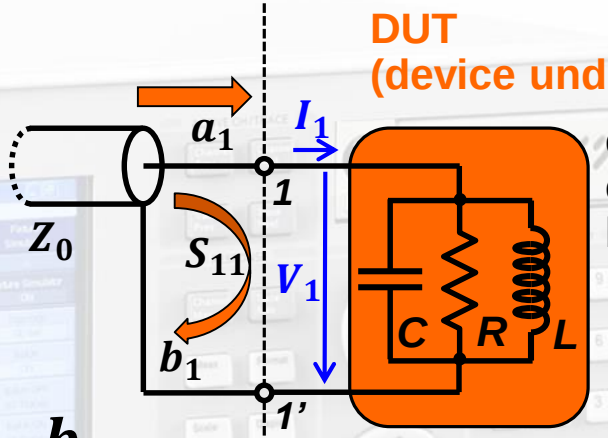
S-Parameters – Introduction (1)

- **Light falling on a car window:**
  - Some parts of the incident light is reflected (you see the mirror image)
  - Another part of the light is transmitted through the window (you can still see inside the car)
- **Optical reflection and transmission coefficients of the window glass define the ratio of reflected and transmitted light**
- **Similar:**
  Scattering (S-) parameters of an *n*-port electrical network (DUT) characterize reflected and transmitted (power) waves

S-Parameters – Introduction (2)

- **Electrical networks**
  - 1…*n*-ports circuits
  - Defined by voltages $V_n(\omega)$ or $v_n(t)$ and currents $I_n(\omega)$ or $i_n(t)$ at the ports
  - Characterized by circuit matrices, e.g. $ABCD$ (chain), Z, Y, H, etc.
- **RF networks**
  - 1…*n*-port RF DUT circuit or subsystem, e.g. filter, amplifier, transmission-line, hybrid, circulator, resonator, etc.
  - Defined by incident $a_n(\omega, s)$ and reflected waves $b_n(\omega, s)$ at a reference plane s (physical position) at the ports
  - Characterized by a scattering parameter (S-parameter) matrix of the reflected and transmitted power waves
  - Normalized to a reference impedance $\sqrt{Z_0}$ of typically $Z_0 = 50\ \Omega$

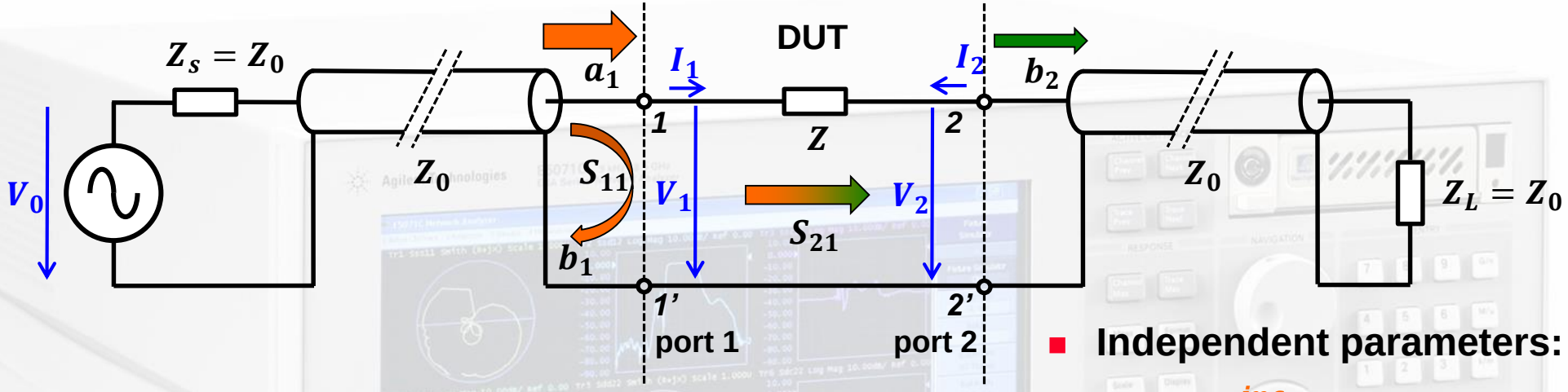
$$S_{11} = \frac{b_1}{a_1} = \Gamma$$



1-port DUT example

- S-Parameters allow to characterize the DUT with the measurement equipment to be located at some distance
- All high frequency effects of distributed elements are taken into account with respect to the reference plane

S-Parameters – Example: 2-port DUT



- **Analysis of the forward S-parameters:**

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \equiv \text{input reflection coefficient}$$

$$(Z_L = Z_0 \Rightarrow a_2 = 0)$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \equiv \text{forward transmission gain}$$

  - Examples of 2-ports DUT: filters, amplifiers, attenuators, transmission-lines (cables), etc.
  - ALL ports ALWAYS need to be terminated in their characteristic impedance!

- Independent parameters:

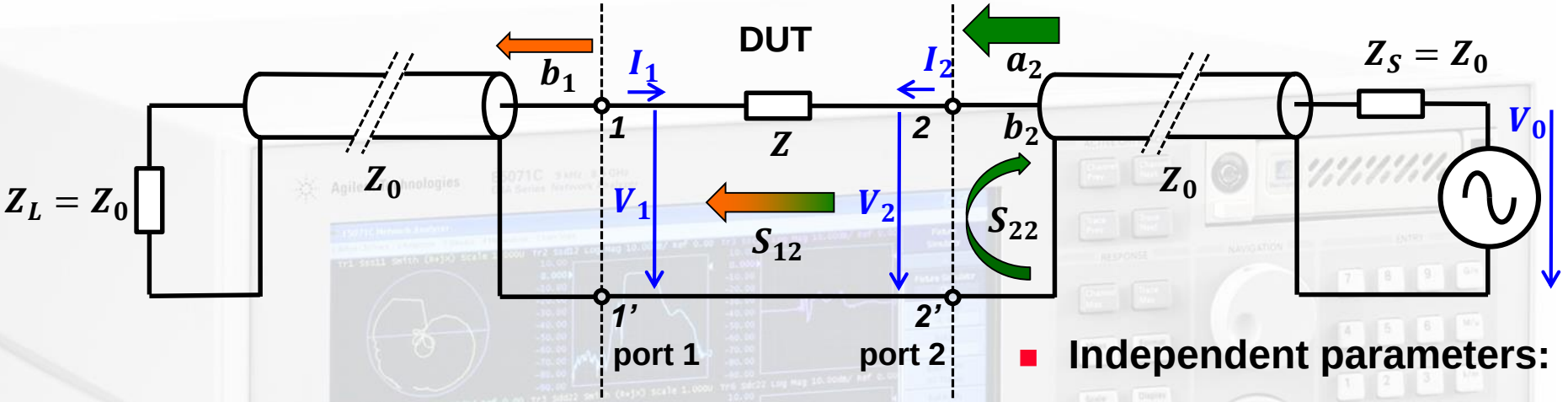
$$a_1 = \frac{V_1^{inc}}{\sqrt{Z_0}} = \frac{V_1 + I_1 Z_0}{2\sqrt{Z_0}}$$

- Dependent parameters:

$$b_1 = \frac{V_1^{refl}}{\sqrt{Z_0}} = \frac{V_1 - I_1 Z_0}{2\sqrt{Z_0}}$$

$$b_2 = \frac{V_2^{refl}}{\sqrt{Z_0}} = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}}$$

S-Parameters – Example: 2-port DUT



- **Analysis of the reverse S-parameters:**

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \equiv \text{output reflection coefficient}$$

$$(Z_L = Z_0 \Rightarrow a_1 = 0)$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \equiv \text{backward transmission gain}$$

  - *n*-port DUTs still can be fully characterized with a 2-port VNA, but again:
    don't forget to terminate unused ports!

- **Independent parameters:**

$$a_2 = \frac{V_2^{inc}}{\sqrt{Z_0}} = \frac{V_2 + I_2 Z_0}{2\sqrt{Z_0}}$$

- **Dependent parameters:**

$$b_1 = \frac{V_1^{refl}}{\sqrt{Z_0}} = \frac{V_1 - I_1 Z_0}{2\sqrt{Z_0}}$$

$$b_2 = \frac{V_2^{refl}}{\sqrt{Z_0}} = \frac{V_2 - I_2 Z_0}{2\sqrt{Z_0}}$$

S-Parameters – Definition (1) [2-port]

- **Linear equations for the 2-port DUT:**

$$b_1 = S_{11}a_1 + S_{12}a_2$$
$$b_2 = S_{21}a_1 + S_{22}a_2$$

- **with:**

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} \equiv \text{input reflection coefficient}$$

$$S_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} \equiv \text{output reflection coefficient}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} \equiv \text{forward transmission gain}$$

$$S_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} \equiv \text{backward transmission gain}$$

impedance measurements

transmission measurements

S-Parameters – Definition (2) [*n*-port]

- **Reflection coefficient and impedance at the n^{th}-port of a DUT:**

$$S_{nn} = \frac{b_n}{a_n} = \frac{\frac{V_n}{I_n} - Z_0}{\frac{V_n}{I_n} + Z_0} = \frac{Z_n - Z_0}{Z_n + Z_0} = \Gamma_n$$

$$Z_n = Z_0 \frac{1 + S_{nn}}{1 - S_{nn}} \text{ with } Z_n = \frac{V_n}{I_n} \text{ being the input impedance at the } n^{th} \text{ port}$$

- **Power reflection and transmission for a *n*-port DUT**

$$|S_{nn}|^2 = \frac{\text{power reflected from port } n}{\text{power incident on port } n}$$

$$|S_{nm}|^2 = \text{transmitted power between ports } n \text{ and } m$$

with all ports terminated in their characteristic impedance Z_0 and $Z_s = Z_0$

Here the US notion is used, where power $= |a|^2$.
European notation (often): power $= |a|^2/2$
These conventions have no impact on the S-parameters, they are only relevant for absolute power calculations

The Scattering Matrix (1)

- **Waves traveling towards the *n*-port:** $(a) = (a_1, a_2, a_2, \ldots a_n)$
- **Waves traveling away from the *n*-port:** $(b) = (b_1, b_2, b_2, \ldots b_n)$

- The relation between a_i and b_i $(i = 1 \ldots n)$ can be written as a **system of n linear equations**
  (a_i = the independent variable, b_i = the dependent variable)

one-port	$b_1 = S_{11}a_1 + S_{12}a_2 + S_{13}a_3 + S_{14}a_4 + \cdots$
two-port	$b_2 = S_{21}a_1 + S_{22}a_2 + S_{23}a_3 + S_{24}a_4 + \cdots$
three-port	$b_3 = S_{31}a_1 + S_{32}a_2 + S_{33}a_3 + S_{34}a_4 + \cdots$
four-port	$b_4 = S_{41}a_1 + S_{42}a_2 + S_{43}a_3 + S_{44}a_4 + \cdots$

  - **in compact matrix form follows**

$$(\boldsymbol{b}) = (\boldsymbol{S})(\boldsymbol{a})$$

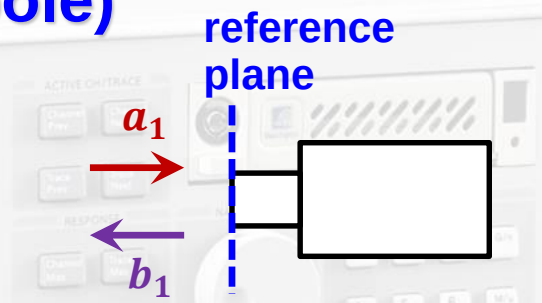
The Scattering Matrix (2)

- The simplest form is a passive **1-port (2-pole)**

$$(S) = S_{11} \quad \Rightarrow \quad b_1 = S_{11} a_1$$

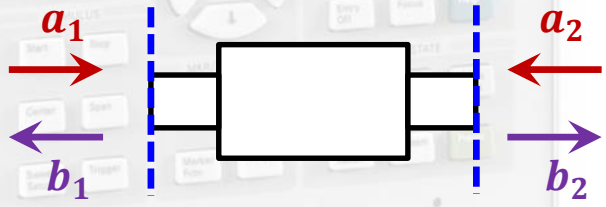
  - with the **reflection coefficient:**

$$\Gamma = S_{11} = \frac{b_1}{a_1}$$



- **2-port (4-pole) DUT:**

$$(S) = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} b_1 &= S_{11} a_1 + S_{12} a_2 \\ b_2 &= S_{21} a_1 + S_{22} a_2 \end{aligned}$$



  - An unmatched load, present at port 2 with a **reflection coefficient** Γ_{load} transfers to the input port as:

$$\Gamma_{in} = S_{11} + \frac{S_{21} \Gamma_{load} S_{12}}{1 - S_{22} \Gamma_{load}}$$

2-Port Examples

Transmission-line of $Z = 50\ \Omega$, length $l = \lambda/4$

$$(S) = \begin{bmatrix} 0 & -j \\ -j & 0 \end{bmatrix} \qquad \begin{aligned} b_1 &= -ja_2 \\ b_2 &= -ja_1 \end{aligned}$$

Attenuator 3 dB, i.e. half output power

$$(S) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{aligned} b_1 &= \frac{1}{\sqrt{2}}a_2 = 0.707a_2 \\ b_2 &= \frac{1}{\sqrt{2}}a_1 = 0.707a_1 \end{aligned}$$

RF Transistor

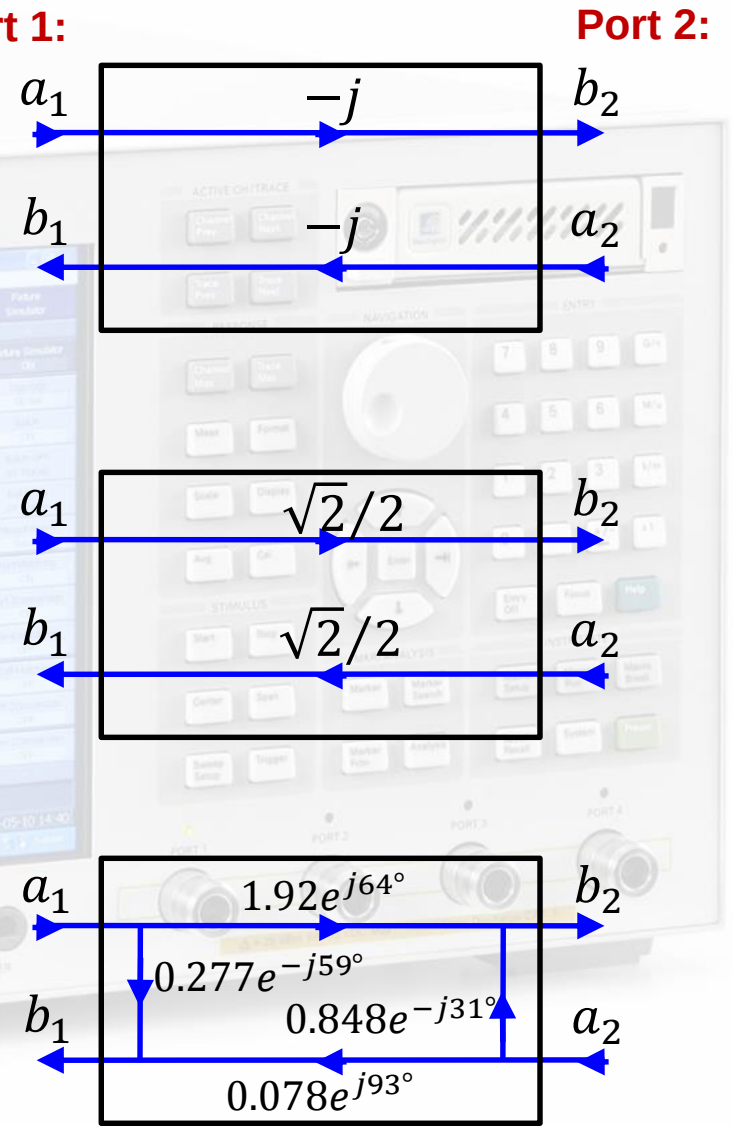
$$(S) = \begin{bmatrix} 0.277e^{-j59^\circ} & 0.078e^{j93^\circ} \\ 1.92e^{j64^\circ} & 0.848e^{-j31^\circ} \end{bmatrix}$$

backward transmission

forward transmission

non-reciprocal since $S_{12} \neq S_{21}$!

= different transmission forwards and backwards

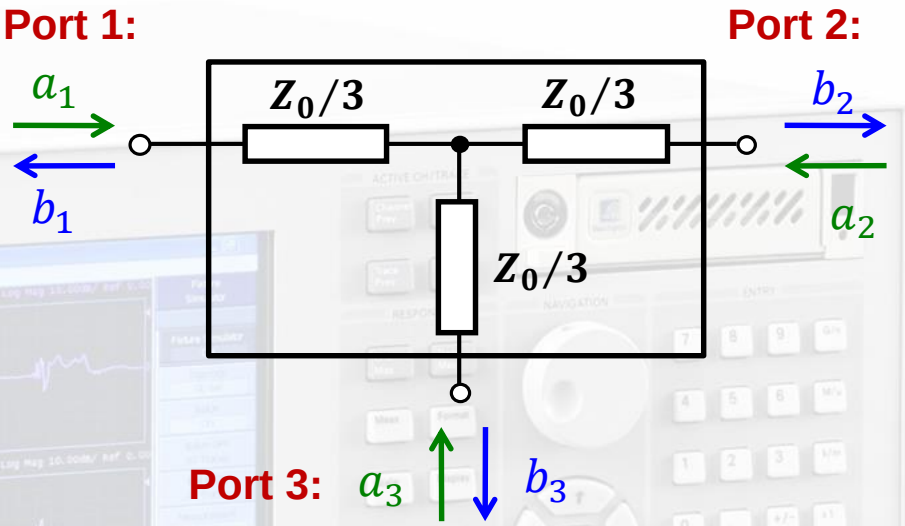


Examples of 3-ports

Resistive power divider

$$(S) = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

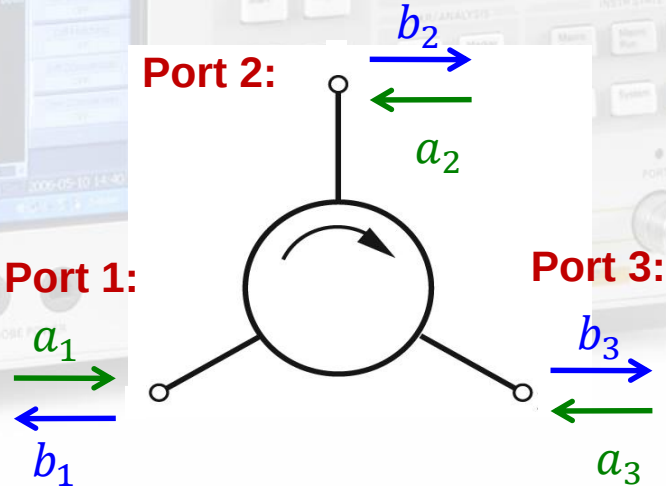
$$b_1 = \frac{1}{2}(a_2 + a_3)$$
$$b_2 = \frac{1}{2}(a_1 + a_3)$$
$$b_3 = \frac{1}{2}(a_1 + a_2)$$



3-port circulator

$$(S) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$b_1 = a_3$$
$$b_2 = a_1$$
$$b_3 = a_2$$



The ideal circulator is lossless, matched at all ports, but not reciprocal. A signal entering the ideal circulator at one port is transmitted exclusively to the next port in the sense of the arrow.

4-Port Examples

Ideal directional coupler

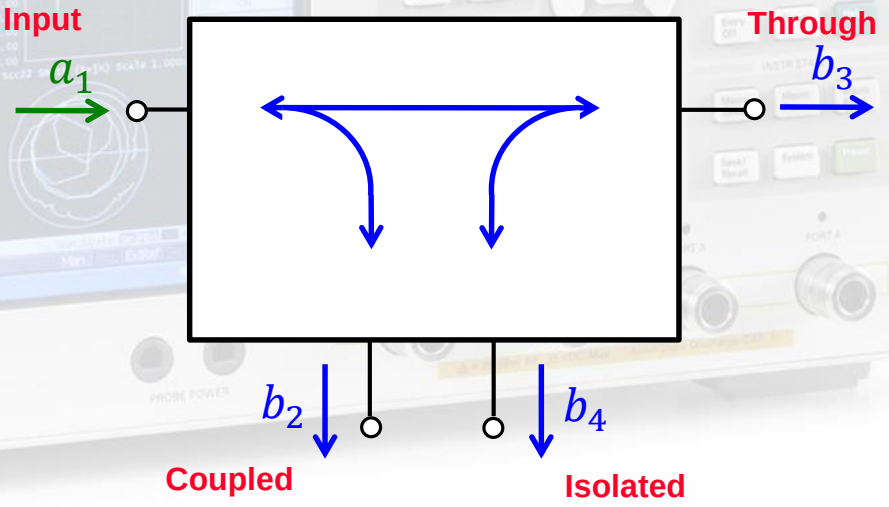
$$(S) = \begin{bmatrix} 0 & jk & \sqrt{1-k^2} & 0 \\ jk & 0 & 0 & \sqrt{1-k^2} \\ \sqrt{1-k^2} & 0 & 0 & jk \\ 0 & \sqrt{1-k^2} & jk & 0 \end{bmatrix} \quad \text{with} \quad k = \left|\frac{b_2}{a_1}\right|$$

To characterize directional couplers, three important figures are used:

the coupling $C = -20 \log_{10} \left|\frac{b_2}{a_1}\right|$

the directivity $D = -20 \log_{10} \left|\frac{b_4}{b_2}\right|$

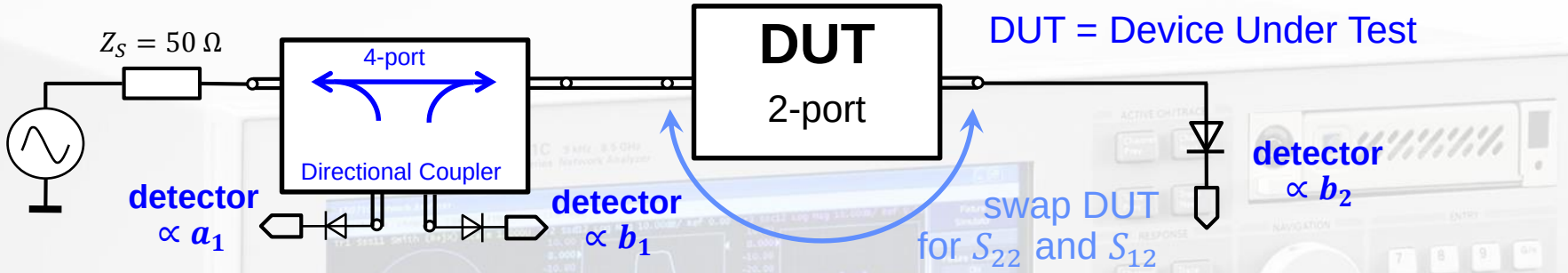
the isolation $I = -20 \log_{10} \left|\frac{a_1}{b_4}\right|$



S-Parameters – Summary

- **Scattering parameters (S-parameters) characterize an RF component or system (DUT) by a matrix.**
  - $n \times n$ matrix for *n*-port device
  - Based on incident (a_n) and reflected (b_n) power waves
- **ALL ports need to be terminated in their characteristic (reference) impedance Z_0**
  - For a proper S-parameter measurement or numerical computation all ports of the Device Under Test (DUT), including the generator port, must be terminated with their characteristic impedance to assure, waves traveling away from the DUT *(b_n-waves) are not reflected twice or multiple times, and convert into a_n-waves.*
    (cannot be stated often enough…!)
- **Typically the S-parameters, and therefore the DUT is ”measured” and characterized in the frequency domain**
  - S-parameters, as wells as DUT circuit elements are described in complex notation with the frequency variable $\omega = 2\pi f$
  - Frequency transformation (iDFT) allows time domain measurements with a ”modern” vector network analyzer (VNA).

How to measure S-Parameters?



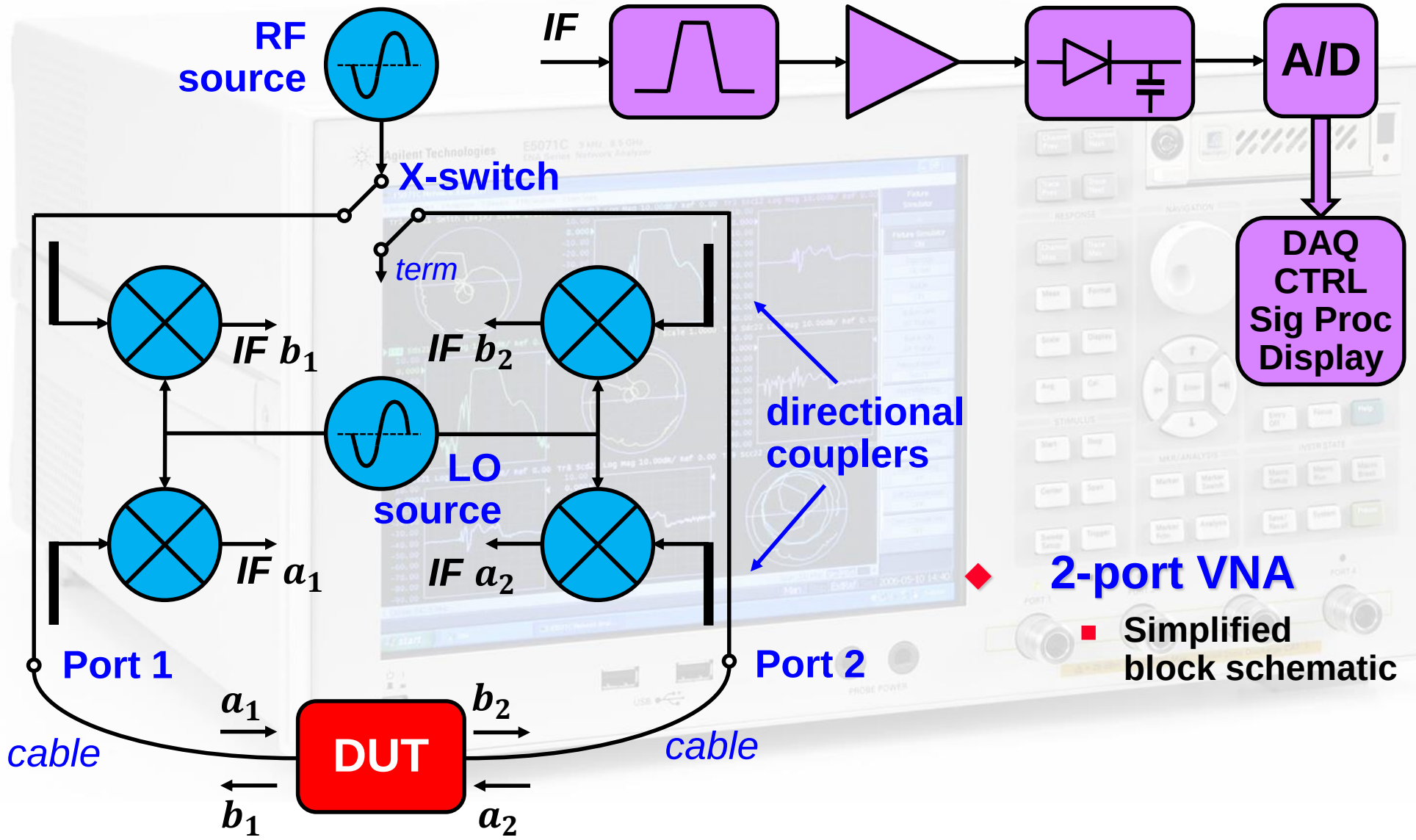
- **Performed in the frequency domain**
  - Single or swept frequency generator, stand-alone or as part of a VNA or SA
  - Requires a directional coupler and RF detector(s) or receiver(s)
- **Evaluate S_{11} and S_{21} of a 2-port DUT**
  - Ensure $a_2 = 0$, i.e. the detector at port 2 offers a well matched impedance
  - Measure incident wave a_1 and reflected wave b_1 at the directional coupler ports and compute for each frequency
  - Measure transmitted wave b_2 at DUT port 2 and compute

$$S_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0}$$

$$S_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0}$$

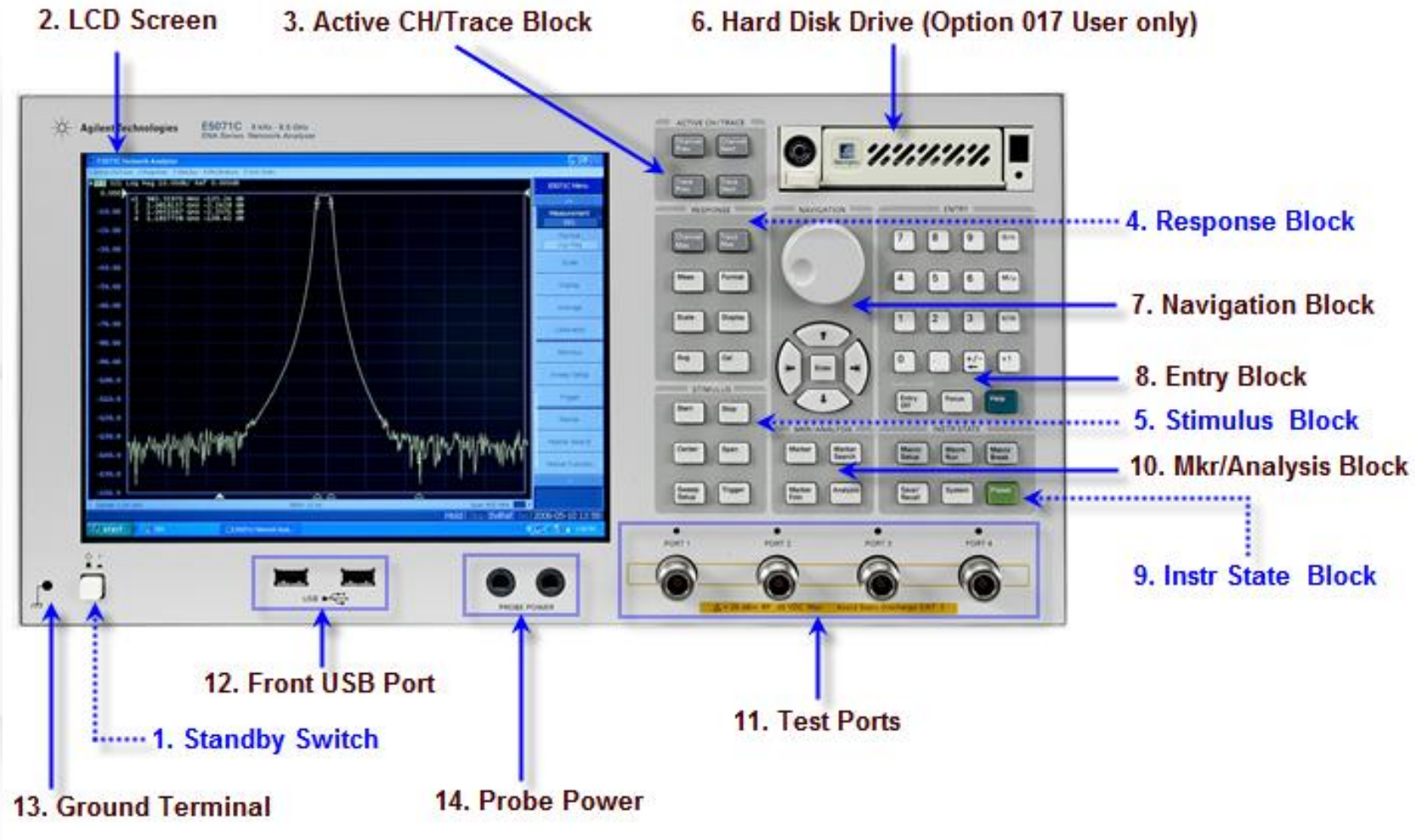
- **Evaluate S_{22} and S_{12} of the 2-port DUT**
  - Perform the same methodology as above by exchanging the measurement equipment on the DUT ports

The Vector Network Analyzer (VNA)



- **2-port VNA**
  - Simplified block schematic

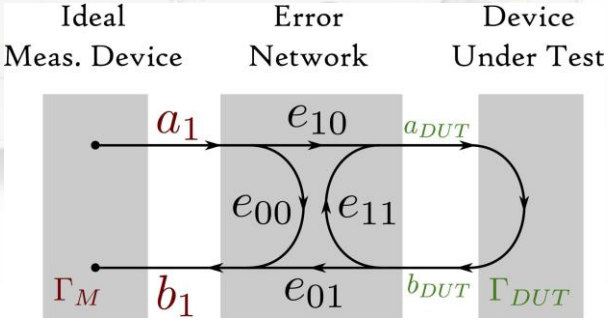
Fun with the VNA!



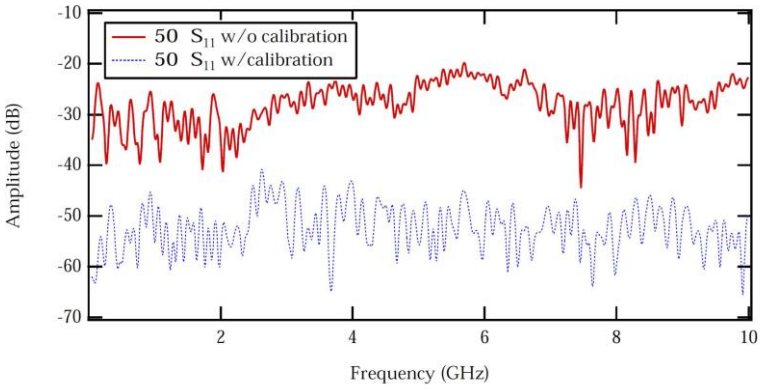
- **VNAs vary between manufacturers and models**
  - **Concepts and operation is still very similar**

VNA Calibration (1)

- **Calibration is not necessary for pure frequency or phase measurements**
- **Before calibrating the VNA measurement setup, perform a brief measurement and chose appropriate VNA settings:**
  - **Frequency range (center, span or start, stop)**
  - **Number of frequency points**
    - **Can be sometimes increased by rearranging the VNA memory (# of channels)**
  - **IF filter bandwidth**
  - **Output power level**
- **Calibrate the setup, preferable with an electronic calibration system if more than 2 ports are used!**
  - **Each port and combination needs to be calibrated, with the cables attached**
  - **Choose the appropriate connector type and sex**
  - **The instrument establishes a correction matrix and displays the ”CAL” status.**



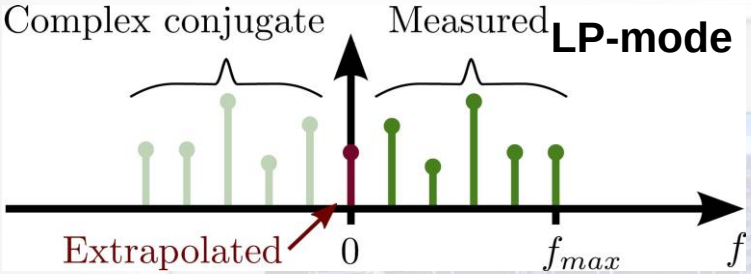
VNA Calibration (2)

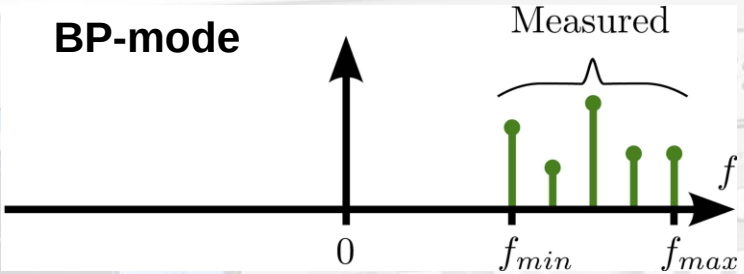


- **Calibration improves the measurement performance**
  - Return loss improvement by typically 20 dB. Enables mdB accuracy measurements
  - Full 2-port or 4-port calibration with manual calibration kits is prone to errors, better use electronic calibration systems.
  - Change VNA settings will cause the instrument to inter- and extrapolate, and the calibration status becomes uncertain.
- **Cables are included in the calibration**
  - However, changing coaxial connector types not.
  - Special VNA cables allows the adaption of different connector types and sex without requiring a re-calibration of the setup!

Synthetic Pulse TD Measurements (1)

- **Based on an inverse discrete *Fourier* transformation (iDFT) option in the VNA**

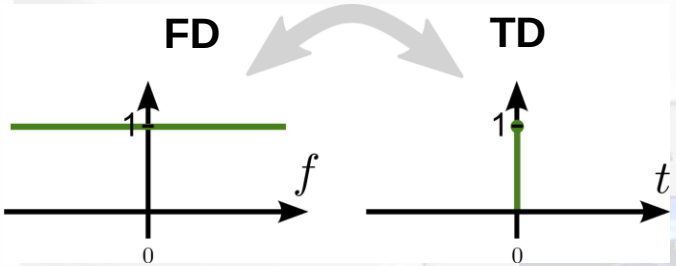




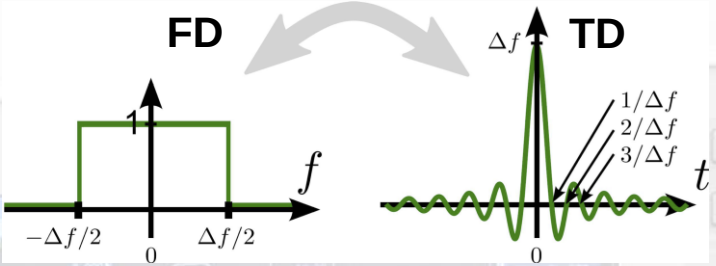
- **Low-pass mode: Impulse or step response, relying on equidistant samples over the extrapolated (to DC) frequency range.**
  - **The VNA does not measure at DC!**
  - **Manually match frequency range and # of points for DC extrapolation, e.g. 1…1000 MHz -> 1001 points, to enable extrapolation exactly to DC, or let the instrument chose the extrapolation settings automatically**
- **Enables time-domain reflectometry (TDR)**
  - **Very useful on portable VNAs, troubleshooting RF cable problems**
- **Band-pass response (no DC extrapolation)**
- **Allows time-domain gating and de-embedding of non-resonant sub-systems, e.g. measurements on a PCB**
- **Limited to linear systems**
- **Select the ”real” format for S_{11} or S_{21} for time-domain transformations!**
  - **or dB magnitude to detect small reflections in TDR analysis**

Synthetic Pulse TD Measurements (2)

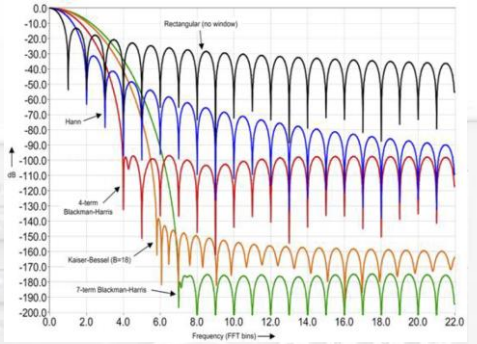
unlimited frequency range



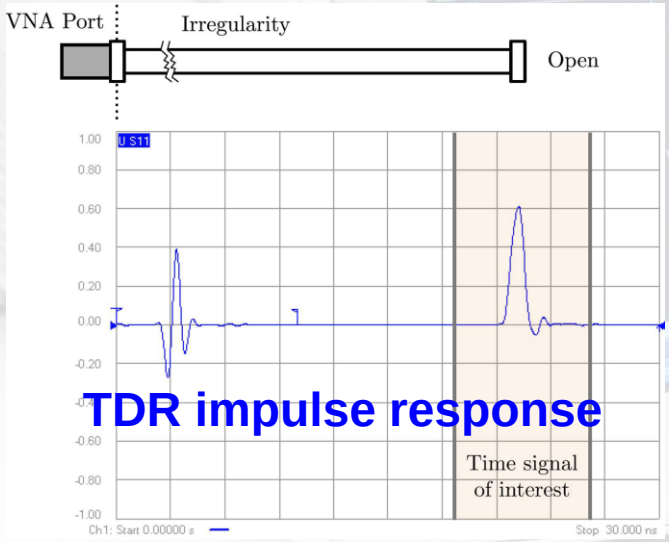
truncated frequency range



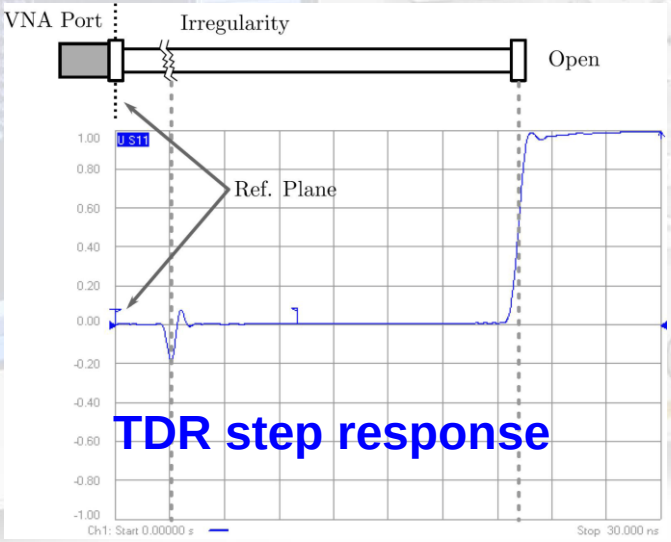
$$\mathrm{rect}\left(\frac{f}{\Delta f}\right) \qquad \frac{\sin(\Delta f \pi t)}{\pi t}$$



smoothing window functions

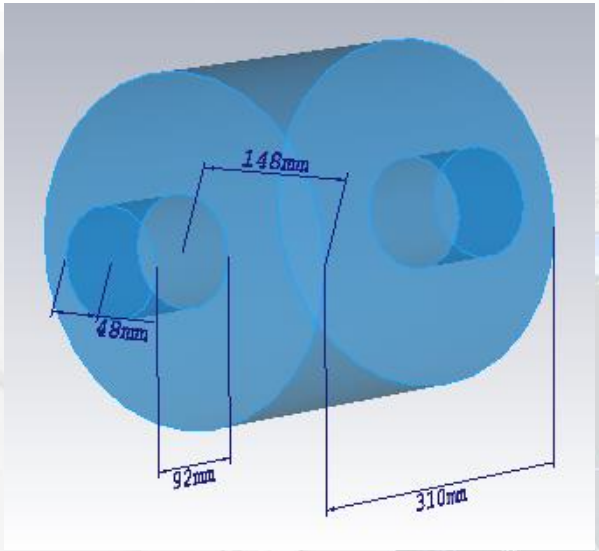


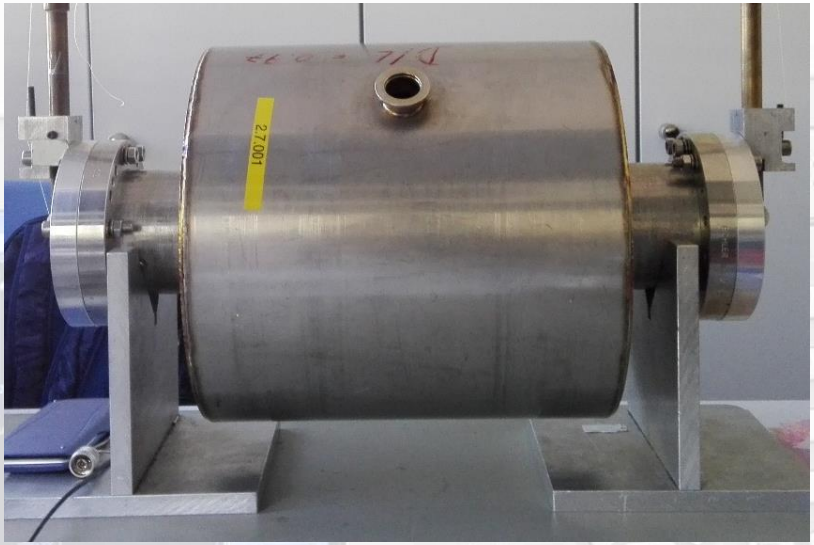
TDR impulse response



TDR step response

An Example – Pillbox* TM_{010} Eigenmode

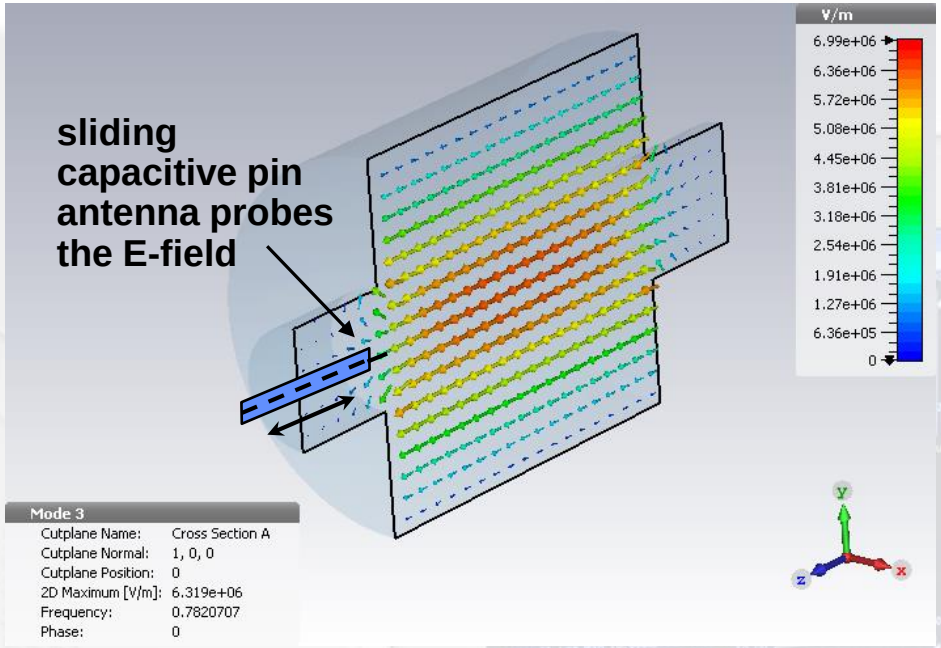


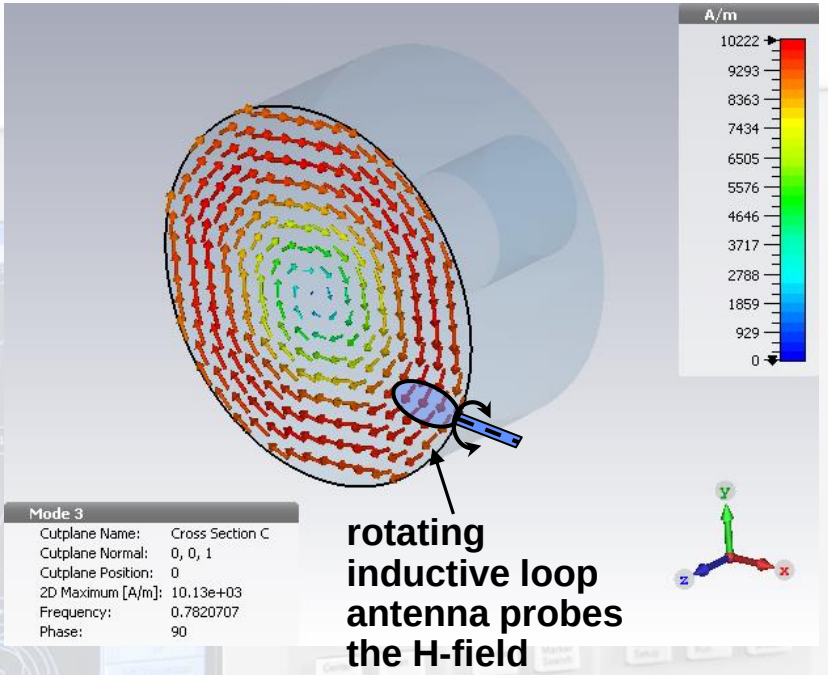


- **Characterize the accelerating TM_{010} mode of a cylindrical cavity with beam ports**
  - **The TM_{010} does not have to be the lowest frequency mode**
- **Compare the measured values of f_{res}, Q_0 and R/Q**
  - **with an analytical analysis of a perfect cylinder (no beam ports)**
  - **with a numerical analysis**

*** normal conducting!**

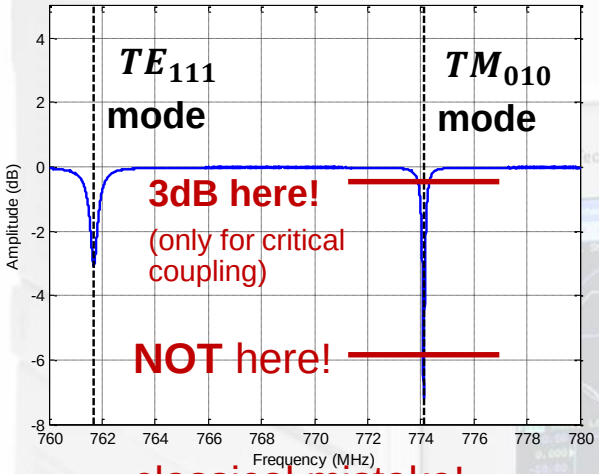
Excite the Modes while Measuring S_{11}



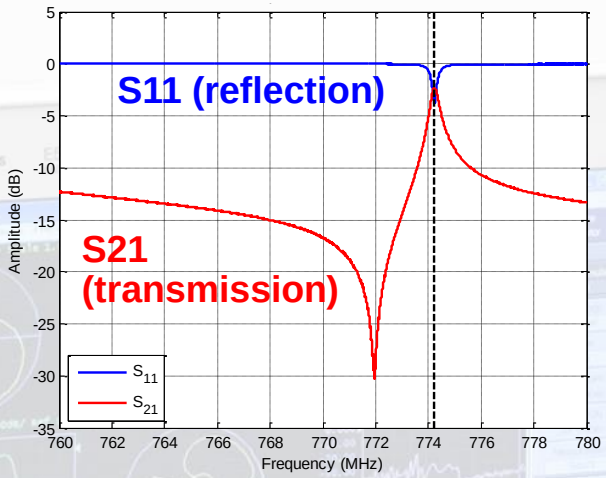


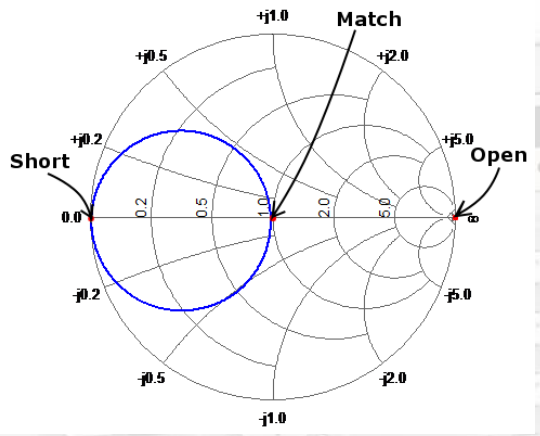
- S_{11} measurement with tunable coupling antenna
  - E-field on z-axis using a capacitive coupling pin
    - Center pin, e.g. of semi-rigid coaxial cable
  - H-field on the cavity rim using a inductive coupling loop
    - Bend the center conductor to a closed loop connected to ground

Measurement of Frequency and Q-value



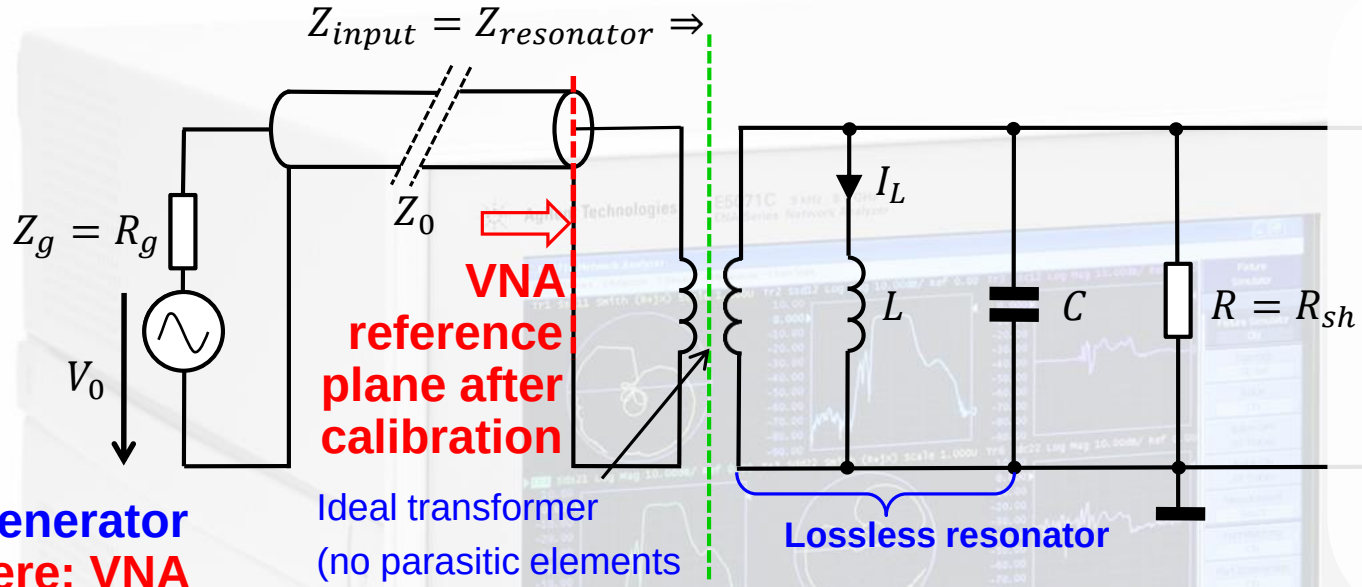
classical mistake!





- **Identify the correct (TM_{010}) mode frequency**
  - **Introduce a small perturbation, e.g. metallic rod or wire on the z-axis and observe the shift of the mode frequencies**
- **Calibrate the VNA and measure S_{11}**
  - **Tune the coupling loop for critical coupling**
  - **Display the resonant circle in the Smith chart using a sufficient number of points**

The Equivalent Circuit of a Resonant Mode



$\Leftarrow Z_{shunt}$

The generator may deliver power to R_{sh} and to the beam, but also the beam can deliver power to R_{sh} and R_g!

I_{beam}

V_{beam}

The beam is usually considered as a current source with infinite source impedance.

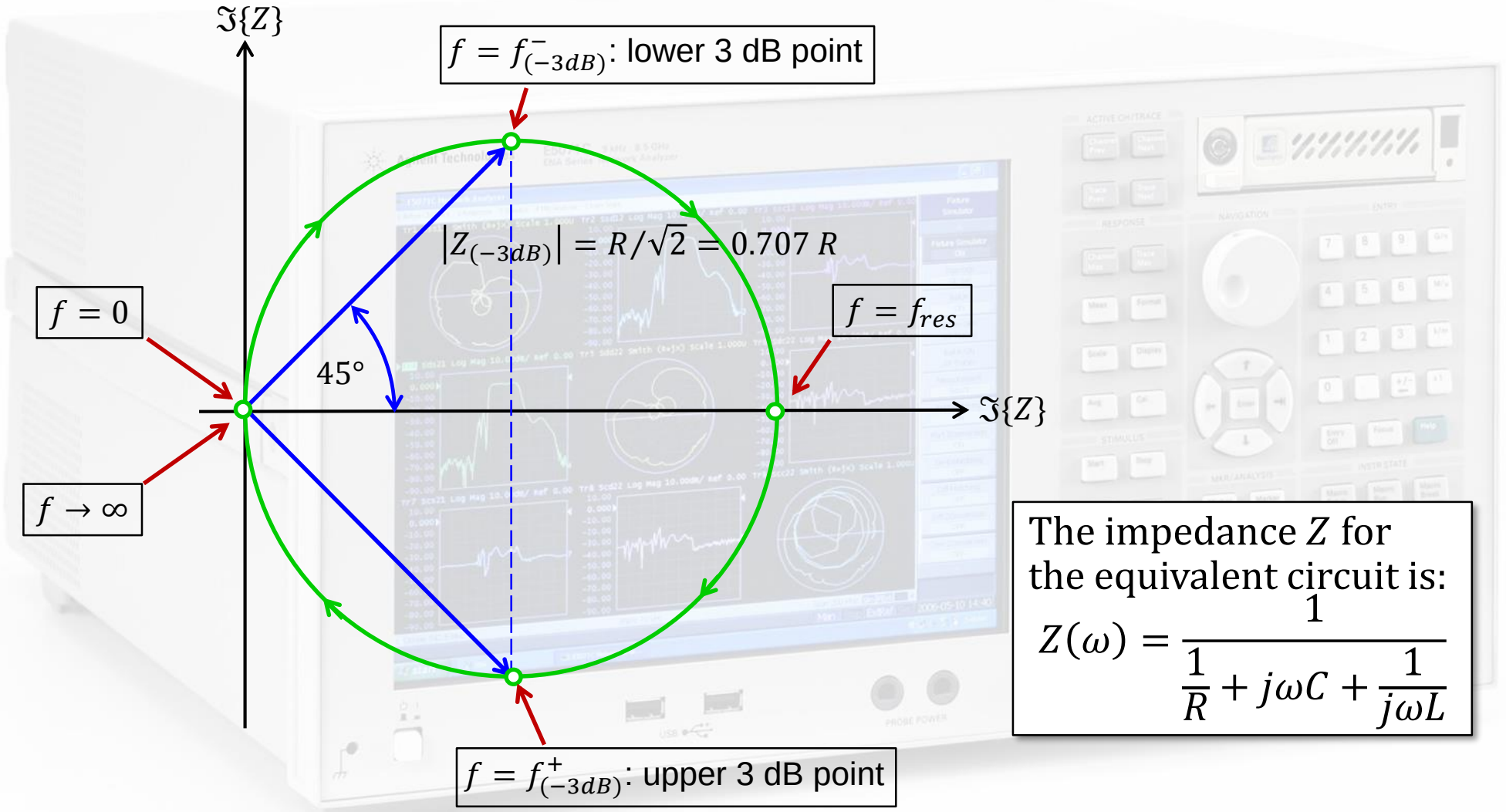
Beam here: not present

$R = R_{sh}$: shunt resistor, representing the losses of the resonator

We have resonance condition, when $\omega L = \frac{1}{\omega C}$

→ Resonance frequency: $\omega_{res} = 2\pi f_{res} = \frac{1}{LC} \Rightarrow f_{res} = \frac{1}{2\pi\sqrt{LC}}$

Input Impedance in the Complex Z-Plane



The impedance Z for the equivalent circuit is:

$$Z(\omega) = \frac{1}{\frac{1}{R} + j\omega C + \frac{1}{j\omega L}}$$

Useful Formulas of the Equivalent Circuit

- **Characteristic impedance ”R over Q”**

$$X = \frac{R}{Q} = \omega_{res} L = \frac{1}{\omega_{res} C} = \sqrt{L/C}$$

- **Stored energy at resonance**

$$U = \frac{C\,V_C}{2} = \frac{L\,I_L}{2}$$

V_C … Voltage at the capacitor

I_L … Current in the inductor

- **Dissipated power**

$$P = \frac{V^2}{2\,R}$$

- **Q-factor**

$$Q = \frac{R}{X} = \frac{\omega_{res}\,U}{P}$$

U … stored energy

P … dissipated power over 1 period

- **Shunt impedance (circuit definition)**

$$R = \frac{V^2}{2\,P}$$

$$\frac{R}{Q} = \frac{V^2}{2\omega_{res}\,U}$$

- **Tuning sensitivity**

$$\frac{\Delta f}{f} = \frac{1}{2}\frac{\Delta C}{C} = -\frac{1}{2}\frac{\Delta L}{L}$$

- **Coupling parameter (shunt impedance over generator or feeder impedance)**

$$k^2 = \frac{R}{R_{input}}$$

tune for critical coupling

The Quality Factor (Q-Value)

- **The quality (Q) factor of a resonant circuit is defined as ratio of the stored energy U over the energy dissipated P in one oscillation cycle:**

$$Q = 2\pi \frac{\text{energy stored}}{\text{energy dissipated in 1 cycle}} = \frac{\omega_{res} U}{P}$$

- **The Q-factor of an impedance loaded resonator:**
  - Q_0: unloaded Q-value of the unperturbed system
  - Q_L: loaded Q-value, e.g. measured with the impedance of the connected generator
  - Q_{ext}: external Q-factor, representing the effects of the external circuit (generator and coupling circuit)

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_{ext}}$$

tune k for critical coupling:

$$Q_0 = Q_{ext}$$

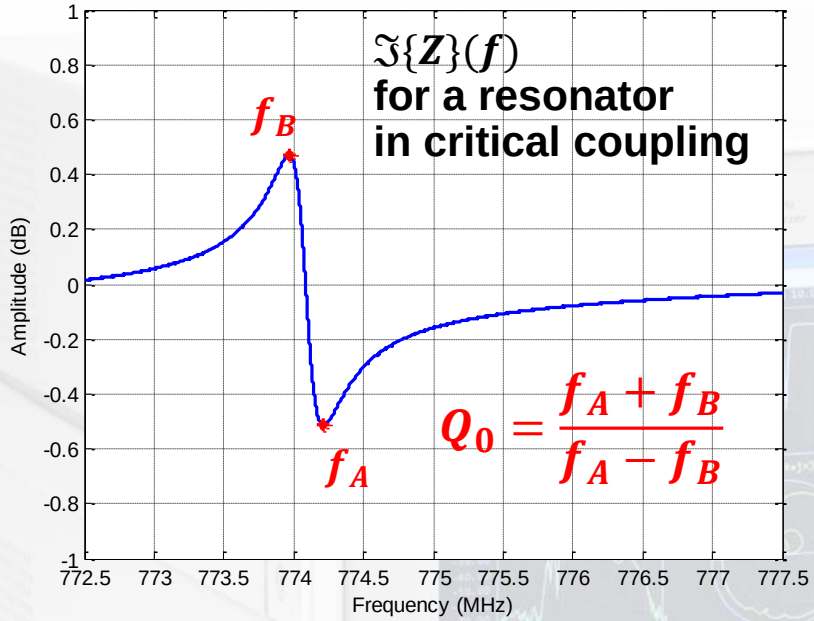
$$\Rightarrow Q_0 = 2\,Q_L$$

With Q_L being our measured Q-value

- **Q-factor and bandwidth**
  - This is how we actually ”measure” the Q-factor!

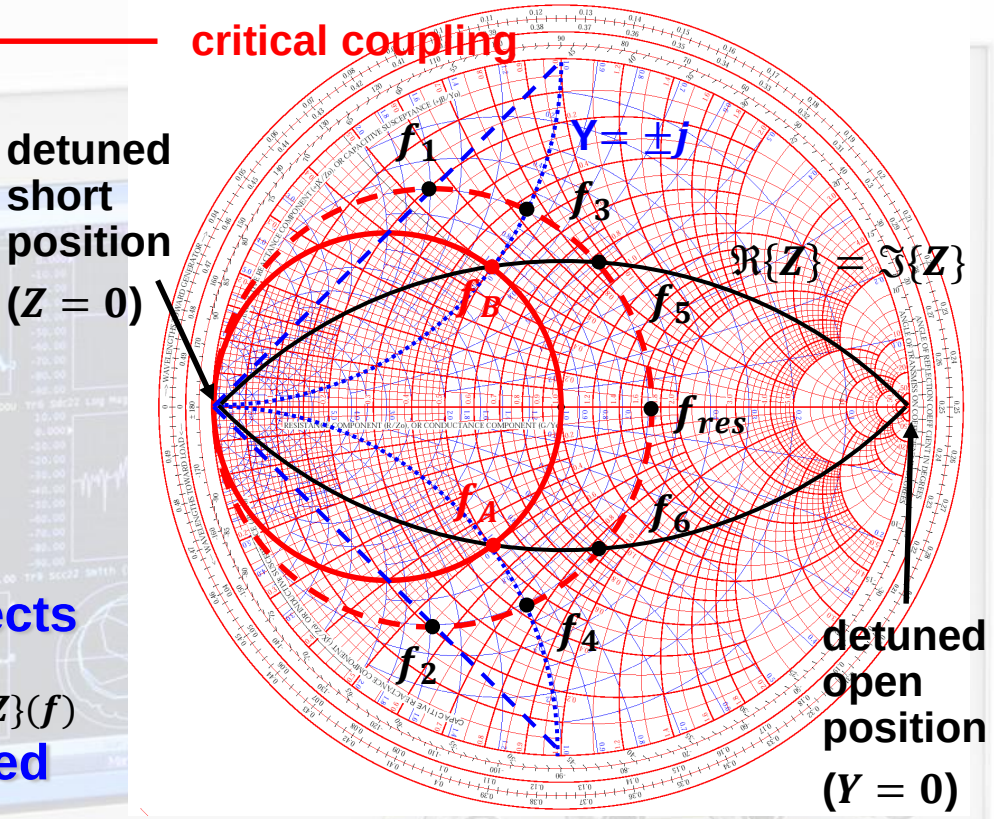
$$Q = \frac{f_{res}}{f_{BW}} \quad \text{with:} \quad f_{BW} = f_{+3dB} - f_{-3dB}$$

Q-factor from S_{11} Measurement



arbitrary coupling (here: over-critical)

critical coupling



- **Correct for the uncompensated effects of the coupling loop**
  - Electrical length adjustment: ”straight” $\Im\{Z\}(f)$
- **Adjust the locus circle to the detuned short location**
  - Phase offset
- **Verify no evanescent fields penetrating outside the beam ports**
  - i.e. no frequency shifts if the boundaries at the beam ports are altered

Frequency marker points in the Smith chart:

- $f_{1,2}$: $\Im\{S_{11}\} = max.$ to calculate Q_L
- $f_{3,4}$: $Y = \Re \pm j$ to calculate Q_{ext}
- $f_{5,6}$: $\Re\{Z\} = \Im\{Z\}$ to calculate Q_0

R/Q Measurement – Slater's Theorem

- **Remember from the equivalent circuit:**

$$\frac{R}{Q} = \frac{V_{acc}^2 \big/ 2P_d}{\omega_{res} U \big/ P_d} = \frac{V_{acc}^2}{2\omega_{res} U}$$

with:

$$V_{acc} = \left| \int E_z(z) \cos\left(\frac{\omega z}{\beta c_0}\right) dz \right|$$

transit time related

  - V_{acc} is based on the integrated longitudinal E-field component E_z along the z-axis ($x = y = 0$)

- **Based on Slater's perturbation theorem:**

$$\frac{\Delta f}{f_{res}} = \frac{1}{U}\left[\mu\left(k_{\parallel}^{H}\left|H_{\parallel}\right|^2 + k_{\perp}^{H}|H_{\perp}|^2\right) - \varepsilon\left(k_{\parallel}^{E}\left|E_{\parallel}\right|^2 + k_{\perp}^{E}|E_{\perp}|^2\right)\right]$$

  - Resonance frequency shift due to a small perturbation object, expressed in longitudinal and transverse E and H field components
  - k: coefficients proportional to the electric or magnetic polarizability of the perturbation object (here: only $k_{\parallel}^{E}$ for a longitudinal metallic object)

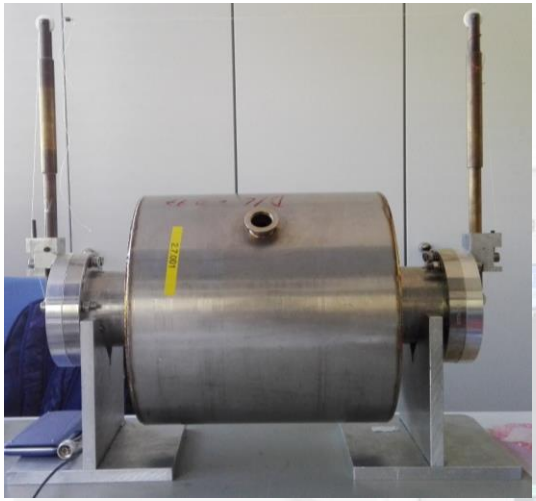
- **E-field characterization along the z-axis**

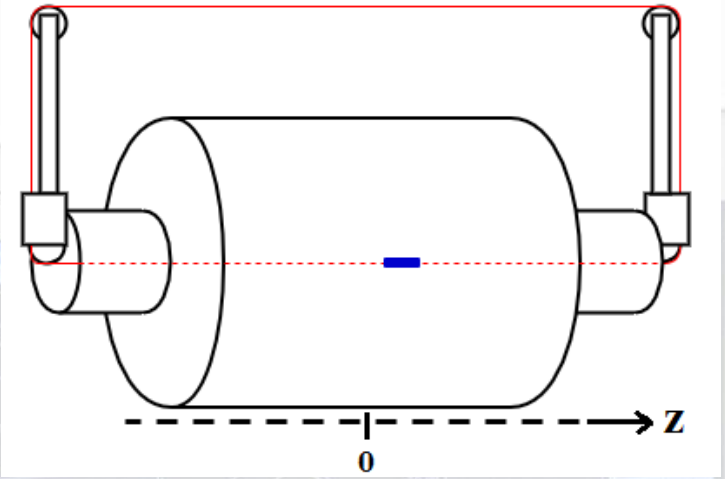
$$E(z) = \sqrt{\frac{\Delta f(z)}{f_{res}} \cdot \frac{-1}{k_{\parallel}^{E} \varepsilon_0}}$$

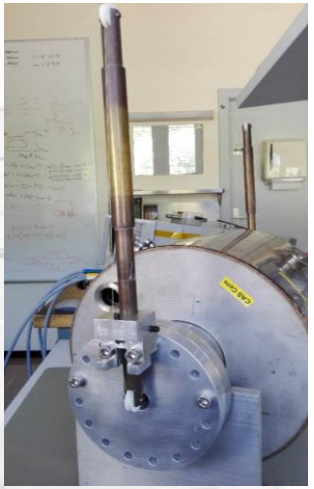
with: $k_{\parallel}^{E} = \frac{\pi}{3} l^3 \left[\sinh^{-1}\left(\frac{2}{3\pi}\frac{l}{a}\right)\right]^{-1}$

(metallic ellipsoid, e.g. syringe needle of half length l and radius a)

R/Q Measurement - Bead Pull Method





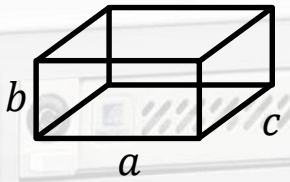


$$\frac{\Delta f}{f_{res}} = \frac{1}{2\,Q_0}\tan\phi$$

- **E-field characterization by evaluating**
  - **The frequency shift (S_{11} reflection measurement with a single probe) or**
  - **The phase shift ϕ at f_{res} (S_{21} transmission measurement with 2 probes)**
- **Exercise with a manual bead-pull through a known cavity**
  - **requires: fishing wire, syringe needle, ruler and VNA**
  - **Compare the measured E_z at the maximum f or ϕ shift (in the center of the cavity) with the theoretical estimation (e.g. numerical computed value)**

Where to go from here…?!

- **Setup your own mode characterization experiment**
  - **Start with a simple analytically solvable structure, e.g. brick-style or cylindrical (”pillbox”) resonator**
  - **Unfortunately you need to go through the math of the modal expansion of the vector potential Ψ…**



***Laplace* equation:**

$$\Delta\Psi + k_0^2 \varepsilon_r \mu_r \Psi = 0$$

$$k_0^2 = \omega^2 \varepsilon_0 \mu_0 \qquad k_0 = {}^{2\pi}/_{\lambda_0}$$

k_0: free space wave number

λ_0: free space wave length

Product ansatz (*Cartesian* coordinates):

$$\Psi = X(x)Y(y)Z(z)$$

separation condition:

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 \varepsilon_r \mu_r$$

standing waves

travelling waves

General solution (field components):

$$\Psi = \left\{ \begin{matrix} A\cos(k_x x) + B\sin(k_x x) \\ \acute{A}e^{-jk_x x} + \acute{B}e^{-jk_x x} \end{matrix} \right\} \left\{ \begin{matrix} C\cos(k_y y) + D\sin(k_y y) \\ \acute{C}e^{-jk_y y} + \acute{D}e^{-jk_y y} \end{matrix} \right\} \left\{ \begin{matrix} E\cos(k_z z) + F\sin(k_z z) \\ \acute{E}e^{-jk_z z} + \acute{F}e^{-jk_z z} \end{matrix} \right\}$$

Eigen frequencies:

$$f_{mnp} = \frac{c_0}{2\pi\varepsilon_r\mu_r} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2}$$

$$k_x = \frac{m\pi}{a} \qquad k_y = \frac{n\pi}{b} \qquad k_z = \frac{p\pi}{c}$$

Pillbox Eigenmodes

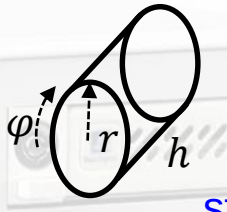
- **Same procedure, but now with cylindrical functions** 😢😢

Product ansatz (cylindrical coordinates):

$$\Psi = R(\rho)F(\varphi)Z(z)$$

separation condition:

$$k_r^2 + k_z^2 = k_0^2 \varepsilon_r \mu_r$$



General solution (field components):

$$\Psi = \begin{Bmatrix} A\,J_m(k_r\rho) + B\,N_m(k_r\rho) \\ \acute{A}H_m^{(2)}(k_r\rho) + \acute{B}H_m^{(2)}(k_r\rho) \end{Bmatrix} \begin{Bmatrix} C\cos(m\varphi) + D\sin(m\varphi) \\ \acute{C}e^{-jm\varphi} + \acute{D}e^{-jm\varphi} \end{Bmatrix} \begin{Bmatrix} E\cos(k_z z) + F\sin(k_z z) \\ \acute{E}e^{-jk_z z} + \acute{F}e^{-jk_z z} \end{Bmatrix}$$

standing waves

travelling waves

$J_m, N_m, H_m^{(1,2)}$: cylindical functions $(Bessel, Hankel, Neumann)$

see *Abramowitz* and *Stegun*

Eigen frequencies:

$$f_{TMmnp} = \frac{c_0}{2\pi\varepsilon_r\mu_r}\sqrt{\left(\frac{j_{mn}}{r}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

$$f_{TEmnp} = \frac{c_0}{2\pi\varepsilon_r\mu_r}\sqrt{\left(\frac{j'_{mn}}{r}\right)^2 + \left(\frac{p\pi}{h}\right)^2}$$

j_{mn} being the n^{th} root of $J_m(x)$ $\quad$ j'_{mn} being the n^{th} root of $J'_m(x)$

Compare your Results

Analytical – Numerical – Measurement

- There is also an analytical expression for the R/Q of the TM_{010} mode of a cylindrical resonator:

$$\frac{R}{Q} = \frac{4\,\eta}{j_{01}^3\,\pi\,J_1^2(j_{01})}\,\frac{\sin^2\left(\frac{j_{01}}{2}\frac{h}{r}\right)}{h/r} \approx 128\,\frac{\sin^2\left(1.2024\ h/r\right)}{h/r} \qquad \eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega$$

- …and for the Q_0 as well: $Q_0 = \frac{r}{\delta}\left[1 + \frac{r}{h}\right]^{-1}$ skin depth: $\delta = \sqrt{\frac{2}{\omega_{res}\sigma\mu}}$
- There are student versions for 3D numerical EM software available
  - Visualize E and H fields to ensure you got the expected results!
- Remember: f_{res} and R/Q are only determined by the geometry of the cavity, the Q-factor however depends on the material properties (conductivity!)
  - The results of the Q-factor may vary between measurement and computation
- Don't forget to compute the values of the equivalent circuit!
- Make use of mode charts to understand the order of the eigen-frequencies
- If the phase-shift method is used to measure the R/Q with the bead-pull, set the VNA to "Zero Span" at f_{TM010} and use a very weak coupling to stay precisely on the eigen-frequency
  - Compare the measured phase before and after the bead-pull! If the value has changed the phase drifted because of temperature effects! Perform the bead-pull at a faster pace!

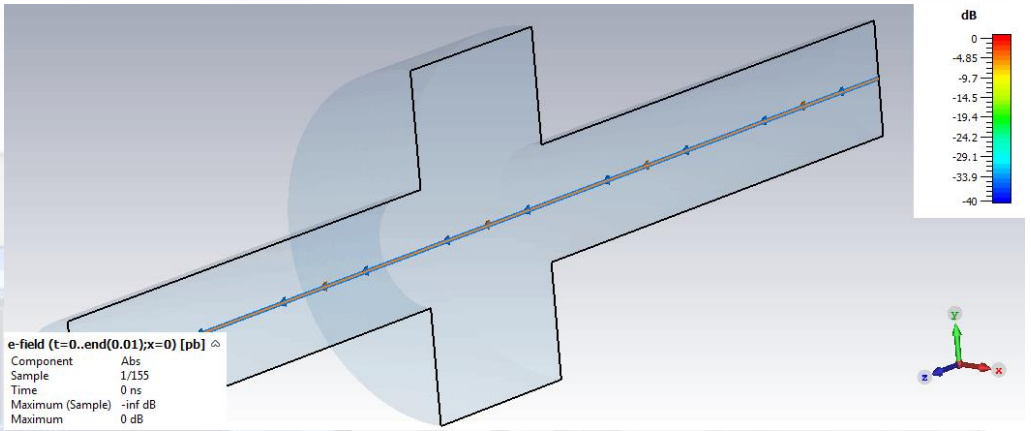
Beam Coupling Impedance

The wake potential

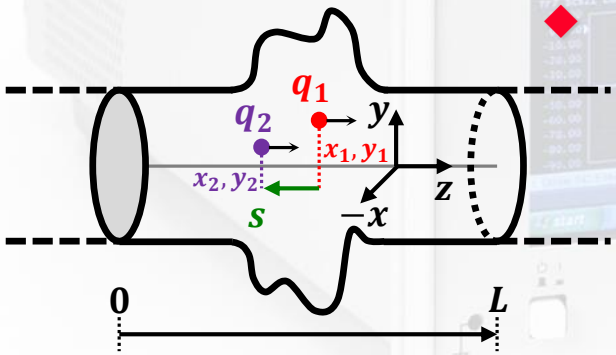
- Lorenz force on q_2 by the wake field of q_1:

$$\vec{F} = \frac{d\vec{p}}{dt} = q_2\left(\vec{E} + c_0\vec{e}_z \times \vec{B}\right)$$

- Wake potential of a structure, e.g. a discontinuity driven by q_1



$$\vec{w}(x_1, y_1, x_2, y_2, s) = \frac{1}{q_1}\int_{-\infty\ (\text{or } 0)}^{+\infty\ (\text{or} L)} dz\left[\vec{E}(x_2, y_2, z, t) + c_0\vec{e}_z \times \vec{B}(x_2, y_2, z, t)\right]_{t=(s+z)/c}$$



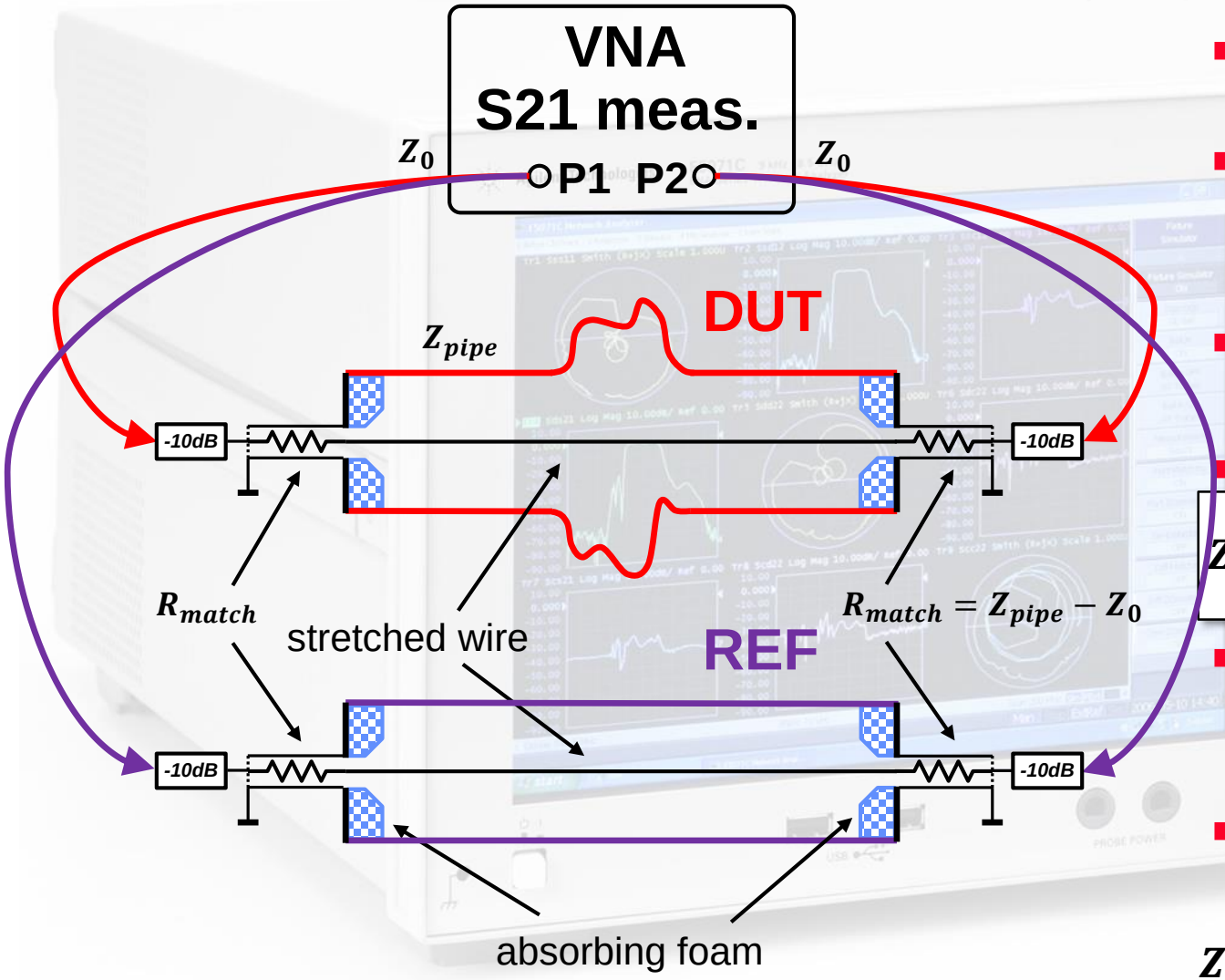
Beam coupling impedance

- Frequency domain representation of the wake potential

$$Z(x_1, y_1, x_2, y_2, \omega) = -\frac{1}{c_0}\int_{-\infty}^{+\infty} ds\, \vec{w}(x_1, y_1, x_2, y_2, s)e^{-j\omega s/c_0}$$

- Can be decomposed in longitudinal $Z_{\parallel}$ and transverse $Z_{\perp}$ components (*Panofsky-Wenzel* theorem)
- Resonant structures, i^{th} mode: $R_{sh,i} = Z_{\parallel,i} = \frac{2k_{loss,i}Q_i}{\omega_i}$

Stretched-Wire $Z_{\parallel}$ Measurement



Formulas:

- **Normalized electrical length:** $\Theta = 2\pi \frac{L}{\lambda}$
- **Lumped impedance formula**

$$Z_{\parallel} = 2Z_{pipe} \frac{1 - S_{21}}{S_{21}} \quad \Theta \leq 1 \quad L < D_{pipe}$$

- **Log formula**

$$Z_{\parallel} = -2Z_{pipe} \ln S_{21}$$

- **Improved log formula**

$$Z_{\parallel} = -2Z_{pipe} \ln S_{21} \left(1 + j \frac{\ln S_{21}}{2\Theta}\right)$$

- **Transmission coefficient**

$$S_{21} = \frac{S_{21,DUT}}{S_{21,REF}}$$

- **Circular beam pipe impedance**

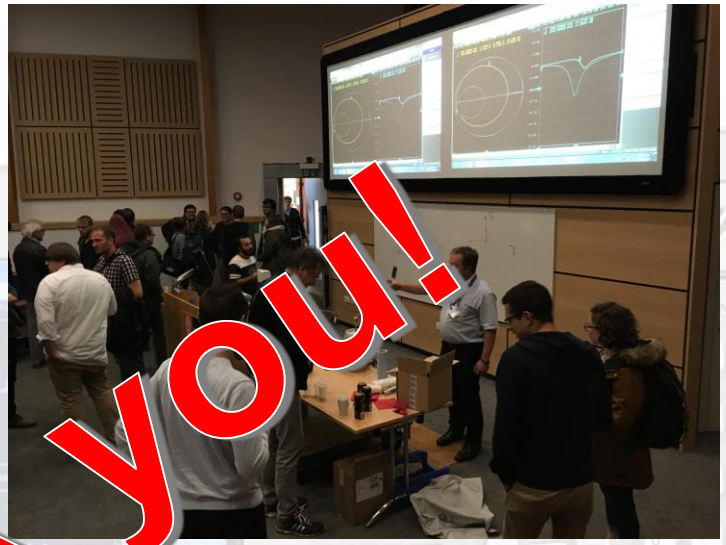
$$Z_{pipe} = \frac{\eta_0}{2\pi\sqrt{\varepsilon_r}} \ln \frac{D}{d} \cong 60\ \Omega \ln \frac{D_{pipe}}{d_{wire}}$$

I could not cover everything…

- **Other VNA "goodies"…**
  - 4-port VNA with "virtual" ports, e.g. differential S-parameters
  - Power sweep for 1 dB compression point RF amplifier analysis
- **Spectrum analysis**
  - Modulation, demodulation
  - Intermodulation measurements (IP3) and other non-linear effects
  - Noise and noise figure analysis
- **More on wakefields and beam impedance measurements**
  - Introduced the stretched wire method, but what about $Z_{\perp}$?
- **Material and surface characterization**
- **Measurements on waveguides and periodic structures**
- **…**

- **A few more examples are found in the Appendix**
  - Also, please study the supplied reports
  - Try to setup and execute some of the CAS RF experiments
  - Watch the movies mentioned on slide 26!

More Options to learn…



Thank you!

This CAS RF practical training

- If you did not sign up for RF, pass by our training room and have a look…

CAS Special Topic: RF

- March/April 2020, Kaunas, Lithuania

JUAS

- Open air tutorial session in Archamps **(nice weather in February!)**

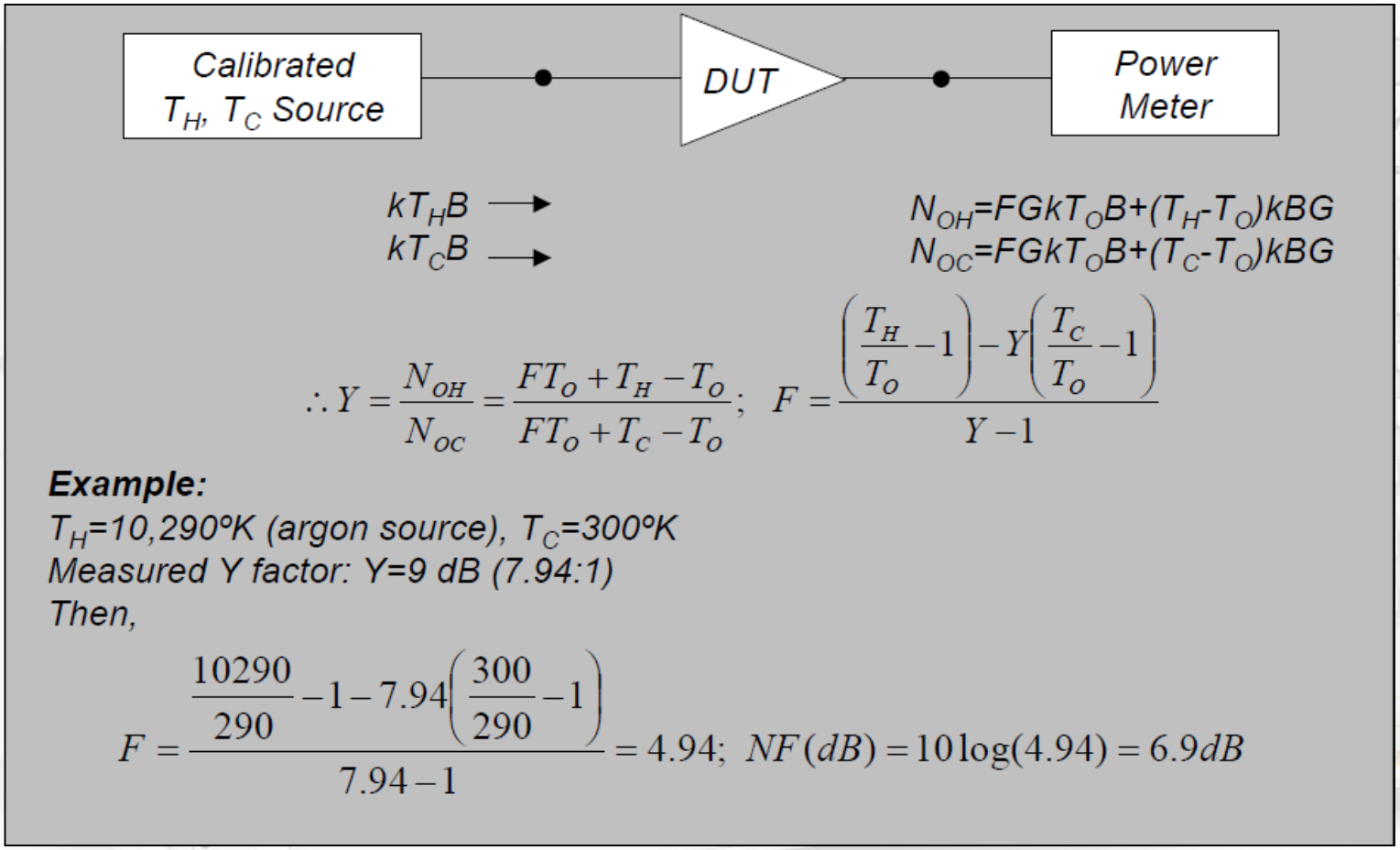
Appendix A: Definition of the Noise Figure

$$F = \frac{S_i / N_i}{S_o / N_o} = \frac{N_o}{GN_i} = \frac{N_o}{GkT_0B} = \frac{GN_i + N_R}{GkT_0B} = \frac{GkT_0B + N_R}{GkT_0B}$$

- F is the *Noise factor* of the receiver
- S_i is the available signal power at input
- $N_i = kT_0B$ is the available noise power at input
- T_0 is the absolute temperature of the source resistance
- N_o is the available noise power at the output , including amplified input noise
- N_r is the noise added by receiver
- G is the available receiver gain
- B is the effective noise bandwidth of the receiver
- If the noise factor is specified in a logarithmic unit, we use the term Noise Figure (*NF*)

$$NF = 10 \lg \frac{S_i / N_i}{S_o / N_o} dB$$

Measurement of Noise Figure (using a calibrated Noise Source)

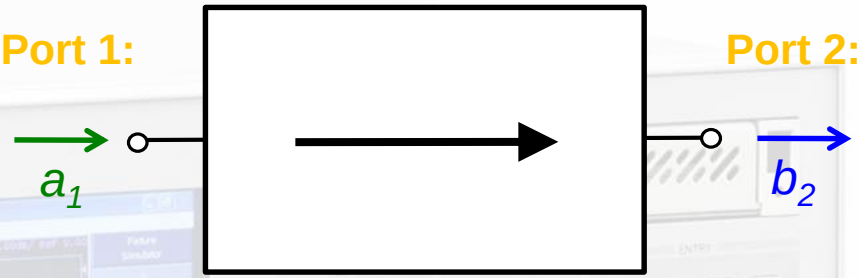


Examples of 2-ports (2)

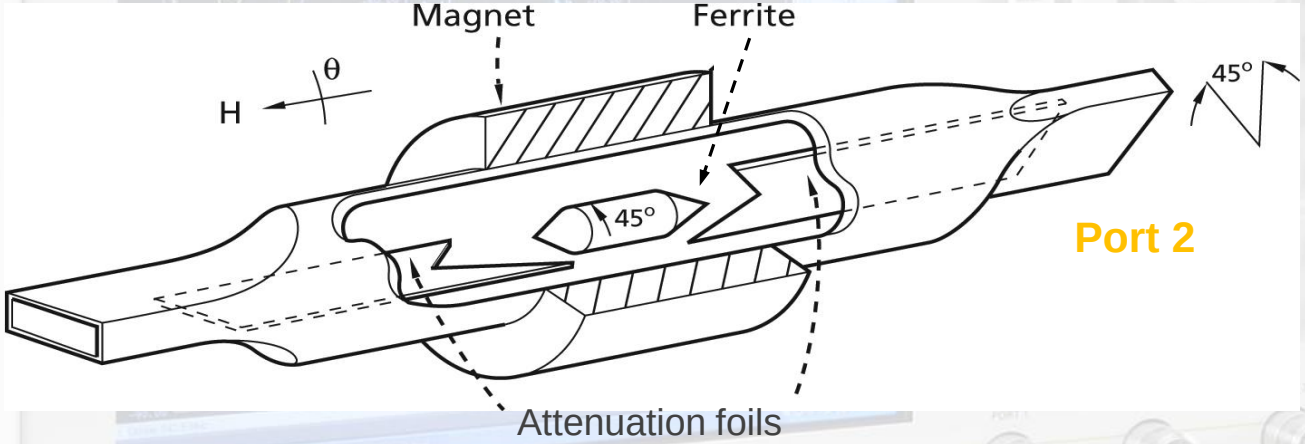
Ideal Isolator

$$(S) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad b_2 = a_1$$

only forward transmission



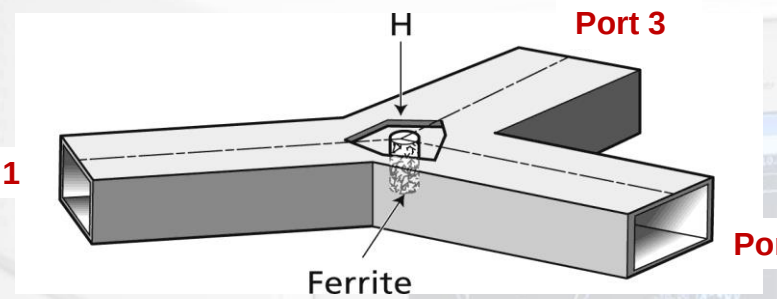
Faraday rotation isolator



The left waveguide uses a TE_{10} mode (=vertically polarized H field). After transition to a circular waveguide, the polarization of the mode is rotated counter clockwise by 45° by a ferrite. Then follows a transition to another rectangular waveguide which is rotated by 45° such that the forward wave can pass unhindered. However, a wave coming from the other side will have its polarization rotated by 45° clockwise as seen from the right hand side.

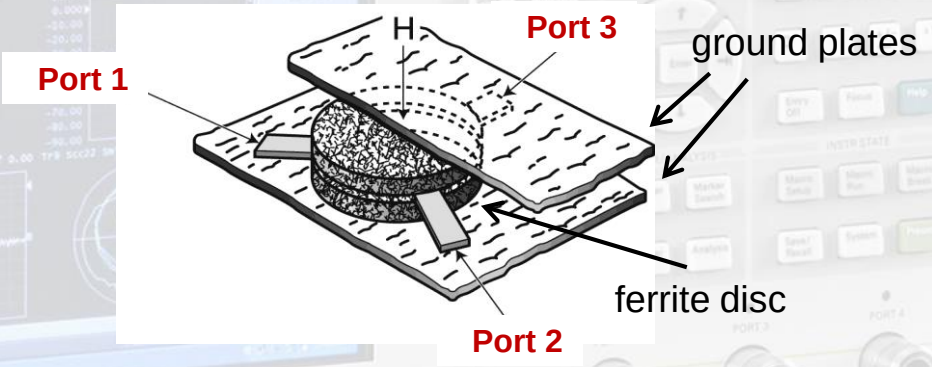
3-Port Examples (2)

Practical implementations of circulators:



Waveguide circulator

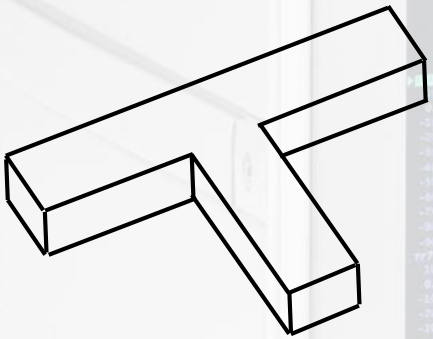
Stripline circulator

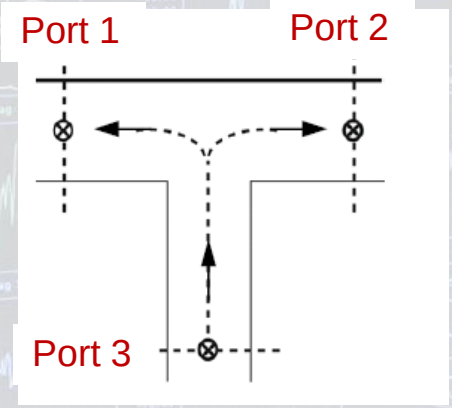


A circulator contains a volume of ferrite. The magnetically polarized ferrite provides the required non-reciprocal properties, thus power is only transmitted from port 1 to port 2, from port 2 to port 3, and from port 3 to port 1.

3-Port Examples (3)

- The T splitter is reciprocal and lossless but not matched at all ports. Using the losslessness condition and symmetry considerations one finds for E and H plane splitters

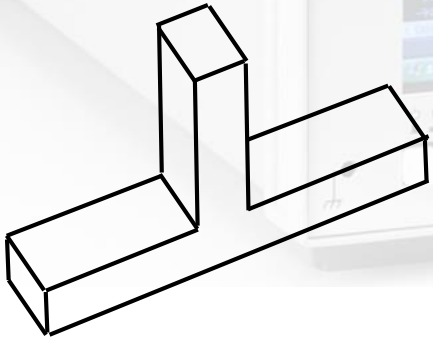


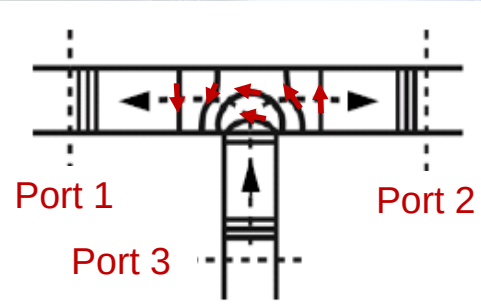


H-plane splitter

$$S_H = \frac{1}{2}\begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

Note: change in sign of wave going left or right





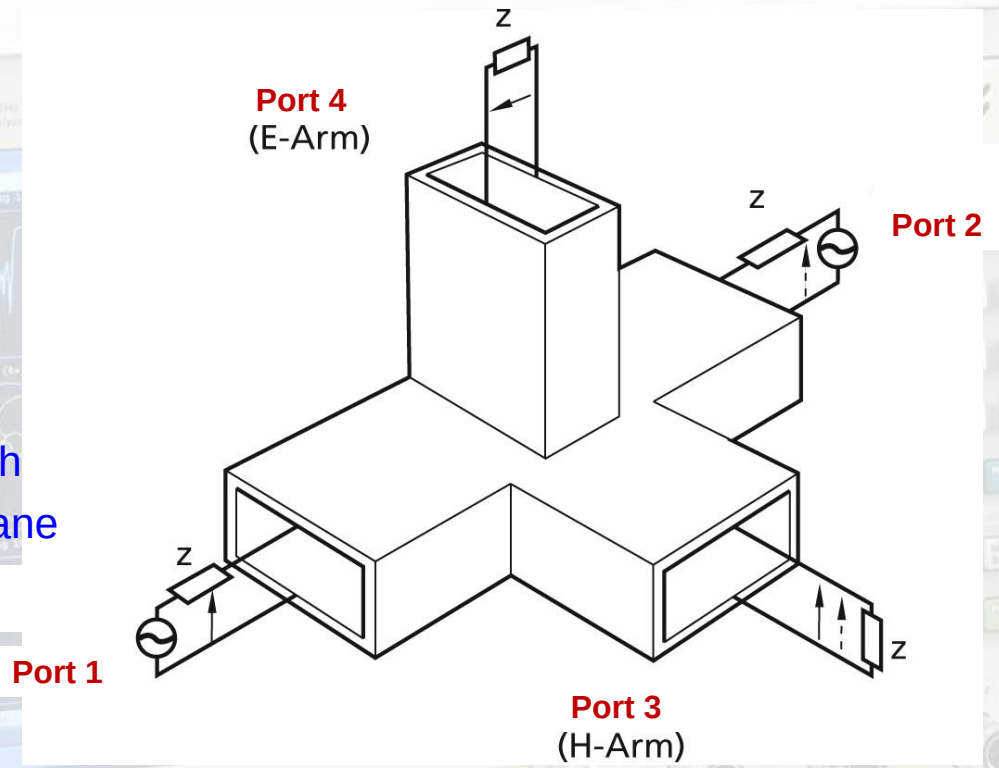
E-plane splitter

$$S_E = \frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix}$$

4-Port Examples (2)

Magic-T also referred to as 180º hybrid:

$$(S) = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$



The H-plane is defined as a plane in which the magnetic field lines are situated. E-plane correspondingly for the electric field.

Can be implemented as waveguide or coaxial version. Historically, the name originates from the waveguide version where you can “see” the horizontal and vertical “T”.

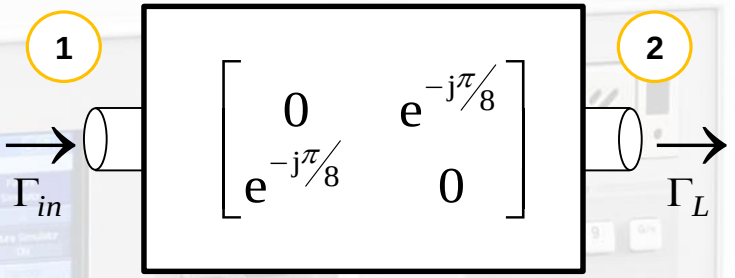
Pathing through a 2-port (1)

In general:

$$\Gamma_{in} = S_{11} + \frac{S_{12}S_{21}\Gamma_L}{1 - S_{22}\Gamma_L}$$

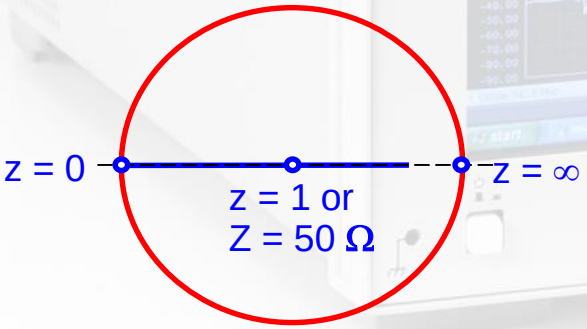
were Γ_{in} is the reflection coefficient when looking through the 2-port and Γ_{load} is the load reflection coefficient.

Line λ/16:

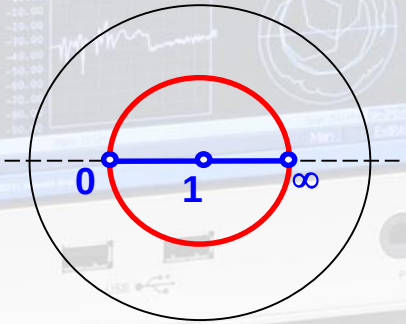


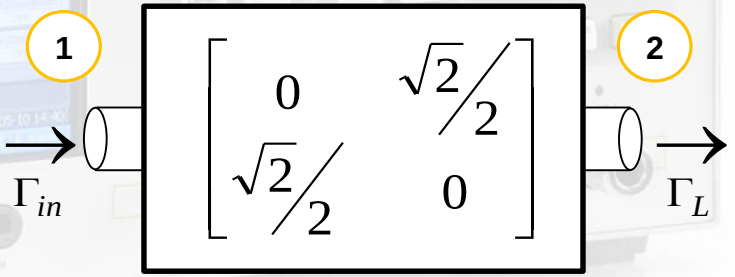
$$\Rightarrow \Gamma_{in} = \Gamma_L \, e^{-j\pi/4}$$

The outer circle and the real axis in the simplified Smith diagram below are mapped to other circles and lines, as can be seen on the right.



Attenuator 3dB:

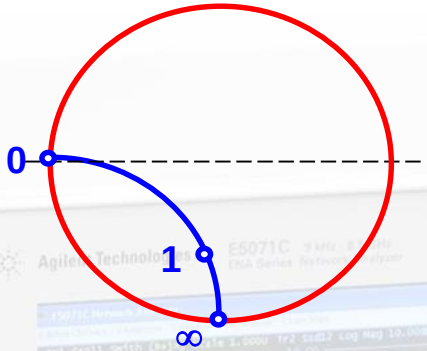




$$\Rightarrow \Gamma_{in} = \Gamma_L / 2$$

Pathing through a 2-port (2)

Lossless Passive Circuit



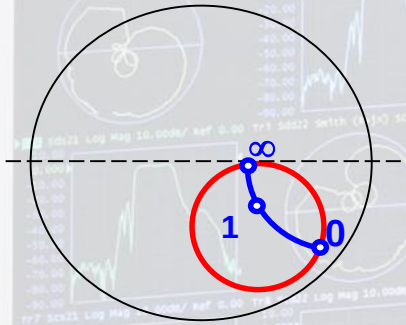
1 — 2

If S is unitary

$$S^*S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

→ Lossless Two-Port

Lossy Passive Circuit



1 — 2

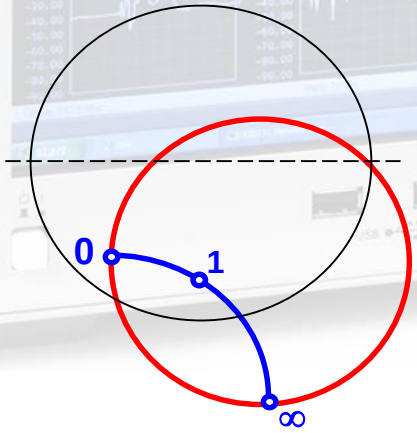
Lossy Two-Port:

If $K_{LINVILL} < 1$

$K_{ROLLET} > 1$

unconditionally stable

Active Circuit



1 — 2

Active Circuit:

If $K_{LINVILL} \geq 1$

$K_{ROLLET} \leq 1$

potentially unstable

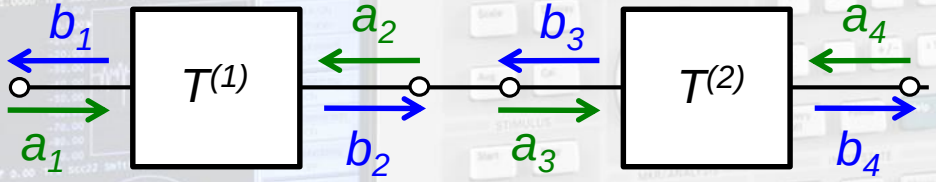
Appendix C: T matrix

The T-parameter matrix is related to the incident and reflected normalised waves at each of the ports.

$$\begin{pmatrix} b_1 \\ a_1 \end{pmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

T-parameters may be used to determine the effect of a cascaded 2-port networks by simply multiplying the individual T-parameter matrices:

$$[T] = [T^{(1)}][T^{(2)}]\ldots[T^{(N)}] = \prod_{N} [T^{(i)}]$$



T-parameters can be directly evaluated from the associated S-parameters and vice versa.

From S to T:

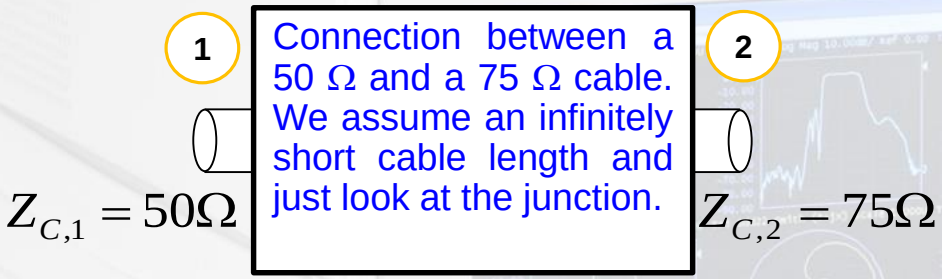
$$[T] = \frac{1}{S_{21}} \begin{bmatrix} -\det(S) & S_{11} \\ -S_{22} & 1 \end{bmatrix}$$

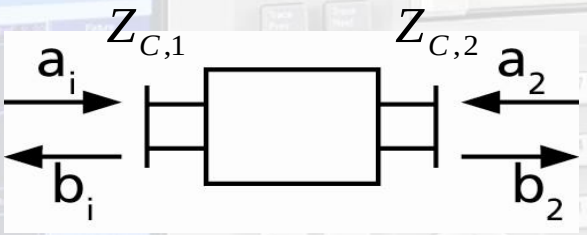
From T to S:

$$[S] = \frac{1}{T_{22}} \begin{bmatrix} T_{12} & \det(T) \\ 1 & -T_{21} \end{bmatrix}$$

Appendix D: A Step in Characteristic Impedance (1)

Consider a connection of two coaxial cables, one with $Z_{C,1}$ = 50 Ω characteristic impedance, the other with $Z_{C,2}$ = 75 Ω characteristic impedance.





Step 1: Calculate the reflection coefficient and keep in mind: all ports have to be terminated with their respective characteristic impedance, i.e. 75 Ω for port 2.

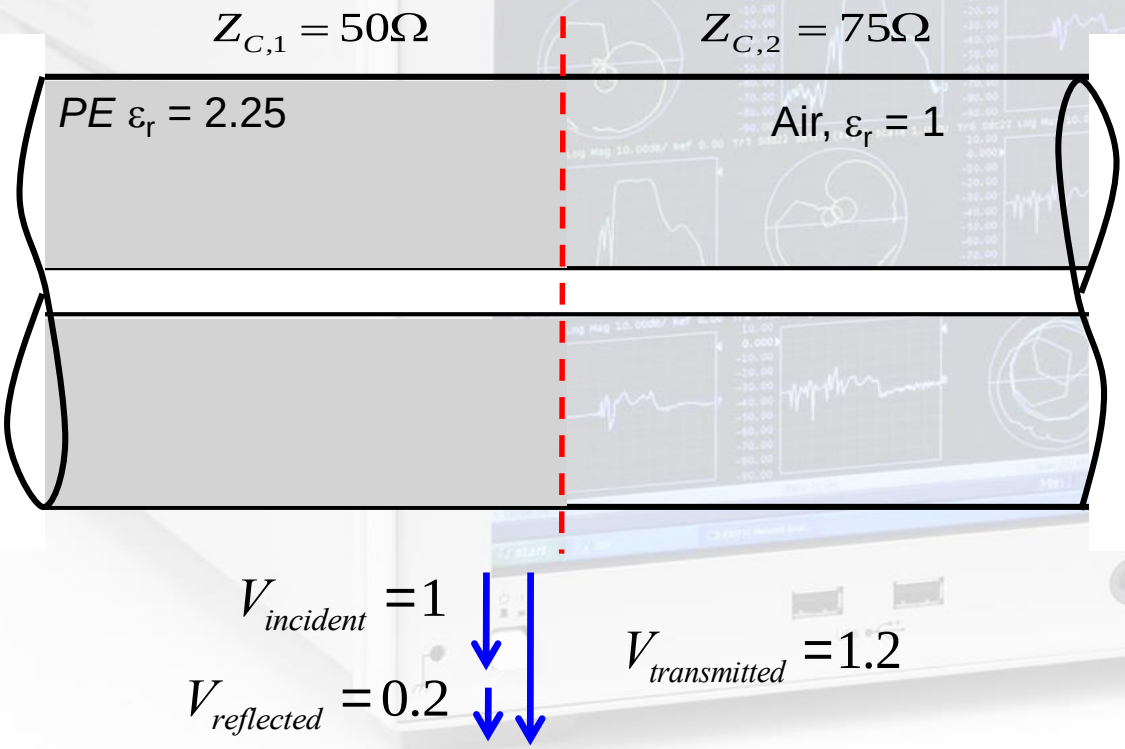
$$\Gamma_1 = \frac{Z - Z_{C,1}}{Z + Z_{C,1}} = \frac{75 - 50}{75 + 50} = 0.2$$

Thus, the voltage of the reflected wave at port 1 is 20% of the incident wave, and the reflected power at port 1 (proportional Γ^2) is 0.2^2 = 4%. As this junction is lossless, the transmitted power must be 96% (conservation of energy). From this we can deduce b_2^2 = 0.96. But: how do we get the voltage of this outgoing wave?

Example: a Step in Characteristic Impedance (2)

Step 2: Remember, *a* and *b* are power-waves, and defined as voltage of the forward- or backward traveling wave normalized to $\sqrt{Z_C}$.

The tangential electric field in the dielectric in the 50 Ω and the 75 Ω line, respectively, must be continuous.



t = voltage transmission coefficient, in this case: $t = 1 + \Gamma$

This is counterintuitive, one might expect 1-Γ. Note that the voltage of the transmitted wave is higher than the voltage of the incident wave. But we have to normalize to $\sqrt{Z_C}$ to evaluate the corresponding S-parameter. $S_{12} = S_{21}$ via reciprocity! But $S_{11} \neq S_{22}$, i.e. the structure is NOT symmetric.

Example: a Step in Characteristic Impedance (3)

Once we have determined the voltage transmission coefficient, we have to normalize to the ratio of the characteristic impedances, respectively. Thus we get for

$$S_{12} = 1.2\sqrt{\frac{50}{75}} = 1.2 \cdot 0.816 = 0.9798$$

We know from the previous calculation that the reflected power (proportional Γ^2) is 4% of the incident power. Thus 96% of the power are transmitted.

Check done $S_{12}{}^2 = 1.44\frac{1}{1.5} = 0.96 = (0.9798)^2$

$$S_{22} = \frac{50-75}{50+75} = -0.2$$ To be compared with S11 = +0.2!

Example: a Step in Characteristic Impedance (4)

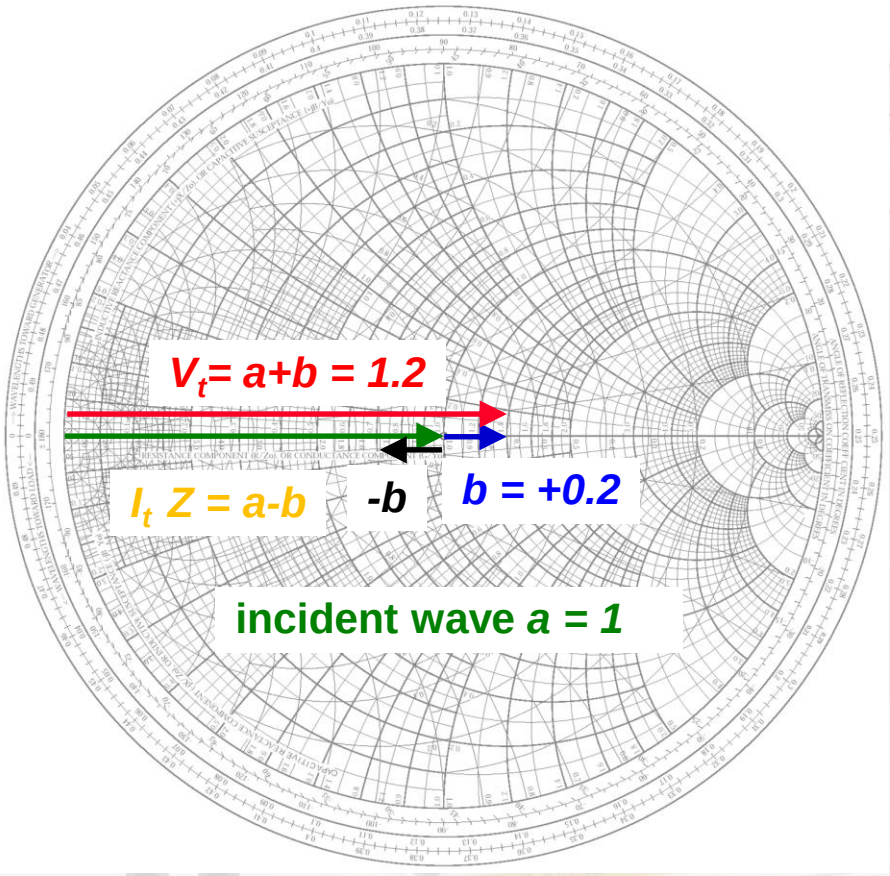
Visualization in the Smith chart:

As shown in the previous slides the voltage of the transmitted wave is

$V_t = a + b$, with $t = 1 + \Gamma$
and subsequently the current is

$I_t Z = a - b$.

Remember: the reflection coefficient Γ is defined with respect to voltages. For currents the sign inverts. Thus a positive reflection coefficient in the normal definition leads to a subtraction of currents or is negative with respect to current.

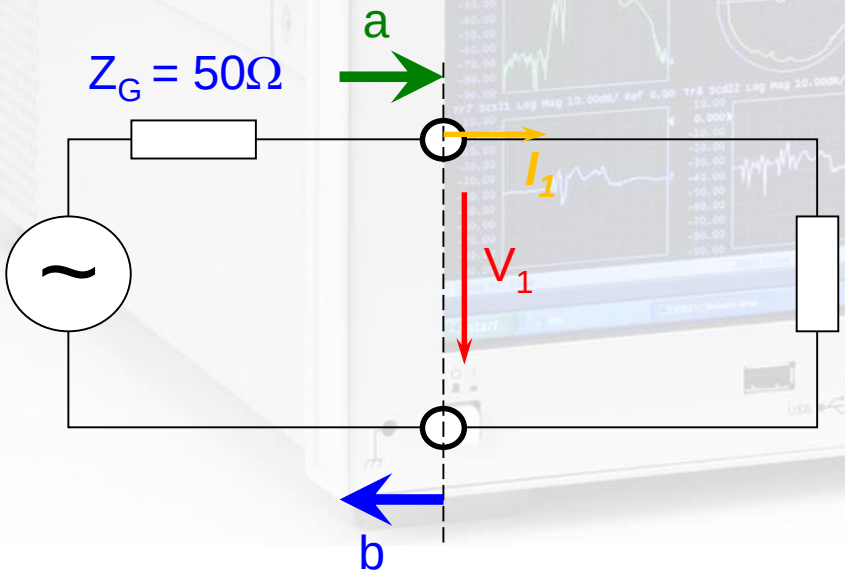


Note: here Z_{load} is real

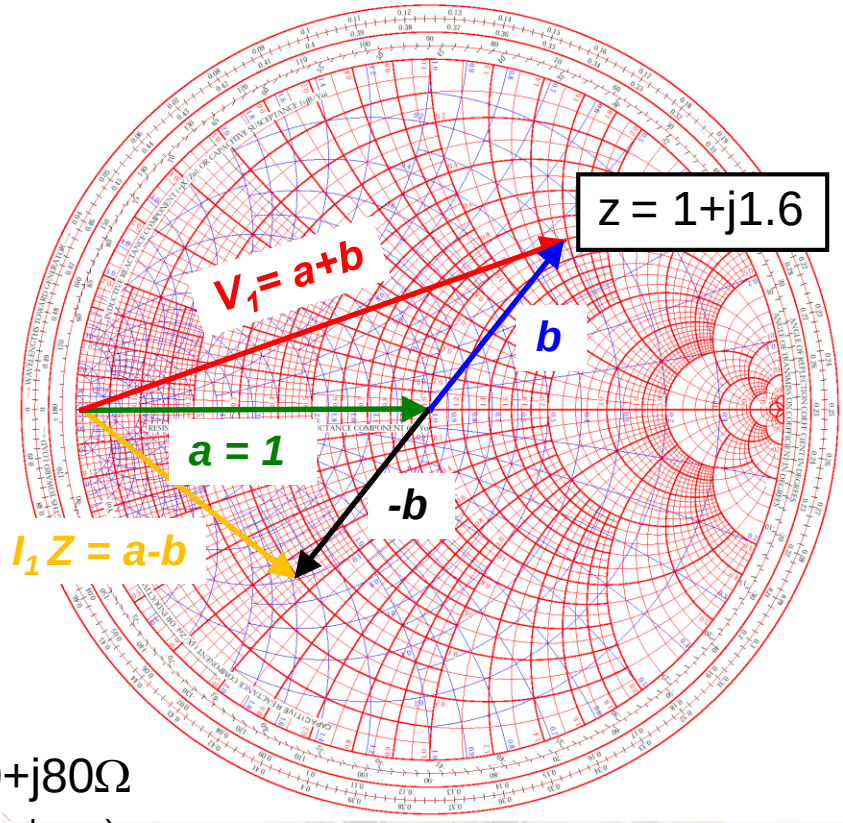
Example: a Step in Characteristic Impedance (5)

General case:

Thus we can read from the Smith chart immediately the amplitude and phase of voltage and current on the load (of course we can calculate it when using the complex voltage divider).



$Z = 50+j80\Omega$
(load impedance)



Appendix E: Navigation in the Smith Chart (1)

This is a “bilinear” transformation with the following properties:

- generalized circles are transformed into generalized circles
  - circle → circle
  - straight line → circle
  - circle → straight line
  - straight line → straight line
- angles are preserved locally

a straight line is nothing else than a circle with infinite radius

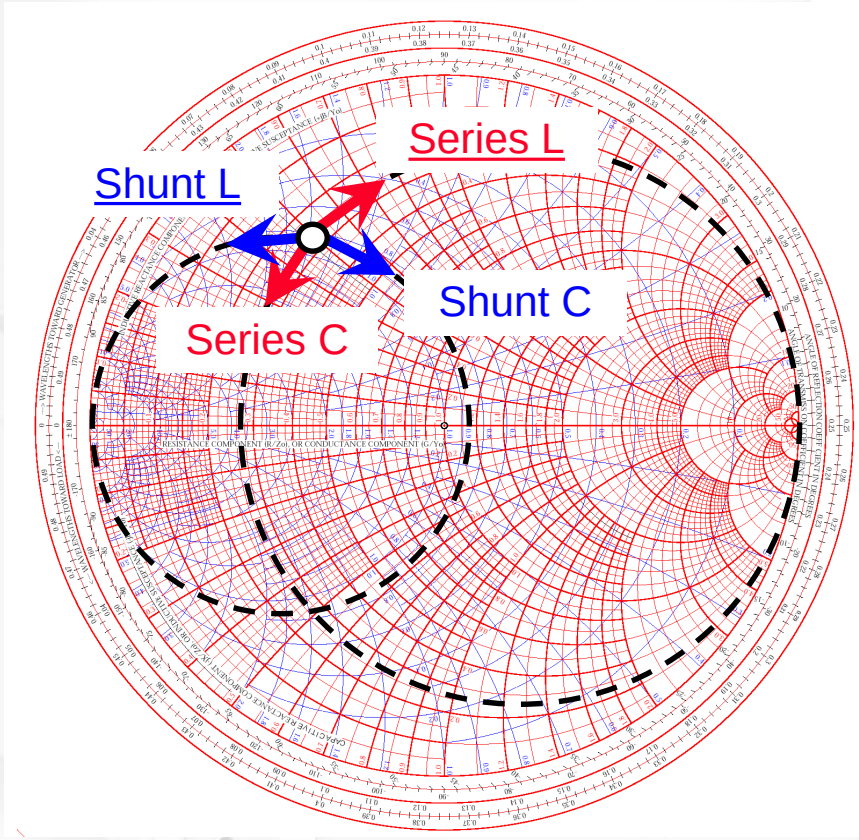
a circle is defined by 3 points

a straight line is defined by 2 points

Navigation in the Smith Chart (2)

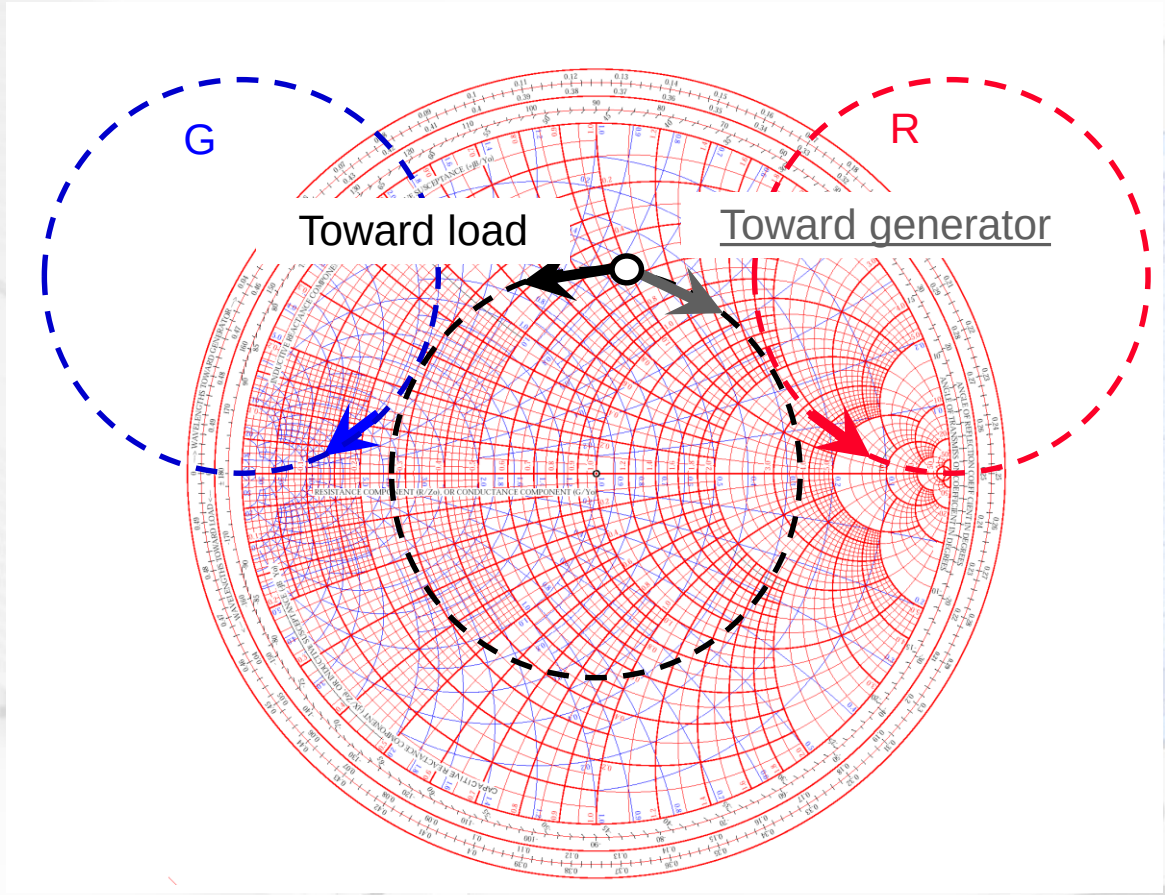
in blue: Impedance plane (=Z)

in red: Admittance plane (=Y)



	Up	Down
Red circles	Series L	Series C
Blue circles	Shunt L	Shunt C

Navigation in the Smith Chart (3)



Red arcs	**Resistance R**
Blue arcs	**Conductance G**
Con-centric circle	**Transmission line going Toward load Toward generator**

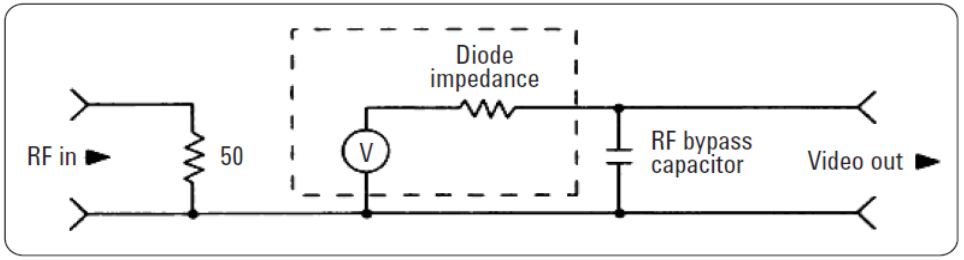
Appendix F: The RF diode (1)

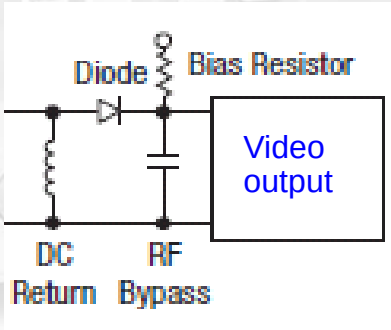
- We are not discussing the generation of RF signals here, just the detection
- Basic tool: fast RF* diode
  (= Schottky diode)
- In general, Schottky diodes are fast but still have a voltage dependent junction capacity
  (metal – semi-conductor junction)

A typical RF detector diode

Try to guess from the type of the connector which side is the RF input and which is the output

Equivalent circuit:





*Please note, in this lecture we will use RF (radio-frequency) for both, the RF and the microwave range, since there is no defined borderline between the RF and microwave regime.

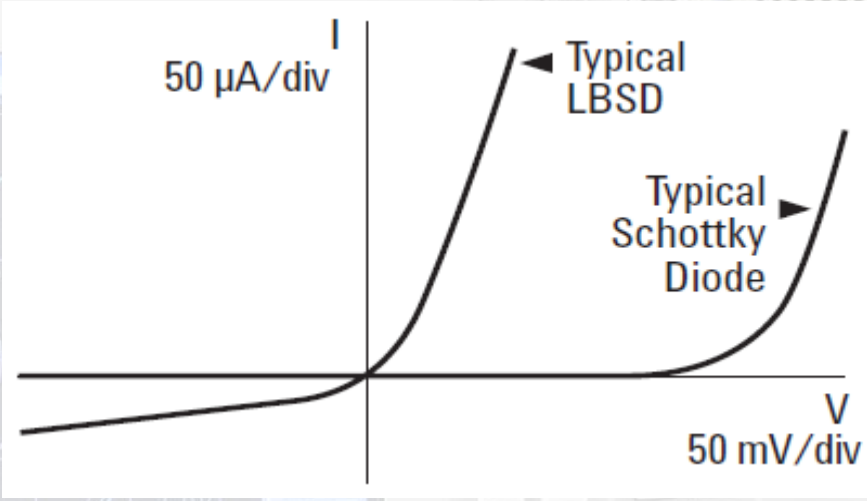
The RF diode (2)

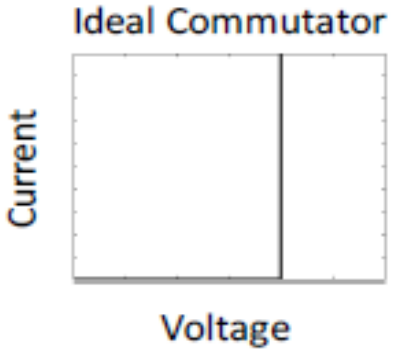
- Characteristics of a diode:

  The current as a function of the voltage for a barrier diode can be described by the Richardson equation:

$$I = AA^{**}{}_2 \exp\left(-\frac{q\phi B}{kT}\right)\left[\exp\left(\frac{qv}{NkT}\right) - 1\right]$$

where
A = area (cm^2)
A** = modified Richardson constant $(amp/oK)^2/cm^2$)
k = Boltzman's Constant
T = absolute temperature (°K)
ϕB = barrier heights in volts
V = external voltage across the depletion layer (positive for forward voltage) - V - IR_S
R_S = series resistance
I = diode current in amps (positive forward current)
n = ideality factor





- The RF diode is NOT an ideal commutator for small signals! We cannot apply big signals otherwise burnout

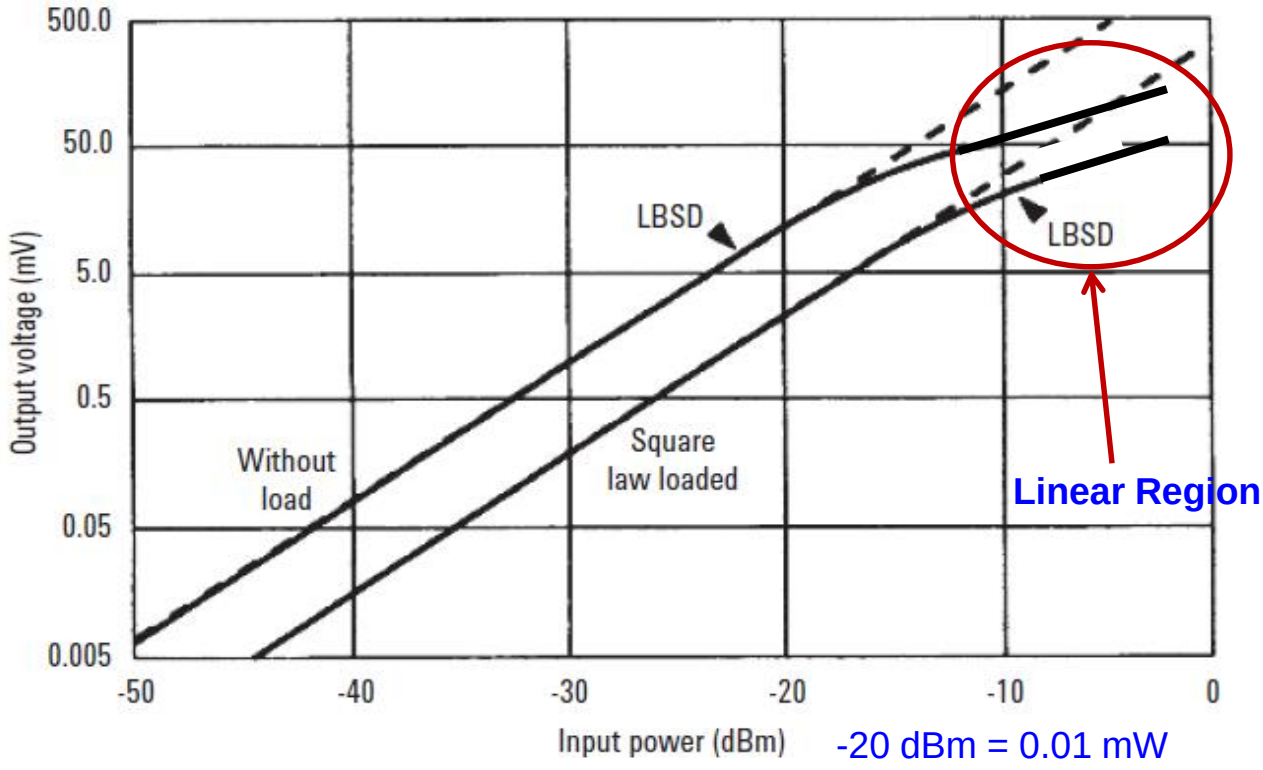
The RF diode (3)

- This diagram depicts the so called square-law region where the output voltage (V_{Video}) is proportional to the input power

Since the input power is proportional to the square of the input voltage (V_{RF}^2) and the output signal is proportional to the input power, this region is called square- law region.

In other words:

$$V_{Video} \sim V_{RF}^2$$

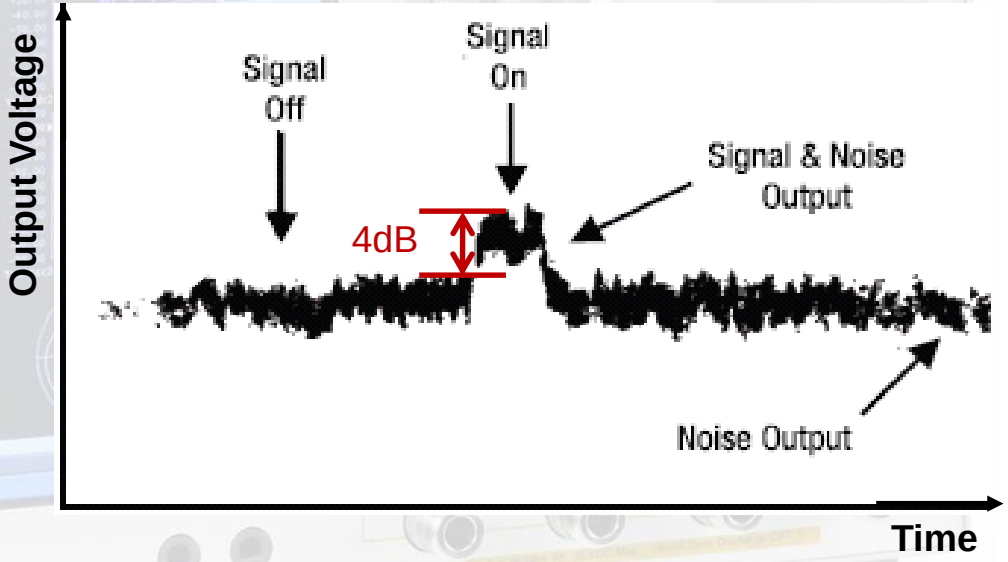


- The transition between the linear region and the square-law region is typically between -10 and -20 dBm RF power (see diagram).

The RF diode (5)

- Due to the square-law characteristic we arrive at the thermal noise region already for moderate power levels (-50 to -60 dBm) and hence the V_{Video} disappears in the thermal noise

- This is described by the term *tangential signal sensitivity* (TSS) where the detected signal (Observation BW, usually 10 MHz) is 4 dB over the thermal noise floor



Appendix G: The RF mixer (1)

- For the detection of very small RF signals we prefer a device that has a linear response over the full range (from 0 dBm (= 1mW) down to thermal noise = -174 dBm/Hz = $4 \cdot 10^{-21}$ W/Hz)
- It is called “RF mixer”, and uses 1, 2 or 4 diodes in different configurations (see next slide)
- Together with a so called LO (local oscillator) signal, the mixer works as a signal multiplier, providing a very high dynamic range since the output signal is always in the “linear range”, assuming the mixer is not in saturation with respect to the RF input signal (For the LO signal the mixer should always be in saturation!)
- The RF mixer is essentially a multiplier implementing the function

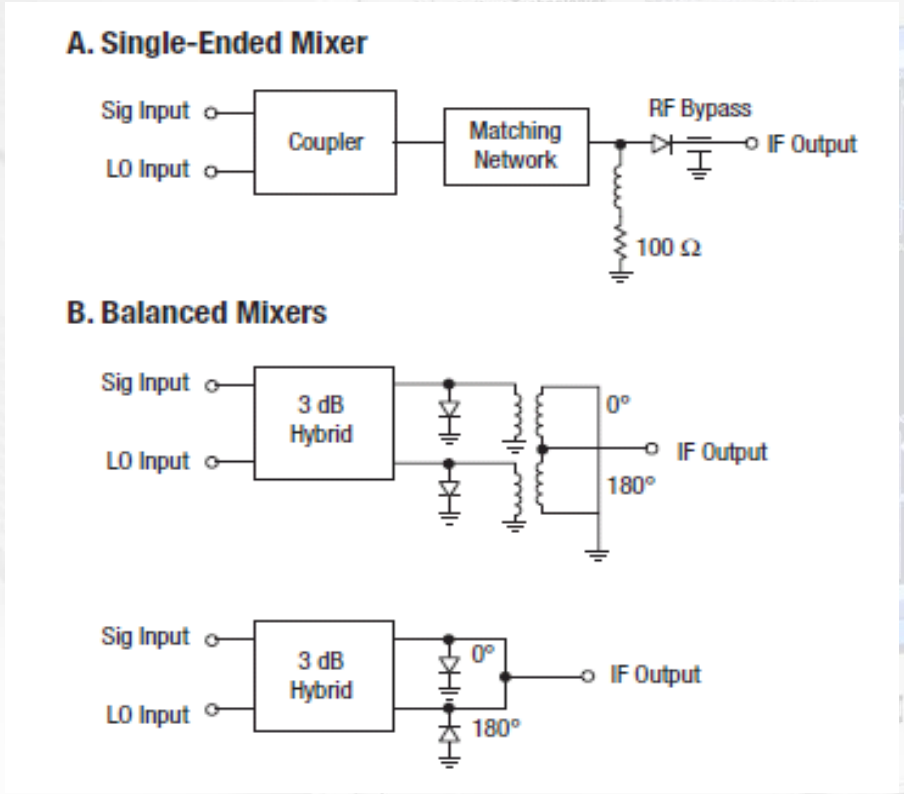
$f_1(t) \cdot f_2(t)$ with $f_1(t)$ = RF signal and $f_2(t)$ = LO signal

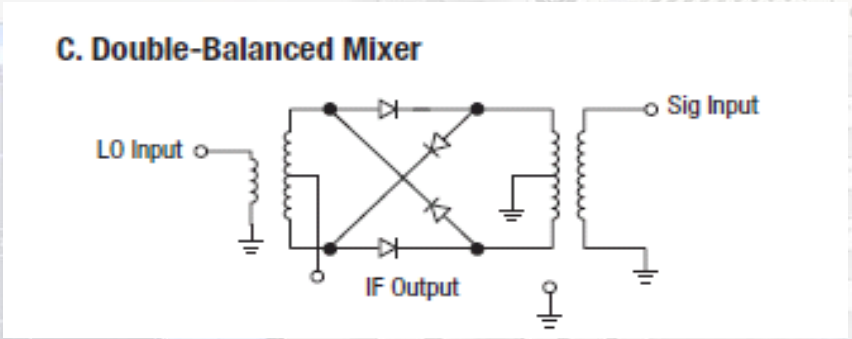
$$a_1 \cos(2\pi f_1 t + \varphi) \cdot a_2 \cos(2\pi f_2 t) = \frac{1}{2} a_1 a_2 [\cos((f_1 + f_2)t + \varphi) + \cos((f_1 - f_2)t + \varphi)]$$

- Thus we obtain a response at the IF (intermediate frequency) port as sum and difference frequencies of the LO and RF signals

The RF mixer (2)

- **Examples of different mixer configurations**





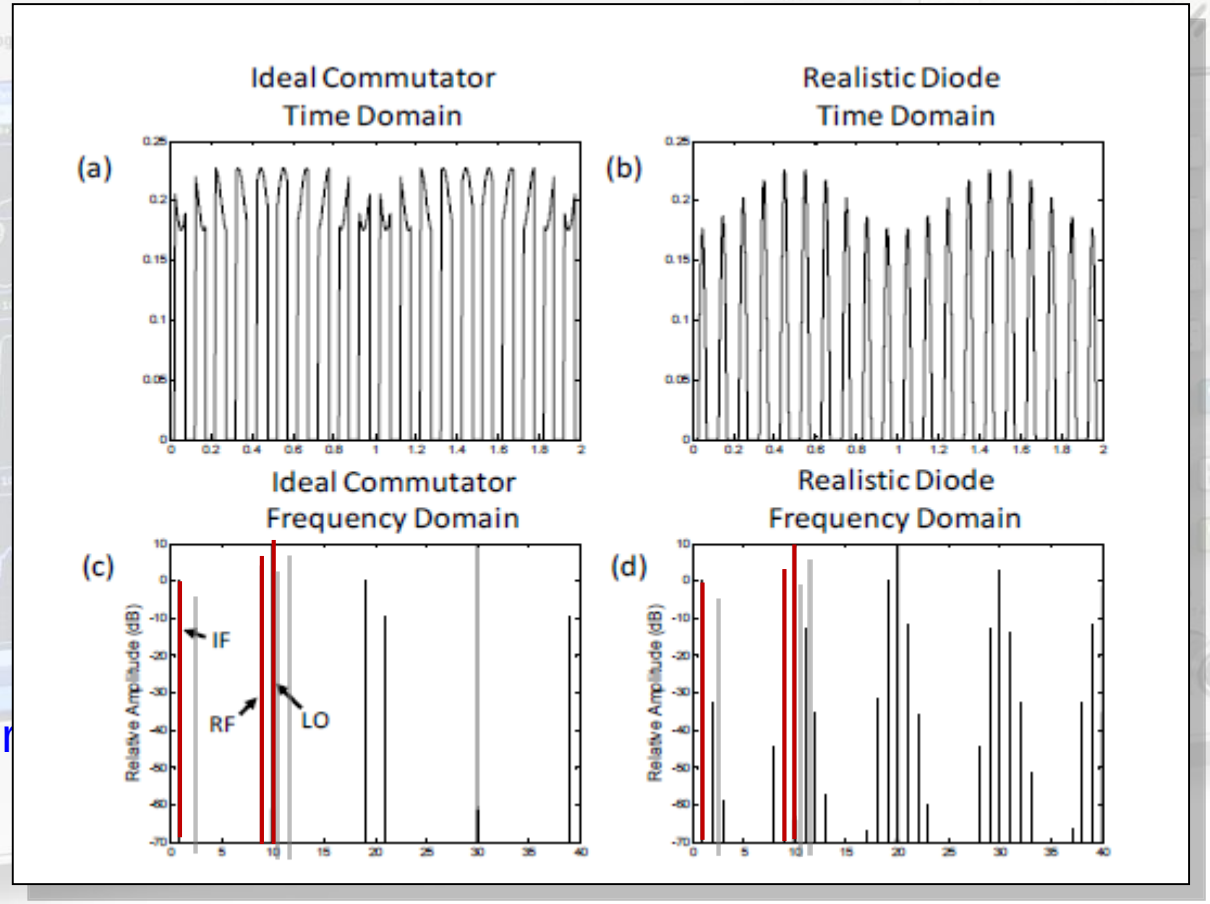
- A typical coaxial mixer (SMA connector)

The RF mixer (3)

- **Response of a mixer in time and frequency domain:**

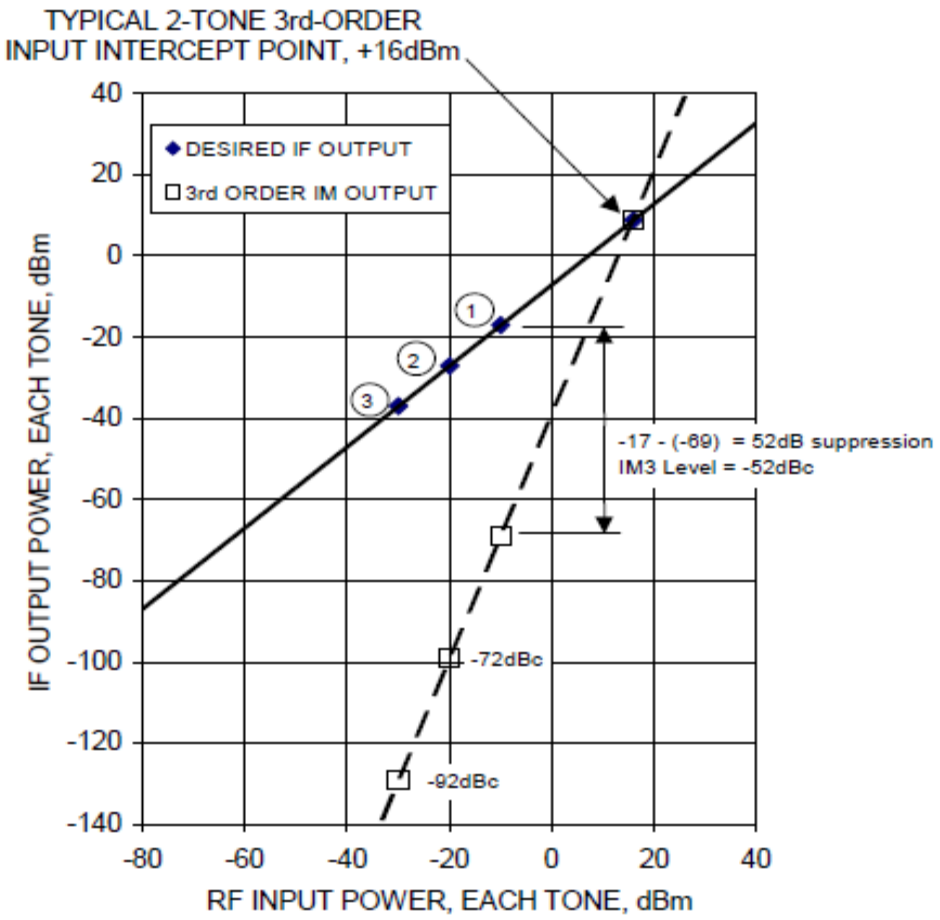
- Input signals here:
- LO = 10 MHz
- RF = 8 MHz

- Mixing products at 2 and 18 MHz and higher order terms at higher frequencies



The RF mixer (4)

Dynamic range and IP3 of an RF mixer

- The abbreviation IP3 stands for *third order intermodulation point*, where the two lines shown in the right diagram intersect. Two signals (f_1,$f_2 > f_1$) which are closely spaced by Δf in frequency are simultaneously applied to the DUT. The intermodulation products appear at + Δf above f_2 and at – Δf below f_1.

- This intersection point is usually not measured directly, but extrapolated from measurement data at much lower power levels to avoid overload and/or damage of the DUT.