PLIN - Note 88-04
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Emittance Measurements for Linac III

Principle

The measurement of the *Courant-Snyder* beam parameters α, β, γ, ϵ is based on the evolution of the beam matrix σ. If the transport is described by the 2 x 2 matrix

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} ,$$

with $R_{11}R_{22} - R_{12}R_{21} = 1$, the beam evolution is given by[1]

$$\sigma = \begin{pmatrix} \epsilon\beta & -\epsilon\alpha \\ -\epsilon\alpha & \epsilon\gamma \end{pmatrix} = R \begin{pmatrix} \epsilon\beta_0 & -\epsilon\alpha_0 \\ -\epsilon\alpha_0 & \epsilon\gamma_0 \end{pmatrix} R^T .$$

$\epsilon\beta$ is the measurable squared beam envelope $\hat{x}^2$ at the end of the transport. It depends linearly on the initial beam matrix elements:

$$\hat{x}^2 = \epsilon\beta = R_{11}^2 \cdot \epsilon\beta_0 - 2R_{11}R_{12} \cdot \epsilon\alpha_0 + R_{12}^2 \cdot \epsilon\gamma_0$$

To obtain $\epsilon\alpha_0$, $\epsilon\beta_0$, $\epsilon\gamma_0$, and $\epsilon^2 = \epsilon\beta_0 \cdot \epsilon\gamma_0 - \epsilon\alpha_0 \cdot \epsilon\alpha_0$, one has set R to three (sufficiently different) values, measure the resulting $\hat{x}^2$, and solve three linear equations.

3-Position Method

One simple example is the simultaneous measurement with three profile harps as forseen for the HEBT. Here the transport matrix

$$R(S) = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$$

describes just a variable drift space. The three settings $S_1 = -L$, $S_2 = 0$, $S_3 = +L$ give the beam transport from a reference position S_2 to two symmetric ones up- and downstream. They lead to the equations

$$\left.\begin{aligned} \hat{x}_1^2 &= \epsilon\beta_2 - 2L \cdot \epsilon\alpha_2 + L^2 \cdot \epsilon\gamma_2 \\ \hat{x}_2^2 &= \epsilon\beta_2 \\ \hat{x}_3^2 &= \epsilon\beta_2 + 2L \cdot \epsilon\alpha_2 + L^2 \cdot \epsilon\gamma_2 \end{aligned}\right\} \text{ with the solution } \left\{\begin{aligned} \epsilon\beta_2 &= \hat{x}_2^2 \\ \epsilon\alpha_2 &= (\hat{x}_3^2 - \hat{x}_1^2)/(4L) \\ \epsilon\gamma_2 &= (\hat{x}_1^2 - 2\hat{x}_2^2 + \hat{x}_3^2)/(2L^2) \\ \epsilon^2 &= \epsilon\beta_2 \cdot \epsilon\gamma_2 - (\epsilon\alpha_2)^2 \end{aligned}\right.$$

The measurements should be done in the vicinity of a beam waist where the curvature due to $\epsilon\gamma$ is most noticeable. If the beam waist is <u>exactly</u> at S_2, we have $\hat{x}_3 = \hat{x}_1$, $\alpha_2 = 0$. The emittance formula is then reduced to

$$\epsilon = \hat{x}_2\sqrt{\hat{x}_3^2 - \hat{x}_2^2}/L \qquad (\text{for } \hat{x}_2 = \textit{minimum})$$

[1]see K.L.Brown et al., CERN 80-04 (TRANSPORT manual)

$\beta \cdot \gamma - \alpha^2 = 1$

3-Gradient Method

If the 3-Position method cannot be used, the beam parameters can be measured by varying the gradient of a quadrupole. The system of the quadrupole and a subsequent transport section with <u>constant</u> elements C, S, C' and S' ($CS' - C'S = 1$) is described by

$$R(k) = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} \cos kl & k^{-1}\sin kl \\ -k\sin kl & \cos kl \end{pmatrix} \quad \text{with} \quad \begin{cases} R_{11}(k) = & C\cos kl & -Sk\sin kl \\ R_{12}(k) = & Ck^{-1}\sin kl & +S\cos kl \end{cases}$$

The measurement of $\hat{x}^2$, as defined above, for three different quadrupole settings k yields the linear equations to determine all beam parameters.

Simplified 3-Gradient Method

The function $\hat{x}^2(k)$ is greatly simplified[2] if the quadrupole can be described as a <u>thin</u> lense ($kl \to 0$, $k^2 l \to q = f^{-1}$):

$$\begin{aligned} R_{11}(k) &\to R_{11}(q) = C - q \cdot S \\ R_{12}(k) &\to R_{12}(q) = S \end{aligned}$$

This leads to

$$\begin{aligned} \hat{x}^2(q) &= (C - q \cdot S)^2 \cdot \epsilon\beta_0 - 2S(C - qS) \cdot \epsilon\alpha_0 + S^2 \cdot \epsilon\gamma_0 \\ &= S^2\epsilon^2/\hat{x}_0^2 + \hat{x}_0^2 \cdot (q - q_{min})^2 \ . \end{aligned}$$

with $q_{min} = C/S - \alpha_0/\beta_0$ being the setting for minimal $\hat{x}^2$, and $\hat{x}_0^2 = \epsilon\beta_0$ the squared envelope before the quadrupole. The emittance is then given by

$$\epsilon = \frac{\hat{x}(q_{min})\ \sqrt{\hat{x}^2(q) - \hat{x}^2(q_{min})}}{S^2|q - q_{min}|} \ .$$

The other beam parameters (at the entrance of the quadrupole) are

$$\begin{aligned} \hat{x}_0 &= \sqrt{\hat{x}^2(q) - \hat{x}^2(q_{min})}/(S|q - q_{min}|) \\ \beta_0 &= \hat{x}_0^2/\epsilon \\ \alpha_0 &= \beta_0\ (C/S - q_{min}) \ . \end{aligned}$$

Some comments may be in order:

1. The formula for the emittance is simple enough to be coded in POCAL.

2. The emittance measurement after linac tank I forsees the variation of quad No 52, while No 53 is turned off [3]. The above formalism allows also measurements with No 53 powered (fixed $C' \neq 0$) and also to account for the defocussing by (linear) space charge and by RF acceleration.

3. Suitable quadrupole settings and the quality of the approximations can be investigated by beam transport calculations.

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[2] G. Jacobs, private communication

[3] S.H. Wang, PLIN - Note 88-01 (June 24, 1988)