Non Linear Dynamics - Phenomenology

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CERN Accelerator School

Advanced Accelerator Physics Course 2019

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Summary

- Phase space dynamics – fixed point analysis
- Poincaré map
- Motion close to a resonance
- Onset of chaos
- Chaos detection methods
  - Dynamic Aperture
  - Lyapunov exponent
  - Frequency map analysis
  - Numerical applications

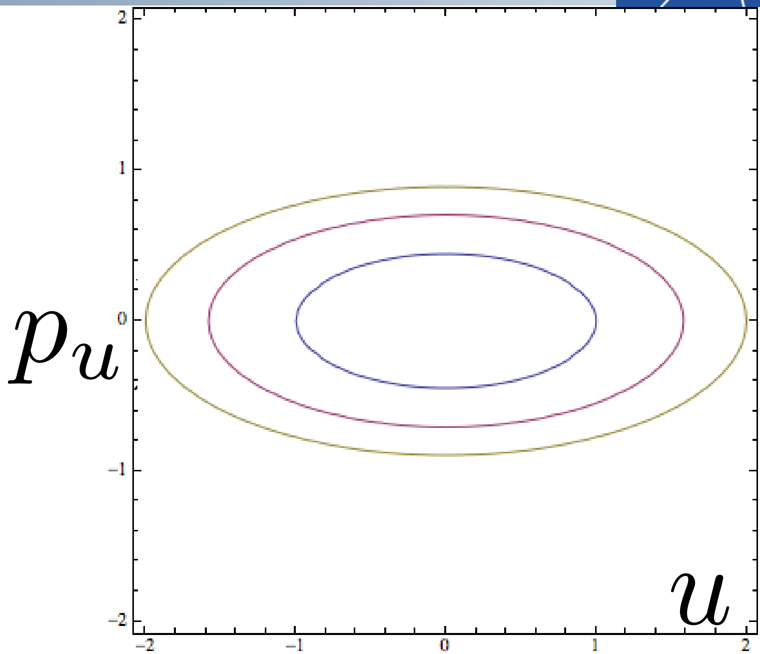
Phase space dynamics - Fixed point analysis

Phase space dynamics

- Valuable description when examining trajectories in **phase space** (u, p_u)
- Existence of integral of motion imposes geometrical constraints on phase flow
- For the simple **harmonic oscillator**

$$H = \frac{1}{2}\left(p_u^2 + \omega_0^2 u^2\right)$$

phase space curves are **ellipses** around the equilibrium point parameterized by the Hamiltonian (energy)



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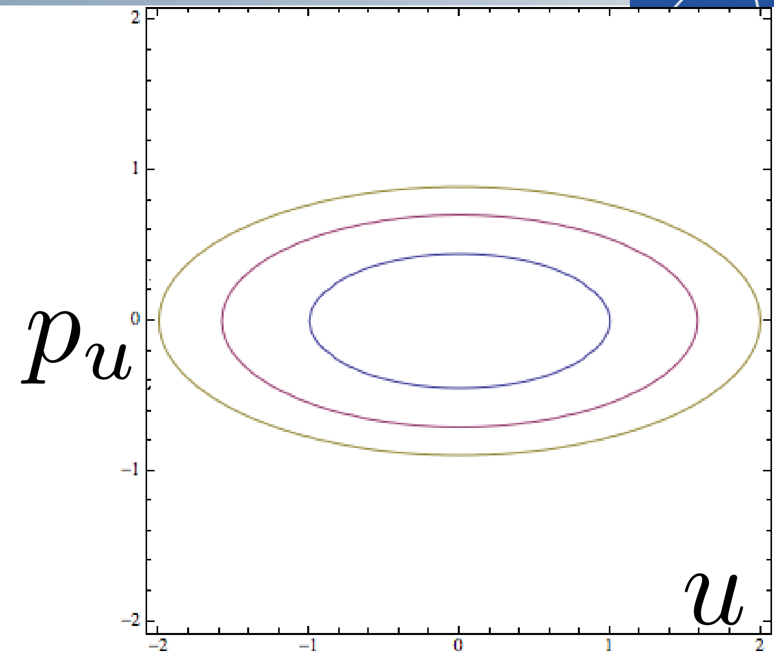
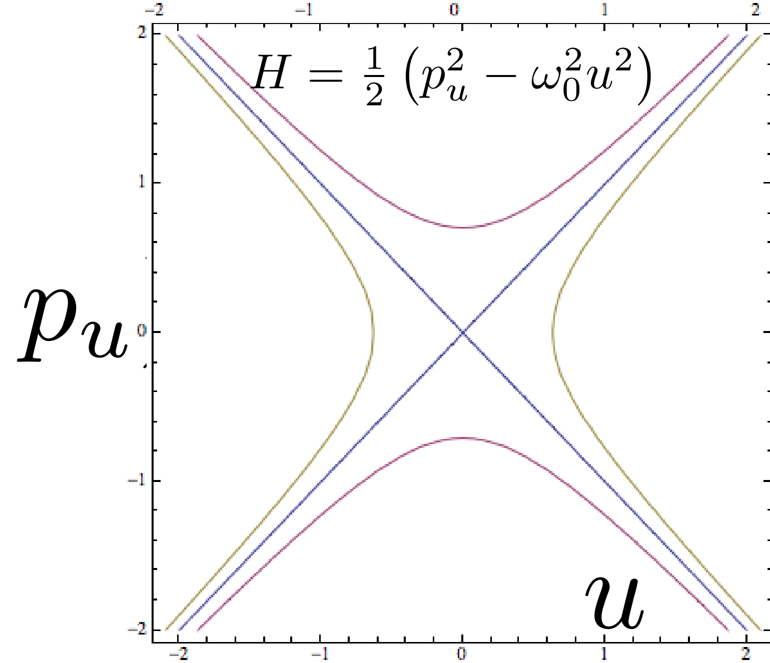
- By simply **changing** the **sign** of the potential in the harmonic oscillator, the phase trajectories become **hyperbolas**, symmetric around the equilibrium point where two straight lines cross, moving towards and away from it





Non-linear oscillators

- Conservative non-linear oscillators have Hamiltonian

$$H = E = \frac{1}{2}p_u^2 + V(u)$$

  with the potential being a general (polynomial) function of positions
- **Equilibrium points** are associated with extrema of the potential

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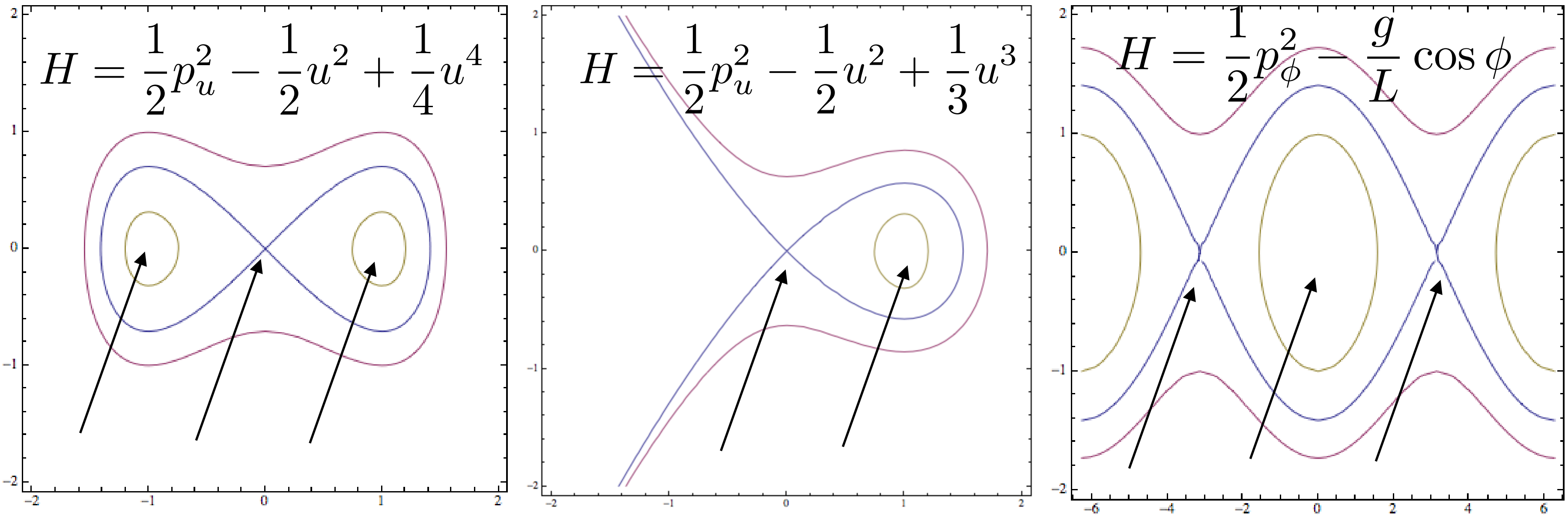

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- **Equilibrium points** are associated with extrema of the potential
- Considering three non-linear oscillators
  - **Quartic** potential (left): two minima and one maximum
  - **Cubic** potential (center): one minimum and one maximum
  - **Pendulum** (right): periodic minima and maxima

Fixed point analysis

- Consider a general second order system

$$\frac{du}{dt} = f_1(u, p_u)$$
$$\frac{dp_u}{dt} = f_2(u, p_u)$$

- Equilibrium or "**fixed**" points $f_1(u_0, p_{u0}) = f_2(u_0, p_{u0}) = 0$ are determinant for topology of trajectories at their vicinity

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- The **linearized equations** of motion at their vicinity are

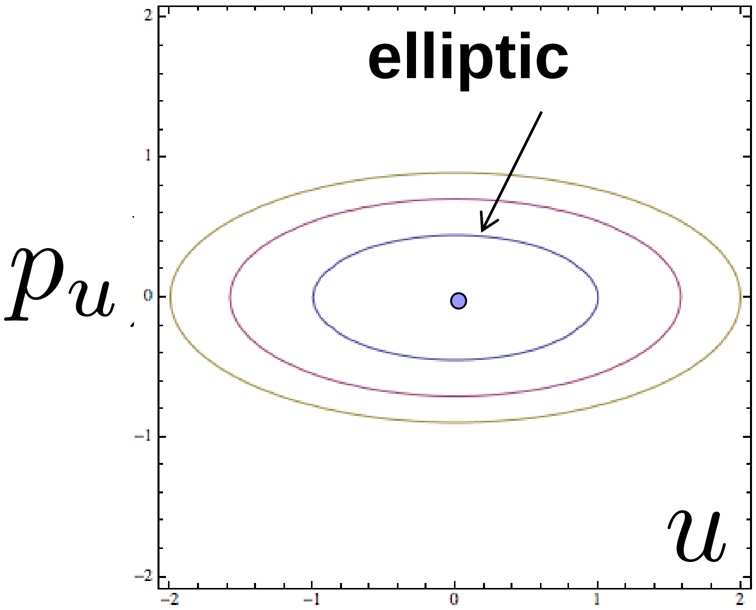
$$\frac{d}{dt}\begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \mathcal{M}_J \begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix} = \underbrace{\begin{bmatrix}\frac{\partial f_1(u_0, p_{u0})}{\partial u} & \frac{\partial f_1(u_0, p_{u0})}{\partial p_u}\\ \frac{\partial f_2(u_0, p_{u0})}{\partial u} & \frac{\partial f_2(u_0, p_{u0})}{\partial p_u}\end{bmatrix}}_{\text{Jacobian matrix}} \begin{bmatrix}\delta u\\ \delta p_u\end{bmatrix}$$

Jacobian matrix

- Fixed point nature is revealed by **eigenvalues** of $\mathcal{M}_J$, i.e. solutions of the characteristic polynomial $\det|\mathcal{M}_J - \lambda \mathbf{I}| = 0$

Fixed point for conservative systems

- For **conservative systems** of 1 degree of freedom, the second order characteristic polynomial for any fixed point has two possible solutions:
  - Two **complex eigenvalues** with opposite sign, corresponding to **elliptic** fixed points. Phase space flow is described by **ellipses**, with particles evolving clockwise or anti-clockwise



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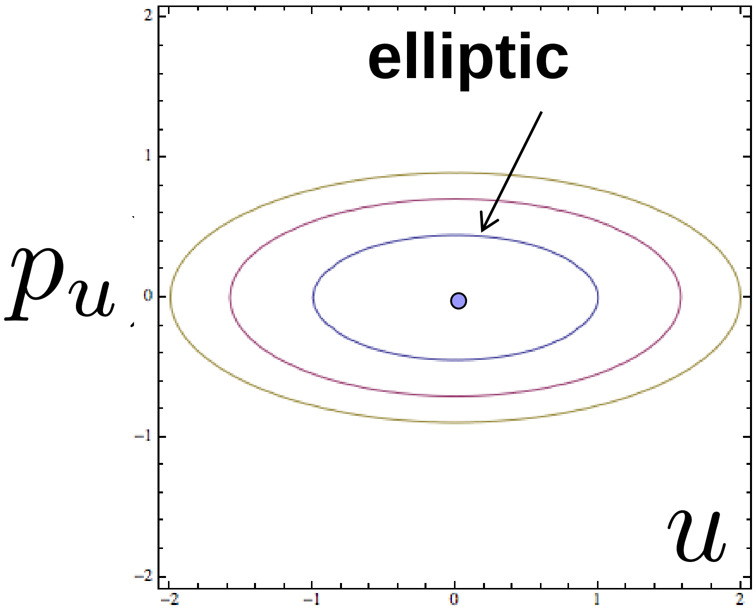
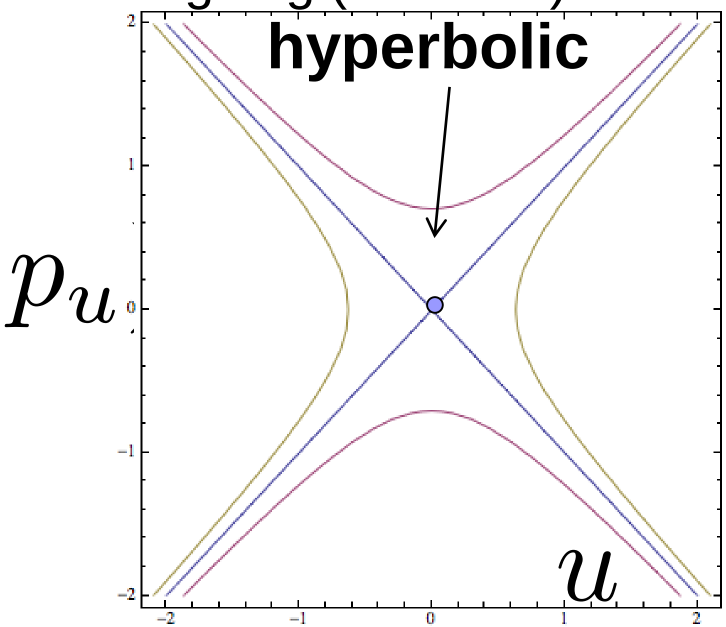
  - Two **real eigenvalues** with opposite sign, corresponding to **hyperbolic** (or saddle) fixed points. Flow described by two lines (or manifolds), incoming (stable) and outgoing (unstable)





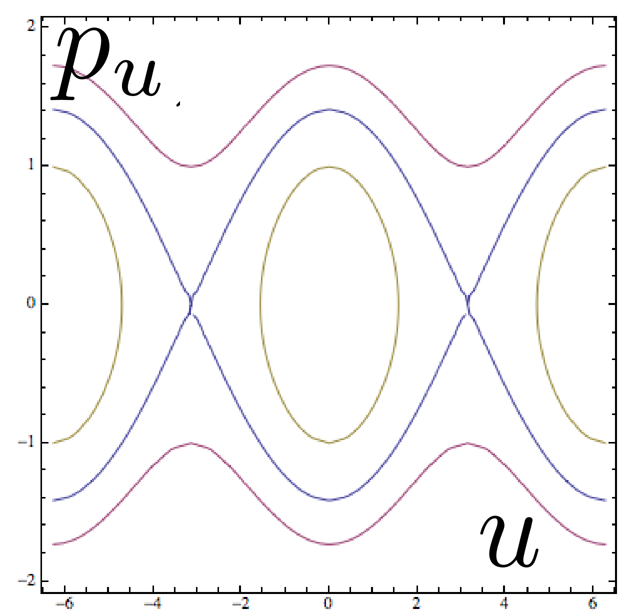
Pendulum fixed point analysis

- The "fixed" points for a pendulum can be found at

$$(\phi_n, p_\phi) = (\pm n\pi, 0)\ ,\ n = 0, 1, 2\ldots$$

- The Jacobian matrix is $\begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\phi_n & 0 \end{bmatrix}$

- The eigenvalues are $\lambda_{1,2} = \pm i\sqrt{\frac{g}{L}\cos\phi_n}$



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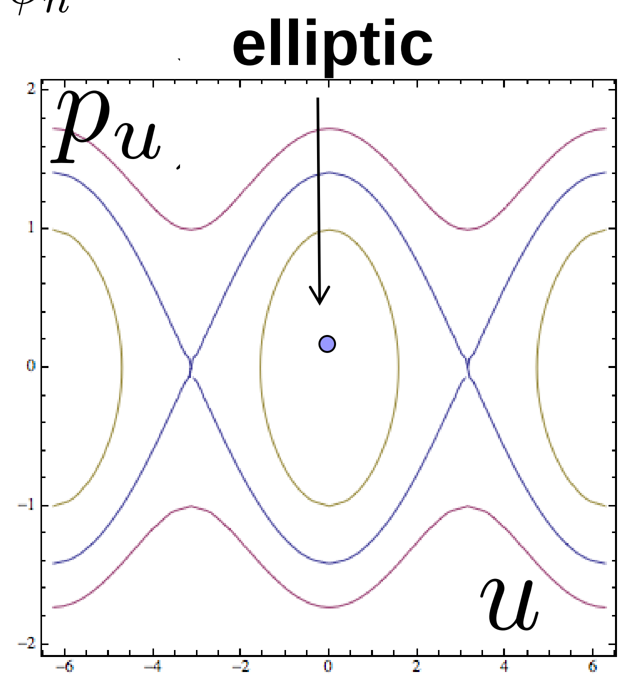

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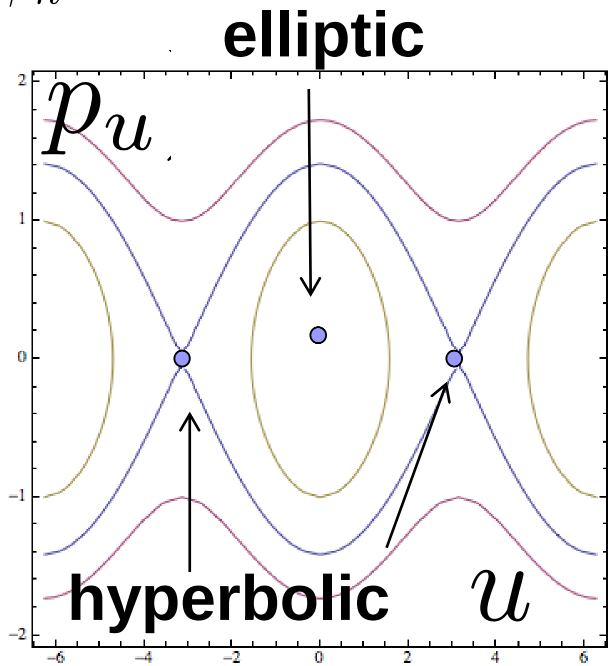
  - $\phi_n = (2n+1)\pi$, for which $\lambda_{1,2} = \pm\sqrt{\frac{g}{L}}$ corresponding to **hyperbolic** fixed points
  - The **separatrix** are the stable and unstable manifolds through the hyperbolic points, separating bounded **librations** and unbounded **rotations**



Phase space for time-dependent systems

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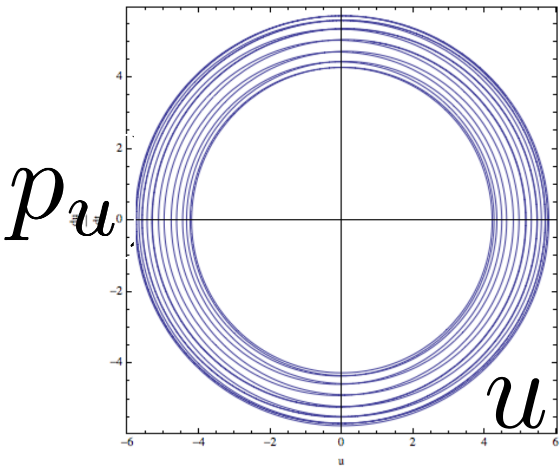

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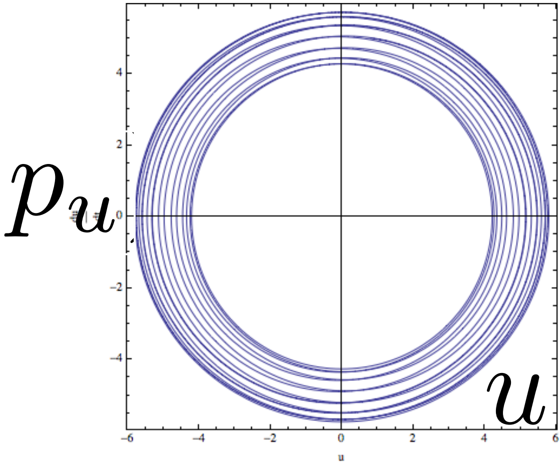

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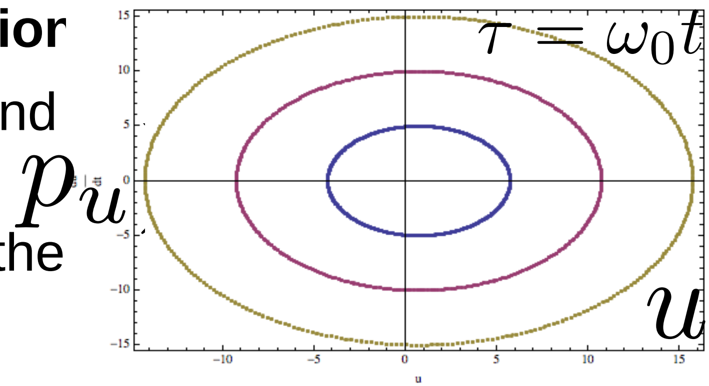

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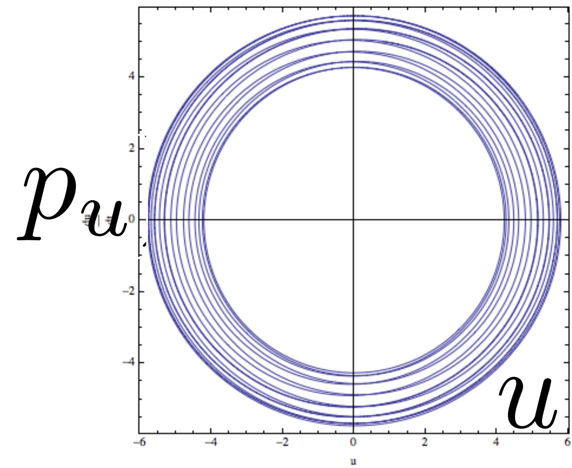
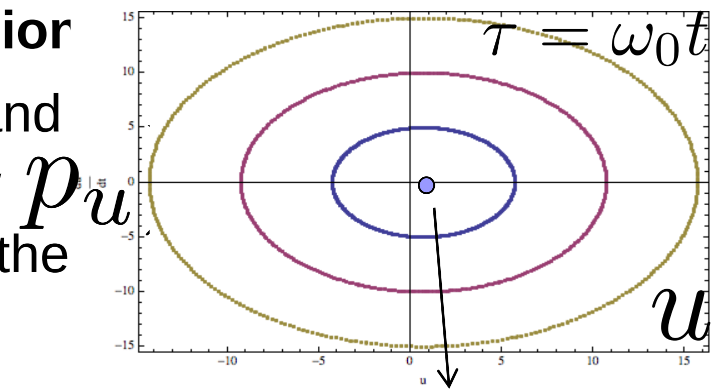
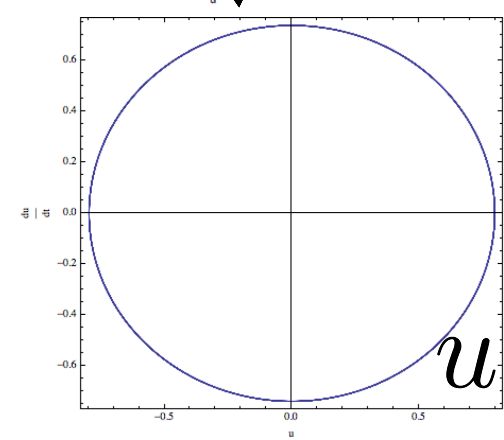
- The **fixed point** in the surface of section is now a periodic orbit



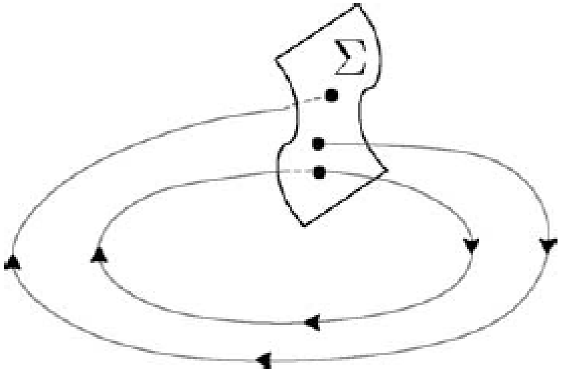




Poincaré map

Poincaré map

- **First recurrence** or **Poincaré map** (or surface of section) is defined by the intersection of trajectories of a dynamical system, with a fixed surface in phase space



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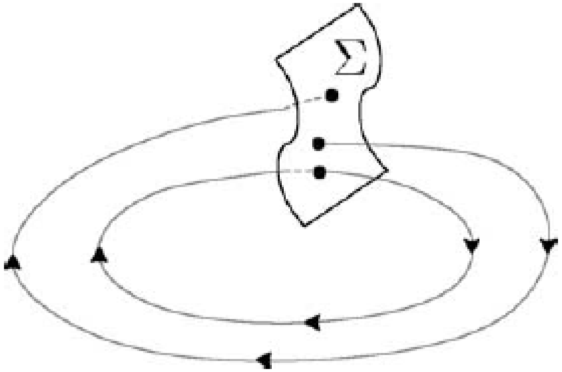

- For an **autonomous** Hamiltonian system $H(\mathbf{q}, \mathbf{p})$ (no **explicit** time dependence), it can be chosen to be any fixed surface in phase space, e.g. $q_i = 0$
- For a **non-autonomous** Hamiltonian system $H(\mathbf{q}, \mathbf{p}, t)$ (explicit time dependence), which is **periodic,** it can be chosen as the period $t = T_0$

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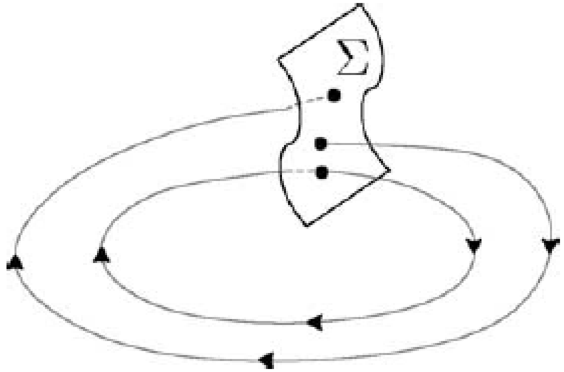

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- For a **non-autonomous** Hamiltonian system $H(\mathbf{q},\mathbf{p},t)$ (explicit time dependence), which is **periodic,** it can be chosen as the period $t = T_0$
- In a system with n degrees of freedom (or $n+1$ including time), the phase space has $2n$ (or $2n+2$) dimensions
- By fixing the value of the Hamiltonian to H_0, the motion on a Poincaré map is reduced to $2n-2$ (or $2n$)

Poincaré map

- Particularly useful for a system with **2 degrees** of **freedom**, or **1 degree** of **freedom + time**, as the motion on Poincaré map is described by 2-dimensional curves
- For continuous system, numerical techniques exist to compute the surface exactly (e.g. M.Henon Physica D 5, 1982)

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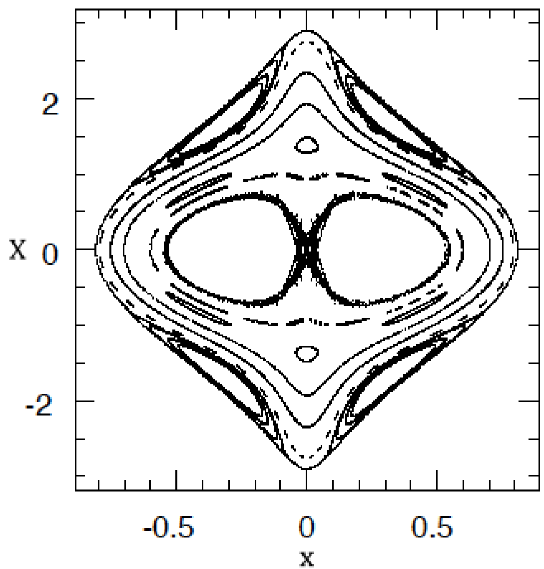
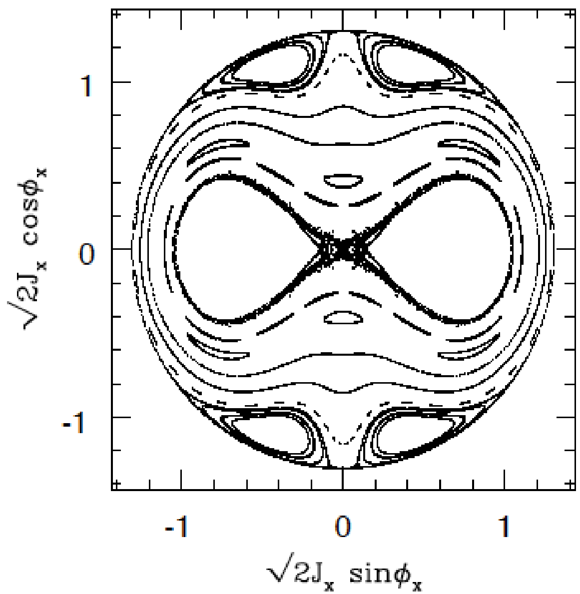
- Example from Astronomy: the logarithmic galactic potential

$$H_q(x,y,X,Y) = \frac{1}{2}(X^2+Y^2) + \ln\left(x^2 + \frac{y}{q^2} + R_c^2\right)$$

$$(x,y,X,Y) \mapsto (\phi_x, \phi_y, J_x, J_y)$$

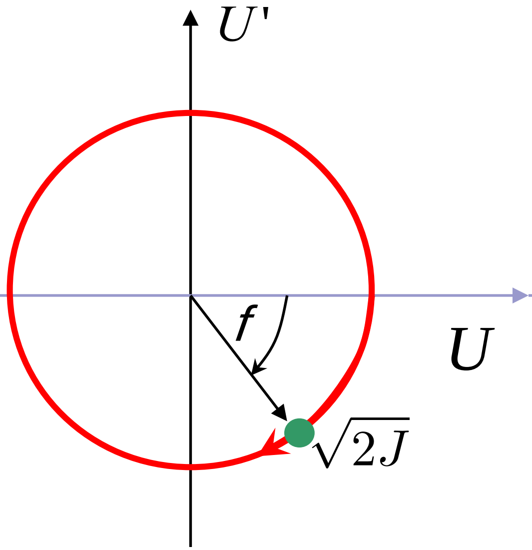
$$y = 0$$

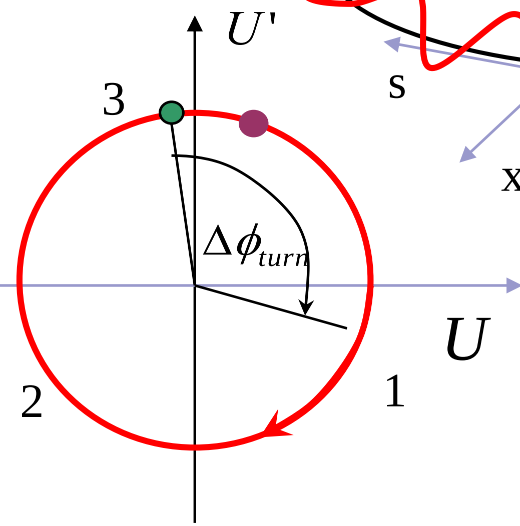




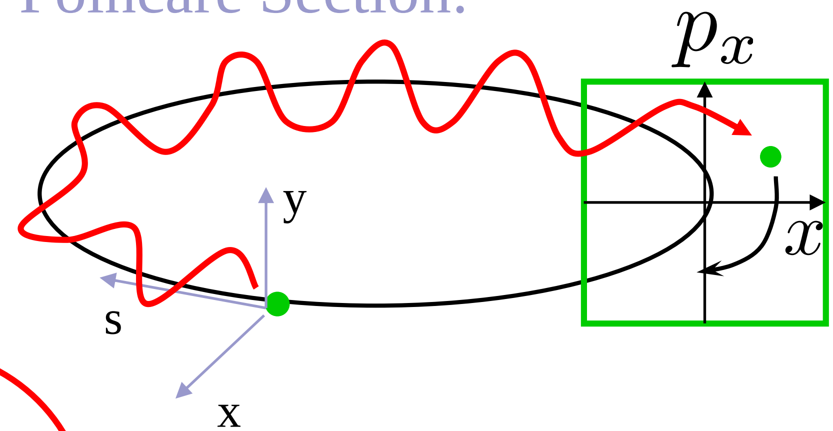
$$\phi_y = 0$$

- Record the particle coordinates at one location in a ring
- Unperturbed motion lies on a circle in normalized coordinates (simple rotation)





Poincaré Section:



Poincaré Section for a ring

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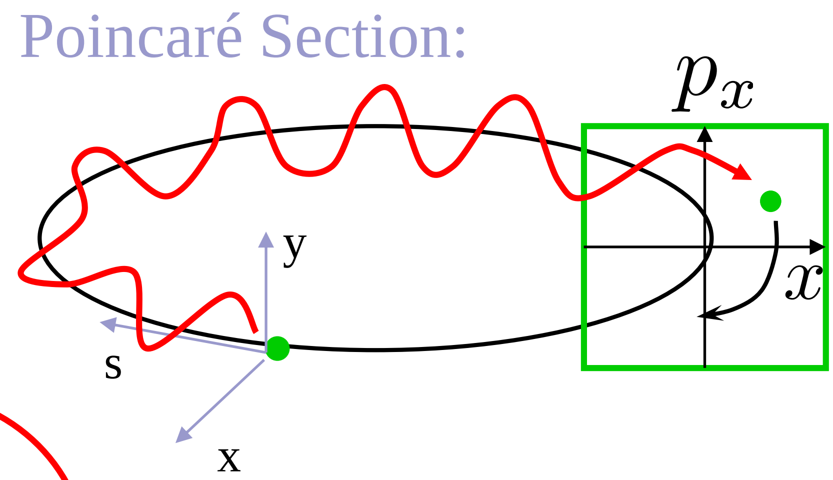
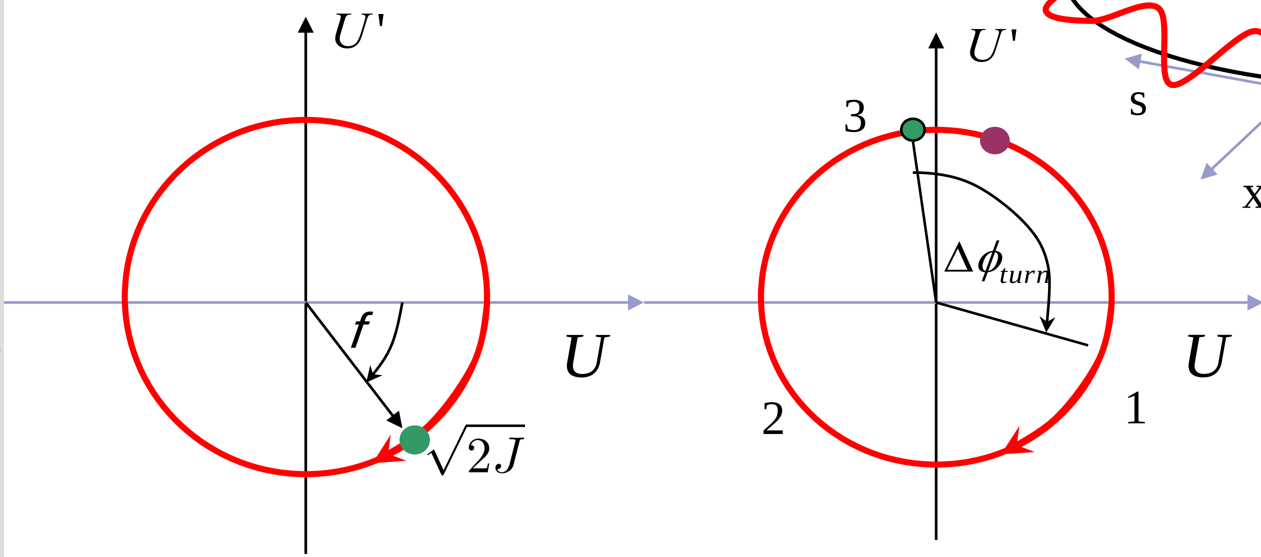
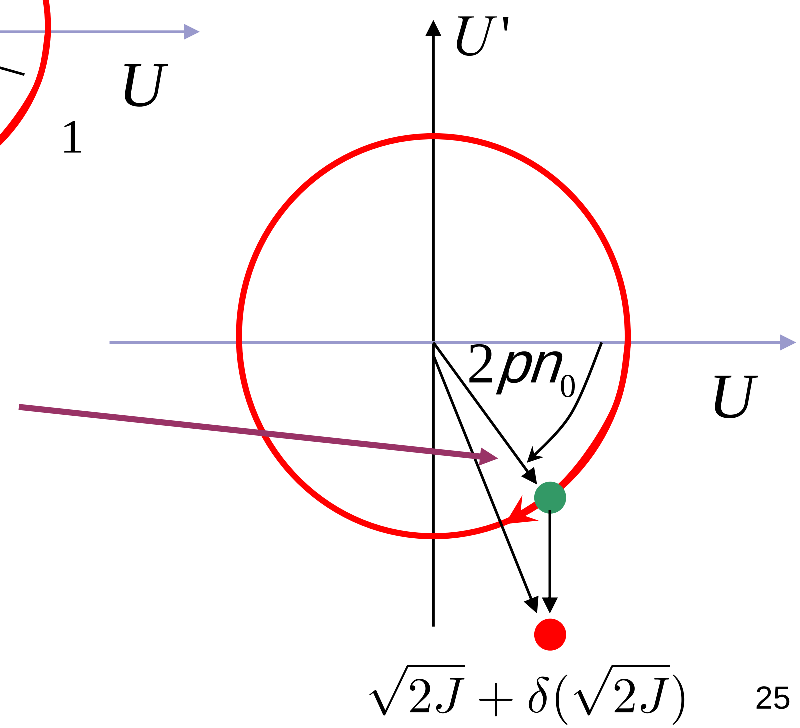




- Resonance condition corresponds to a periodic orbit or fixed points in phase space
- For a non-linear kick, the radius will change by $\delta(\sqrt{2J})$ and the particles stop lying on circles



Example: Single Octupole

- Simple **map** with single octupole kick with integrated strength k_3 + rotation with phase advances (μ_x, μ_y)

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def OctupoleMap(k3,x,px,y,py):
  x1  = x
  px1 = px - k3*(x**3-3*x*y**2)
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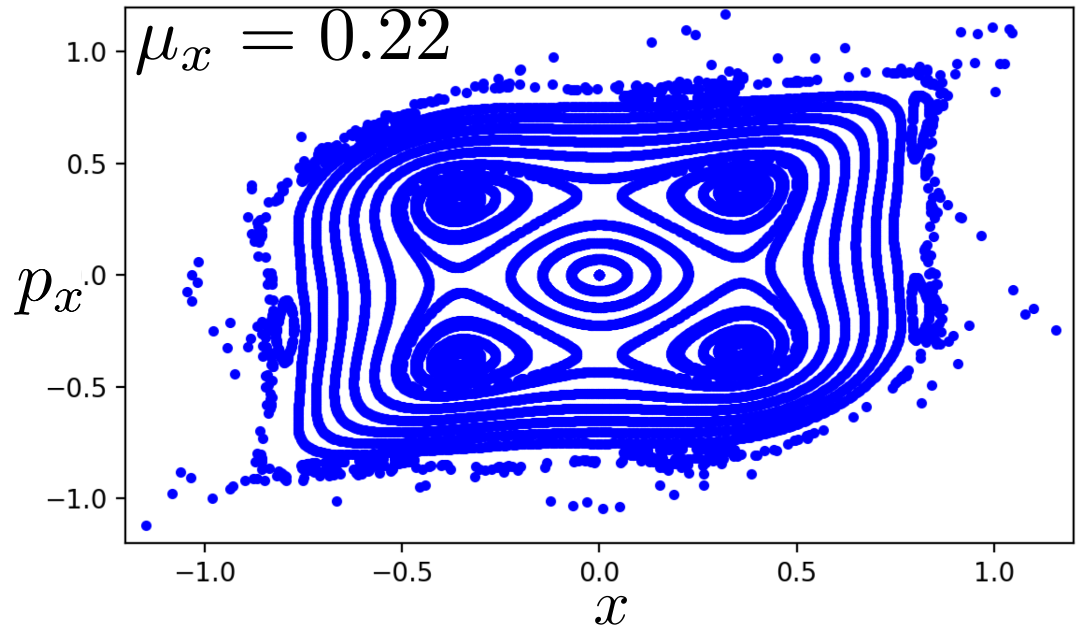

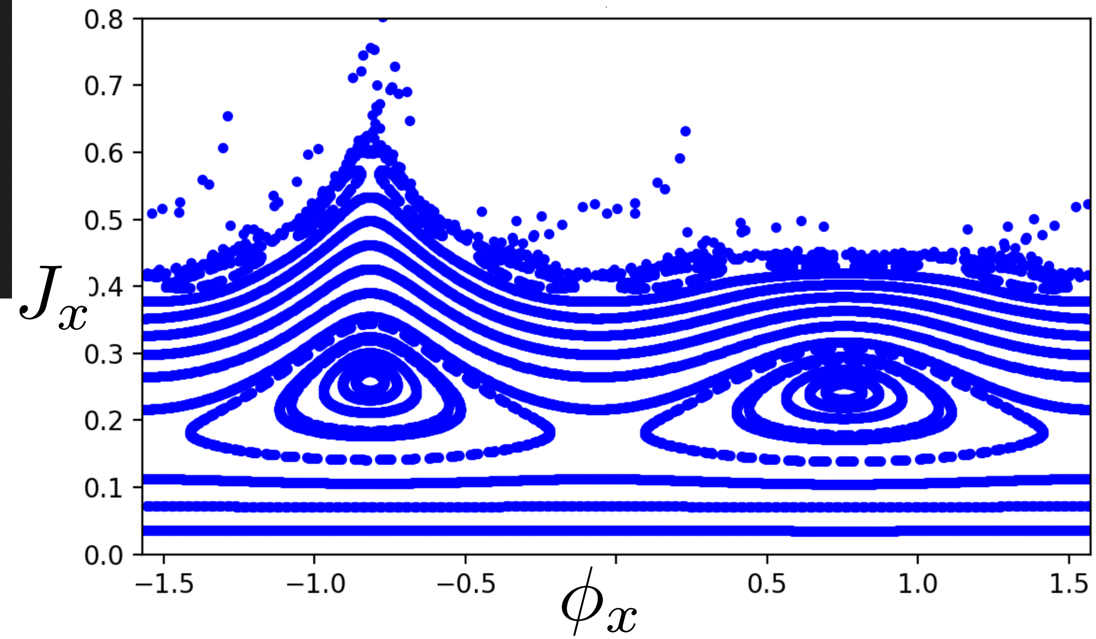

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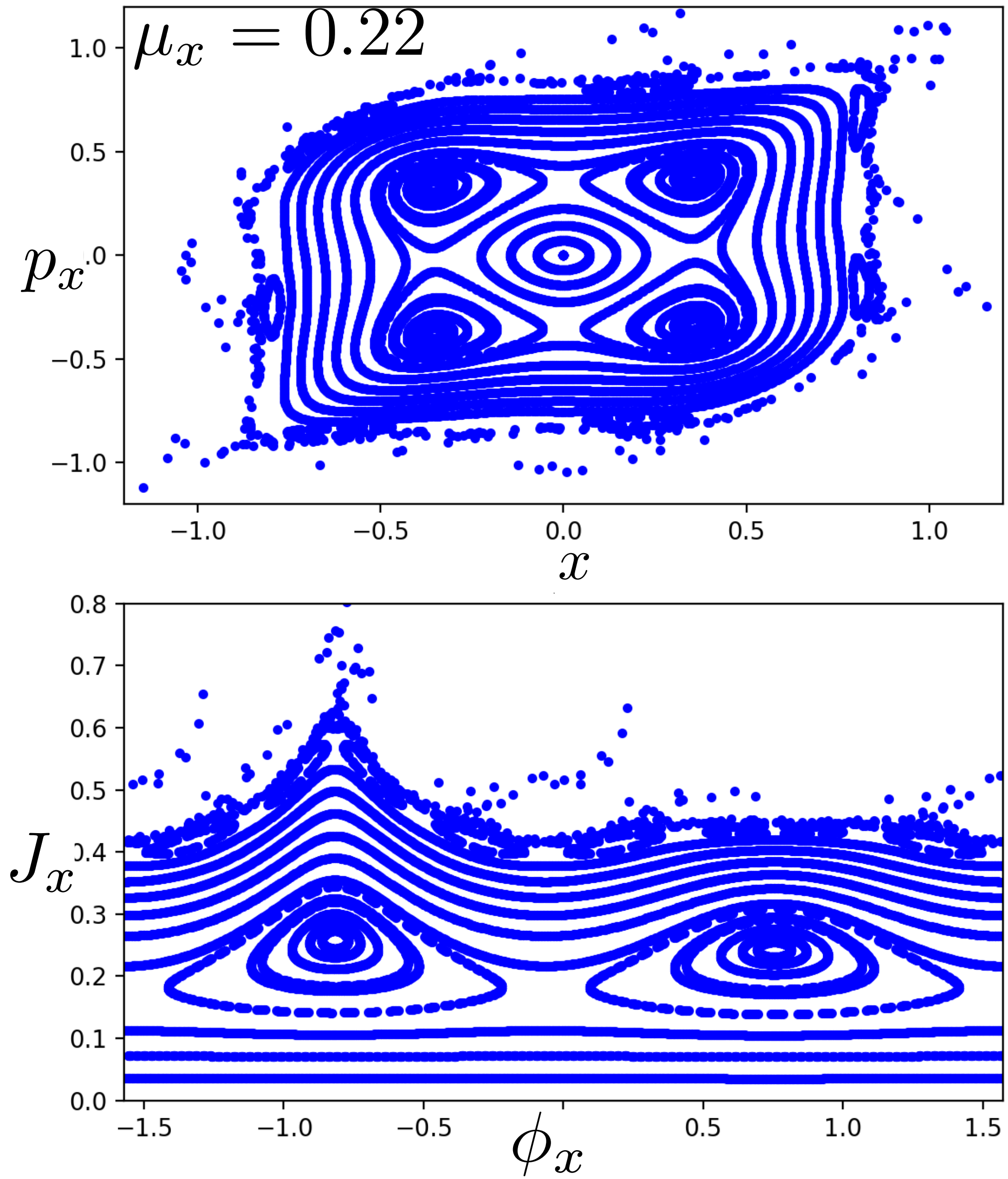
- **Appearance of invariant curves** (“distorted” circles), where “action” is an integral of motion
- **Resonant islands** with stable and separatrices with unstable fixed points
- **Chaotic motion**
- Electromagnetic fields coming from multi-pole expansions (polynomials) do not bound phase space and chaotic trajectories may eventually escape to infinity (**Dynamic Aperture**)
- For some fields like beam-beam and space-charge this is not true, i.e. chaotic motion leads to halo formation



Motion close to a resonance

- The vicinity of a resonance $n_1\omega_1 + n_2\omega_2 = 0$, can be studied through **secular perturbation theory** (see appendix) or transforming the 1-turn map (see Etienne's lectures)
- A canonical transformation is applied such that the new variables are in a frame remaining **on top** of the **resonance**
- If one frequency is slow, one can average the motion and remain only with a **1 degree of freedom Hamiltonian** which looks like the one of the **pendulum**
- Thereby, one can find the location and nature of the fixed points measure the width of the resonance

Fixed points for general multi-pole

- For **any polynomial perturbation** of the form x^k the "resonant" Hamiltonian is written as

$$\hat{H}_2 = \delta J_2 + \alpha(J_2) + J_2^{k/2} A_{kp} \cos(k\psi_2)$$

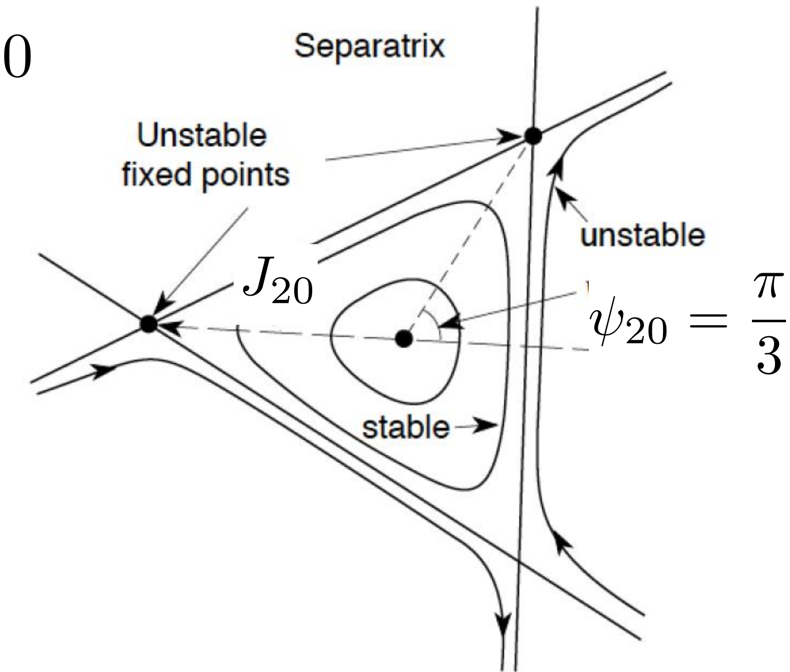
- With the **distance** to the resonance defined as $\nu = \frac{p}{3} + \delta\,, \quad \delta << 1$
- The non-linear shift of the tune is described by the term $\alpha(J_2)$
- The conditions for the fixed points are

$$\sin(k\psi_2) = 0\,, \quad \delta + \frac{\partial\alpha(J_2)}{\partial J_2} + \frac{k}{2} J_2^{k/2-1} A_{kp} \cos(k\psi_2) = 0$$

- There are fixed points for which $\cos(k\psi_{20}) = -1$ and the fixed points are **stable** (elliptic). They are surrounded by ellipses
- There are also fixed points for which $\cos(k\psi_{20}) = 1$ and the fixed points are **unstable** (hyperbolic). The trajectories are hyperbolas

Fixed points for 3rd order resonance

- The Hamiltonian for a sextupole close to a third order resonance is $\hat{H}_2 = \delta J_2 + J_2^{3/2} A_{3p} \cos(3\psi_2)$
- Note the absence of the non-linear tune-shift term (in this 1st order approximation!)
- By setting the Hamilton's equations equal to zero, three fixed points can be found at $\psi_{20} = \frac{\pi}{3}, \frac{3\pi}{3}, \frac{5\pi}{3}, \quad J_{20} = \left(\frac{2\delta}{3A_{3p}}\right)^2$
- For $\frac{\delta}{A_{3p}} > 0$ all three points are unstable
- Close to the elliptic one at $\psi_{20} = 0$ the motion in phase space is described by circles that they get more and more distorted to end up in the "triangular" **separatrix** uniting the unstable fixed points
- The tune separation from the resonance is $\delta = \frac{3A_{3p}}{2} J_{20}^{1/2}$



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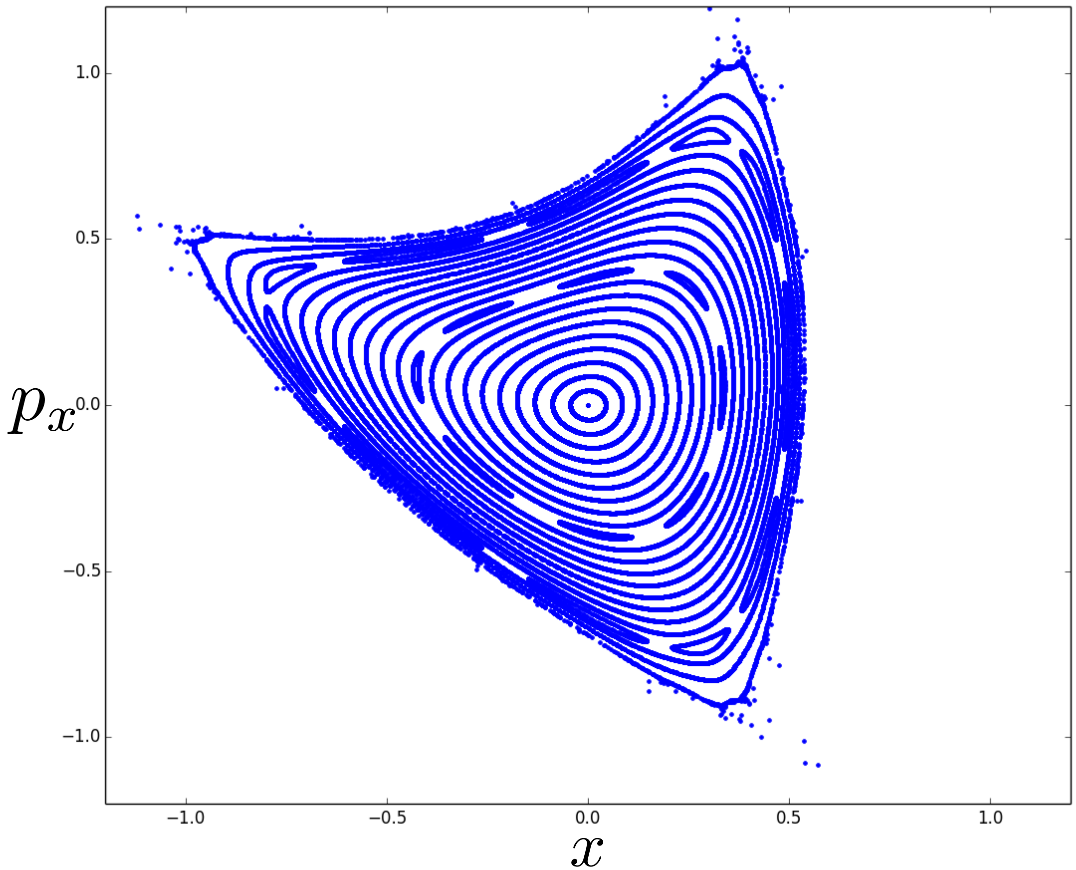
- Simple **map** with single **sextupole kick** with integrated strength k_2 + rotation with phase advances (μ_x, μ_y)

```
def SextupoleMap(k2,x,px,y,py):
  x1  = x
  px1 = px - k2*(x**2-y**2)
  y1  = y
  py1 = py - k2*(-2*x*y)
  return x1,px1,y1,py1

def Rotation(mux,muy,x,px,y,py):
  x1  = cos(mux)*x+sin(mux)*px
  px1 =-sin(mux)*x+cos(mux)*px
  y1  = cos(muy)*y+sin(muy)*py
  py1 =-sin(muy)*y+cos(muy)*py
  return x1,px1,y1,py1
```

- Restrict motion in (x, p_x) plane i.e. $y_0 = p_{y0} = 0$
- Iterate for a number of "turns" (here 1000)

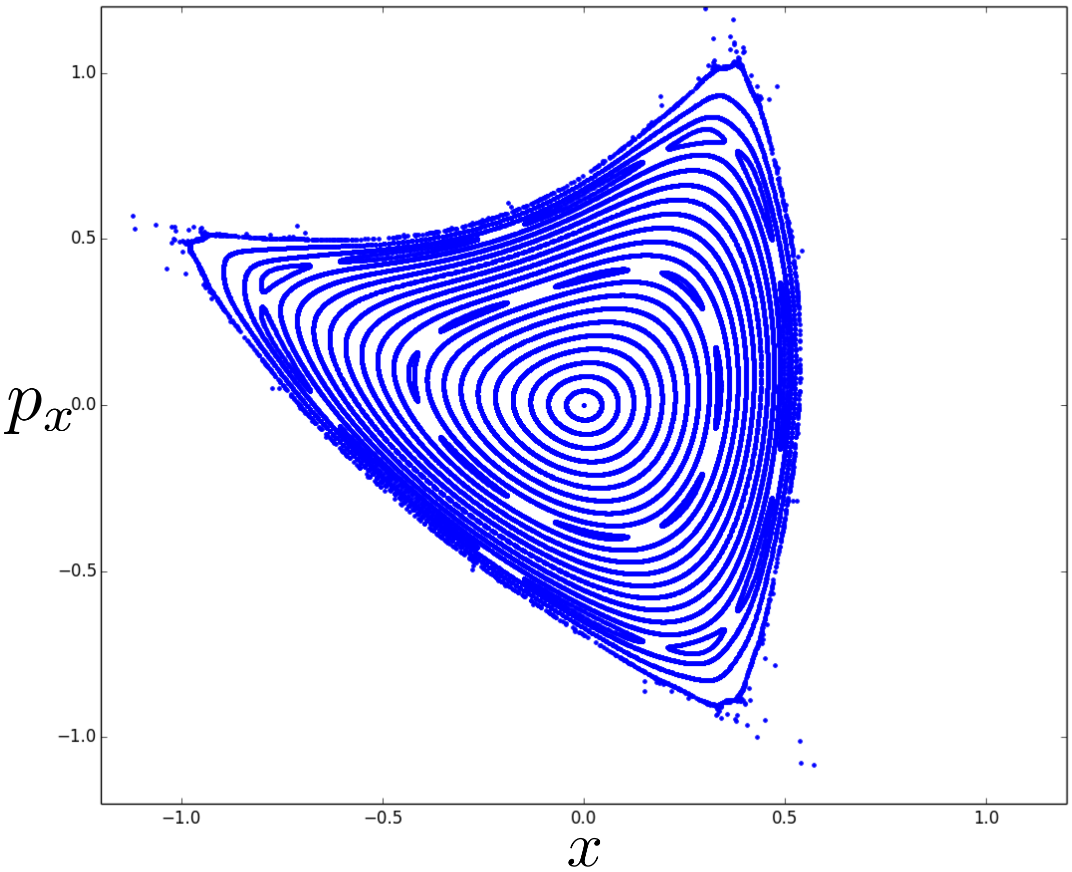
$$\mu_x = 0.38$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance

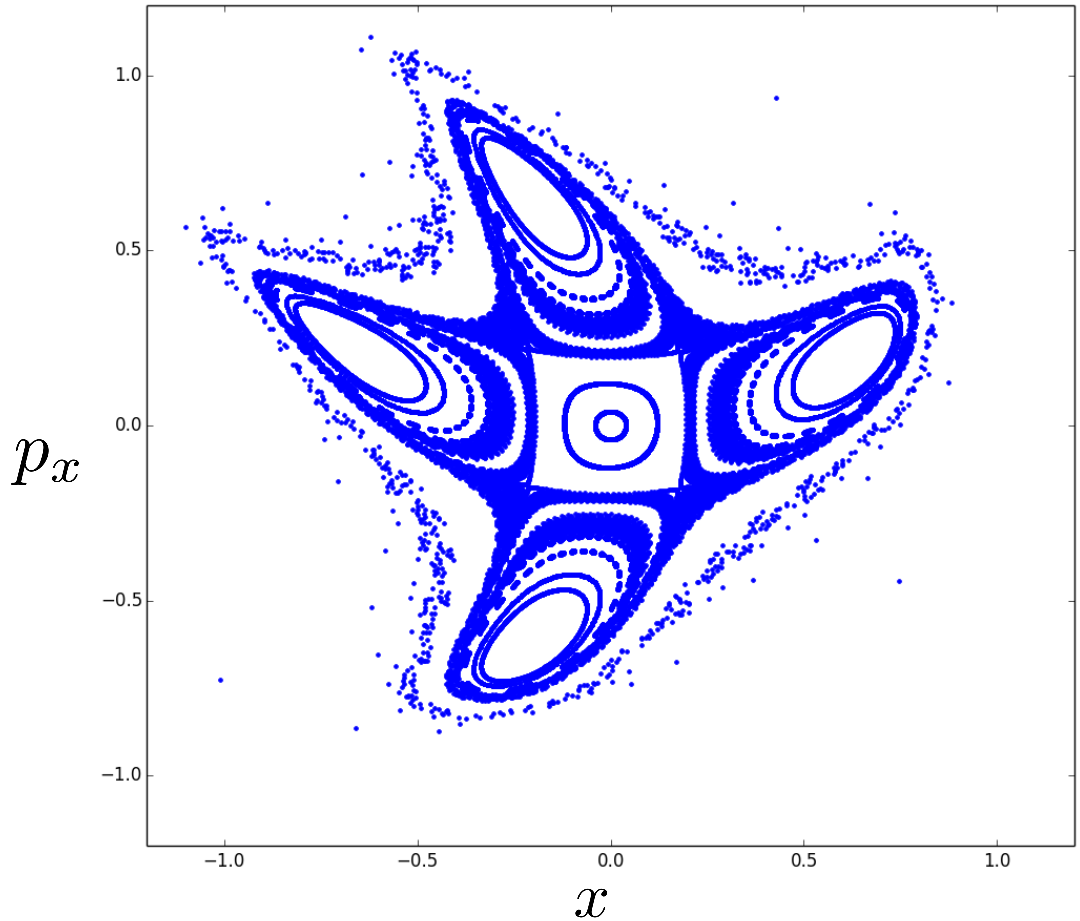
$$\mu_x = 0.38$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance

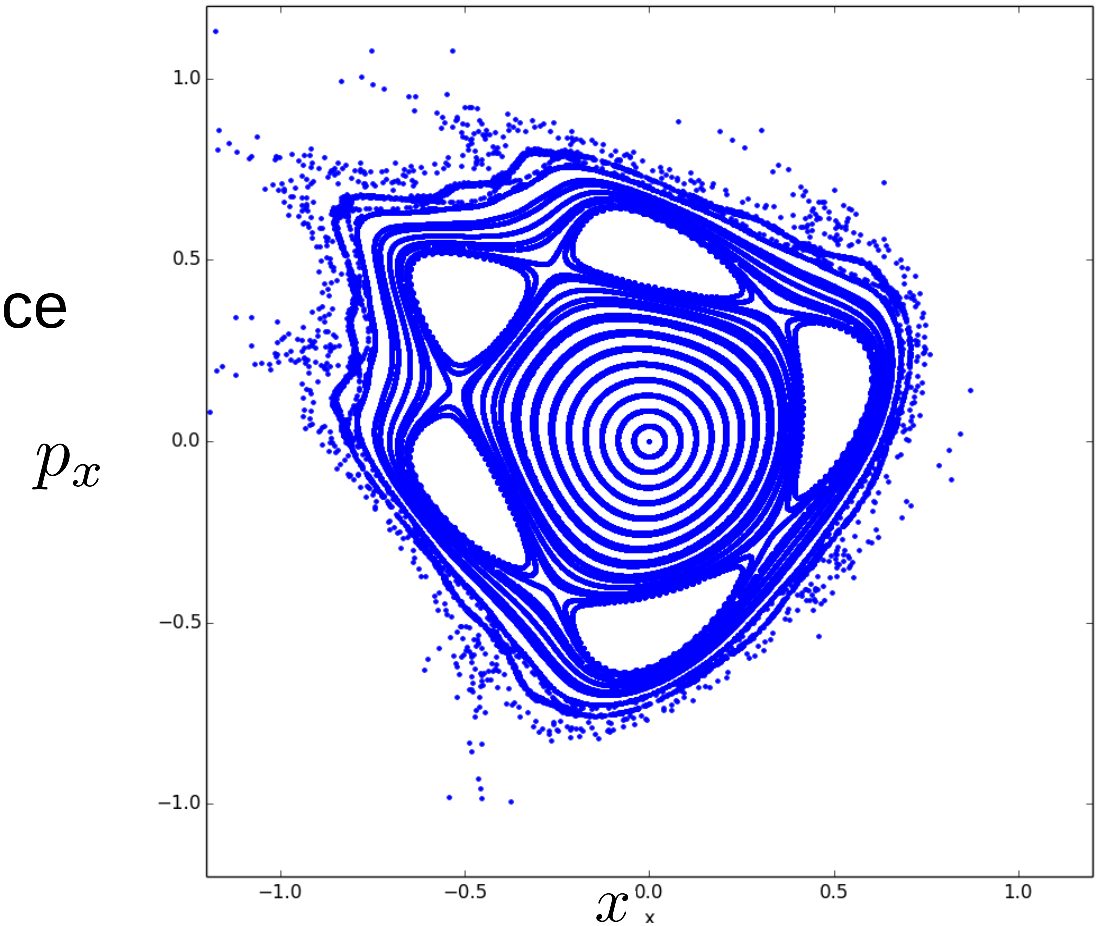
$$\mu_x = 0.253$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance
- … and 5th order resonance

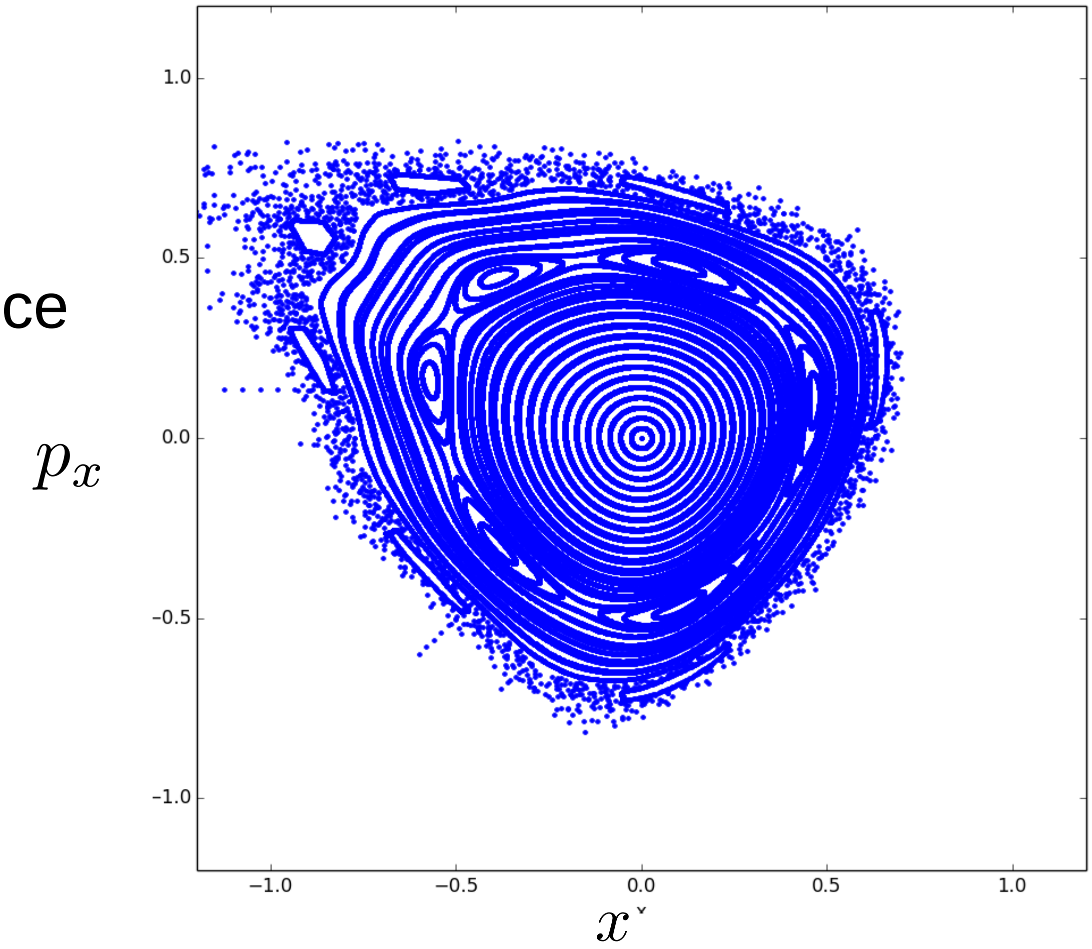
$$\mu_x = 0.21$$



Example: Single Sextupole

- Appearance of 3rd order resonance for certain phase advance
- … but also 4th order resonance
- … and 5th order resonance
- … and 6th order and 7th order and several higher orders…

$$\mu_x = 0.18$$



Fixed points for an octupole

- The resonant Hamiltonian close to the **4th order resonance** is written as

$$\hat{H}_2 = \delta J_2 + cJ_2^2 + J_2^2 A_{4p} \cos(4\psi_2)$$

- The **fixed points** are found by taking the derivative over the two variables and setting them to zero, i.e.

$$\sin(4\psi_2) = 0 \ , \quad \delta + 2cJ_2 + 2J_2 A_{kp} \cos(4\psi_2) = 0$$

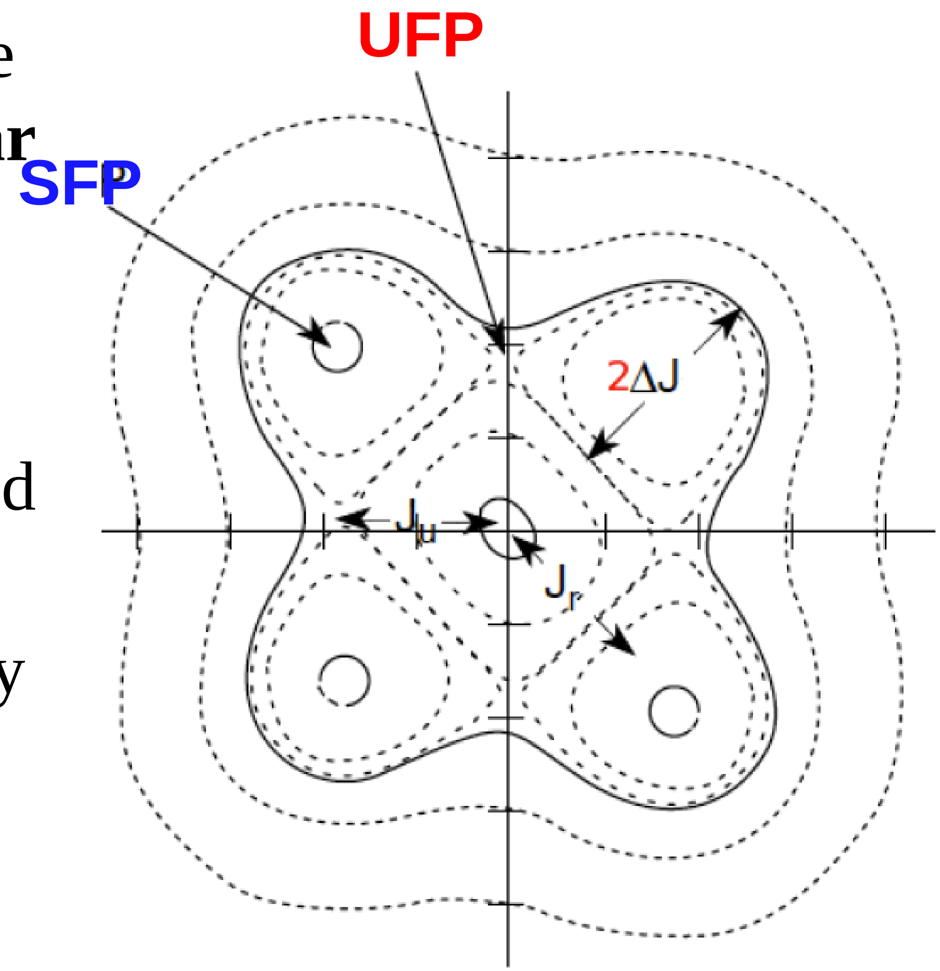
- The fixed points are at

$$\psi_{20} = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$$

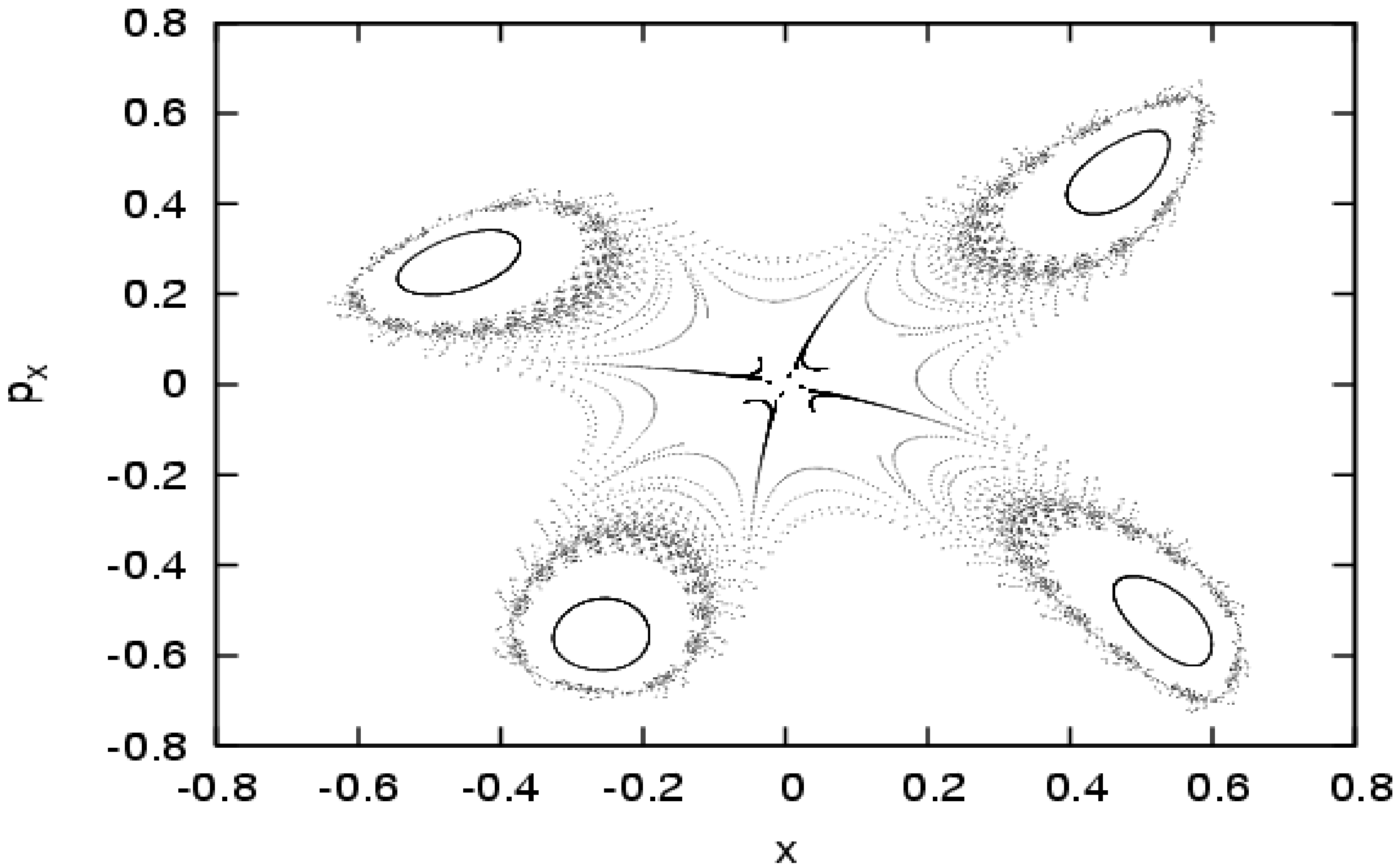
- For **half** of them, there is a minimum in the potential as $\cos(4\psi_{20}) = -1$ and they are **elliptic** and **half** of them they are **hyperbolic** as $\cos(4\psi_{20}) = 1$

Topology of an octupole resonance

- **Regular motion** near the center, with curves getting more deformed towards a **rectangular shape**
- The **separatrix** passes through 4 unstable fixed points, but motion seems well contained
- **Four stable fixed points** exist and they are surrounded by stable motion (**islands of stability**)
- Question: Can the **central fixed point** become **hyperbolic** (answer in the appendix)

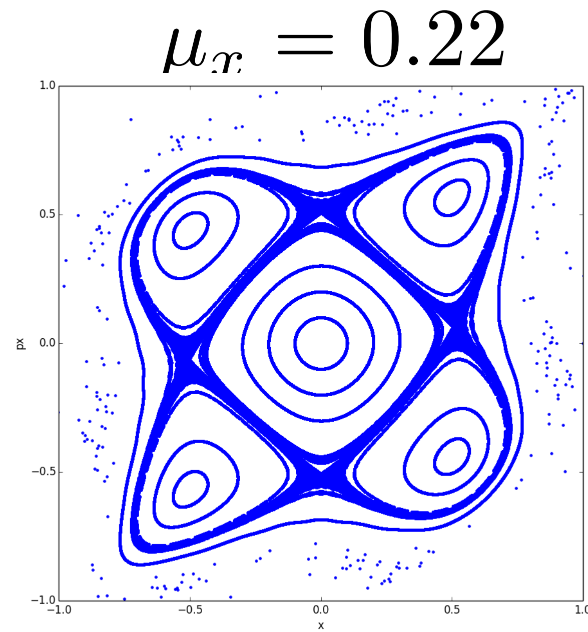


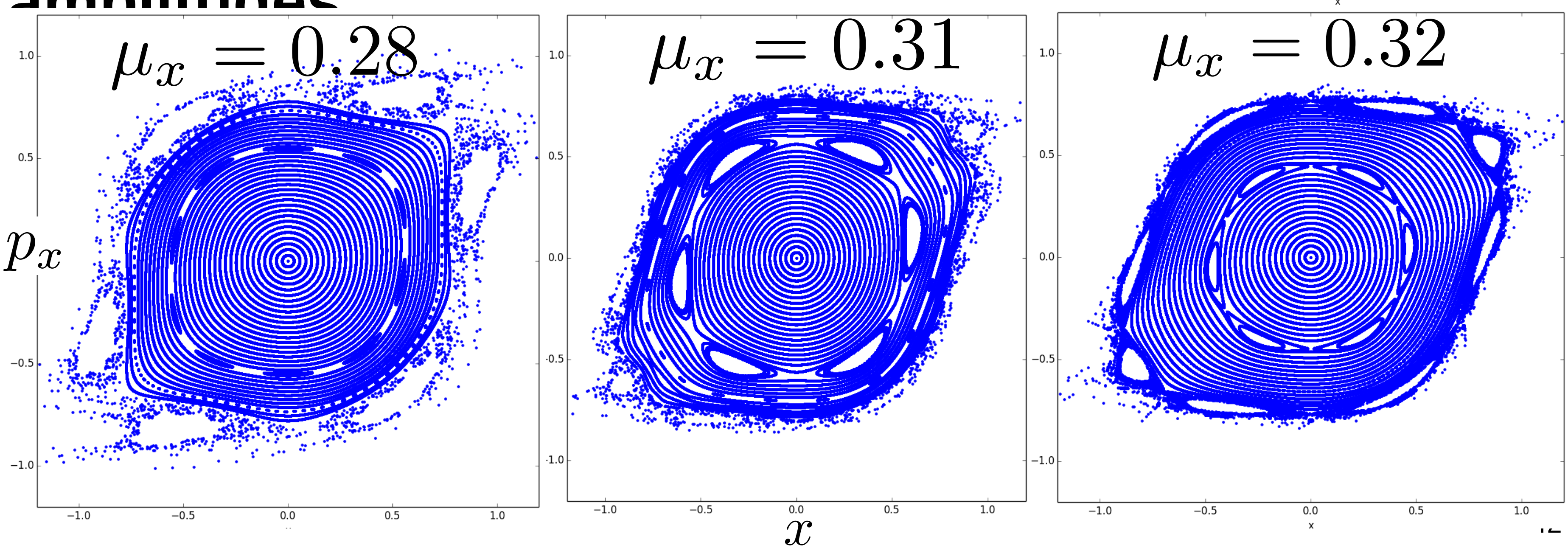
- Now, if $c = 0$ the solution for the action is $J_{20} = 0$
- So there is **no minima** in the potential, i.e. the central fixed point is **hyperbolic**



Single Octupole

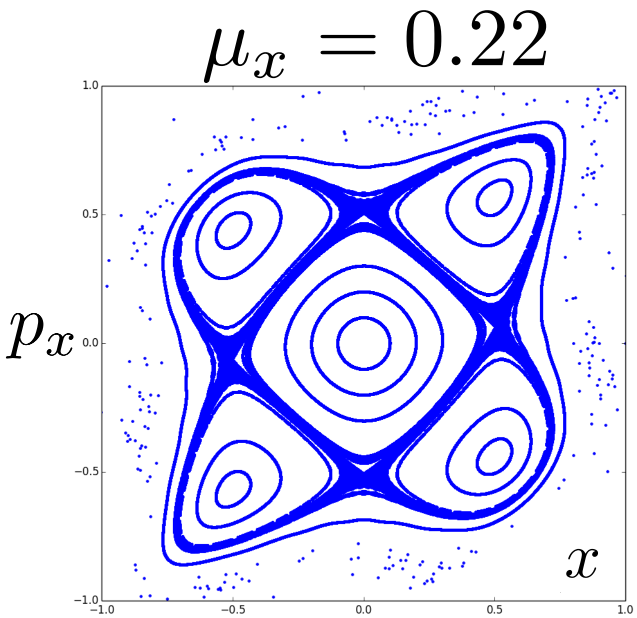
- As for the sextupole, the octupole can excite any resonance
- Multi-pole magnets can excite **any resonance** order
- It depends on the **tunes**, **strength** of the **magnet** and particle **amplitudes**

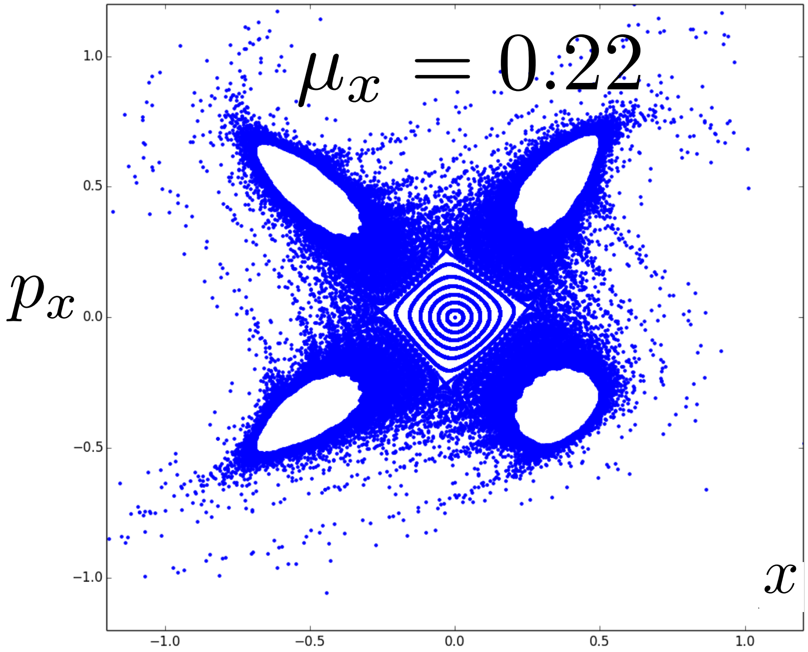




Single Octupole + Sextupole

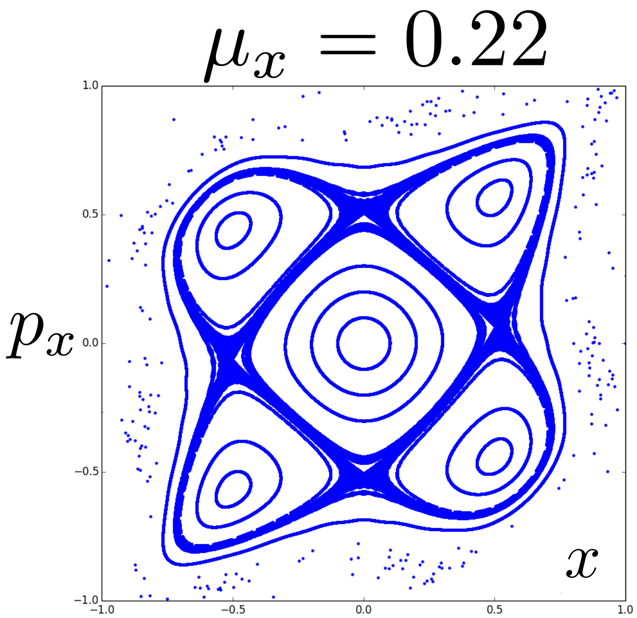
- Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4th order resonance

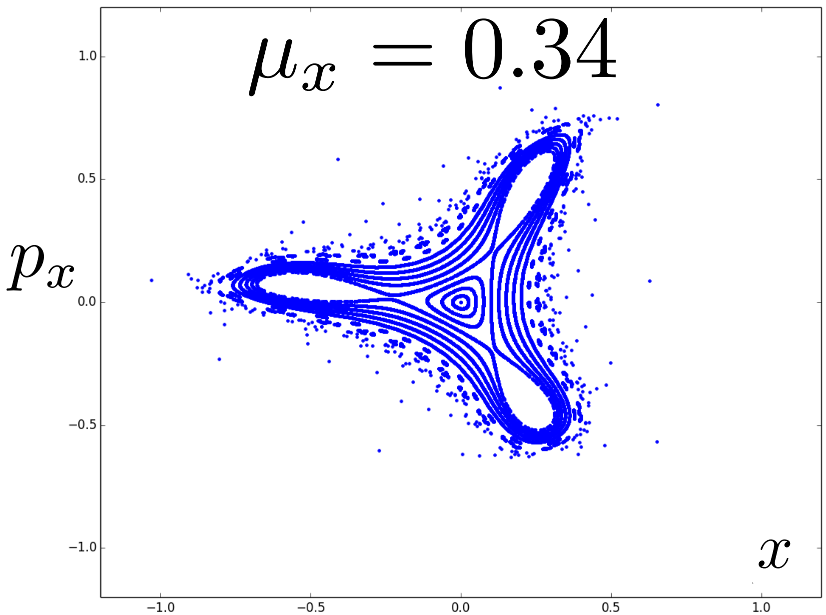


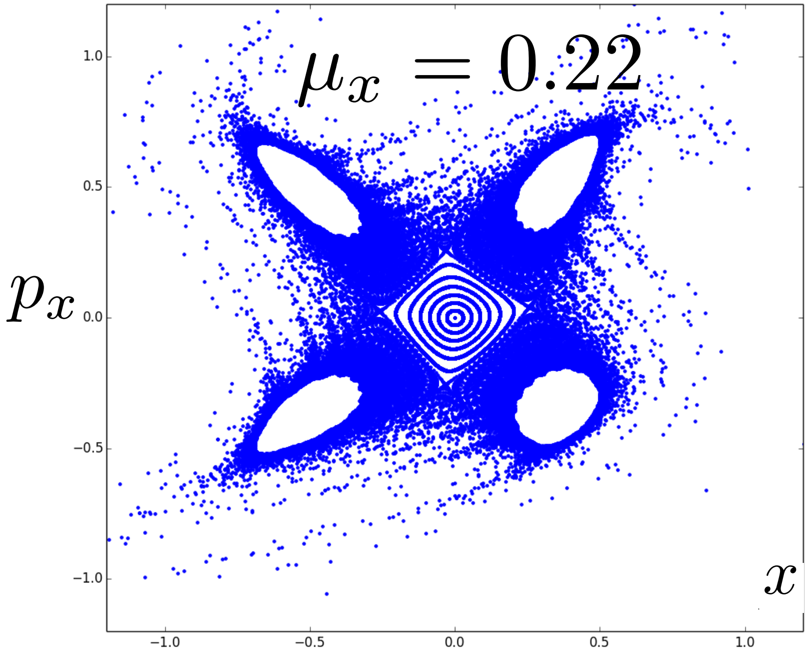


Single Octupole + Sextupole

- Adding a sextupole and an octupole increases the chaotic motion region, when close to the 4th order resonance
- But also allows the appearance of **3rd order resonance stable fixed points**



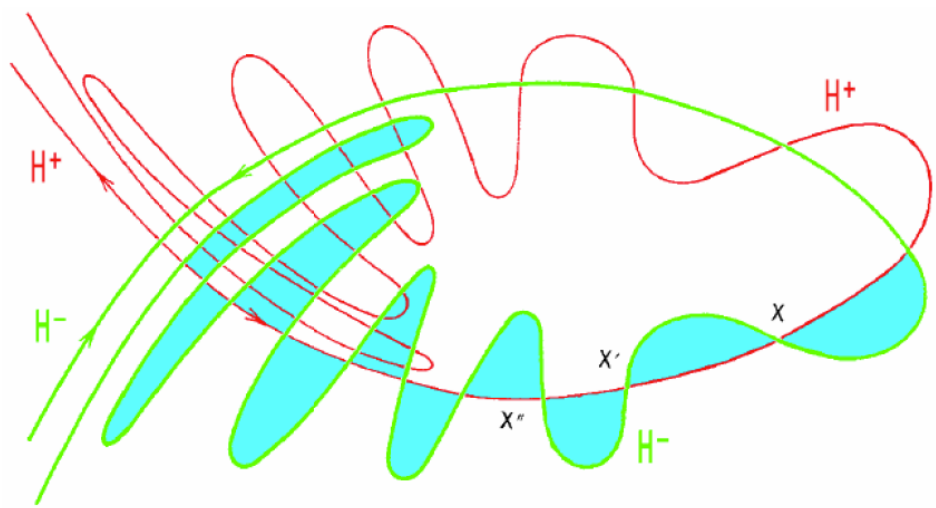


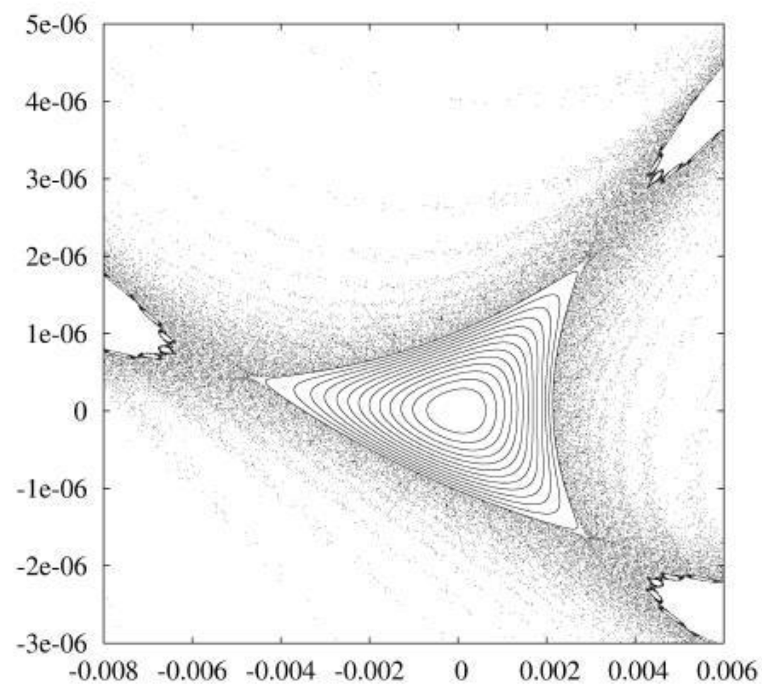


Onset of chaos

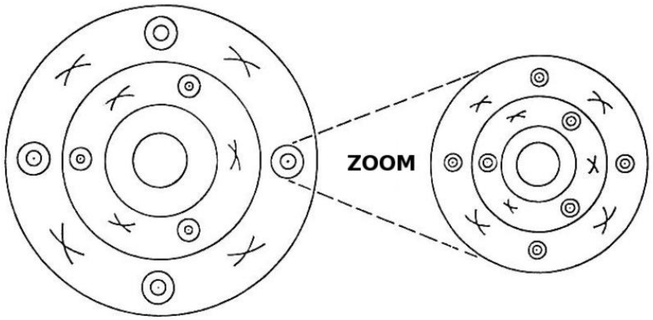
Path to chaos

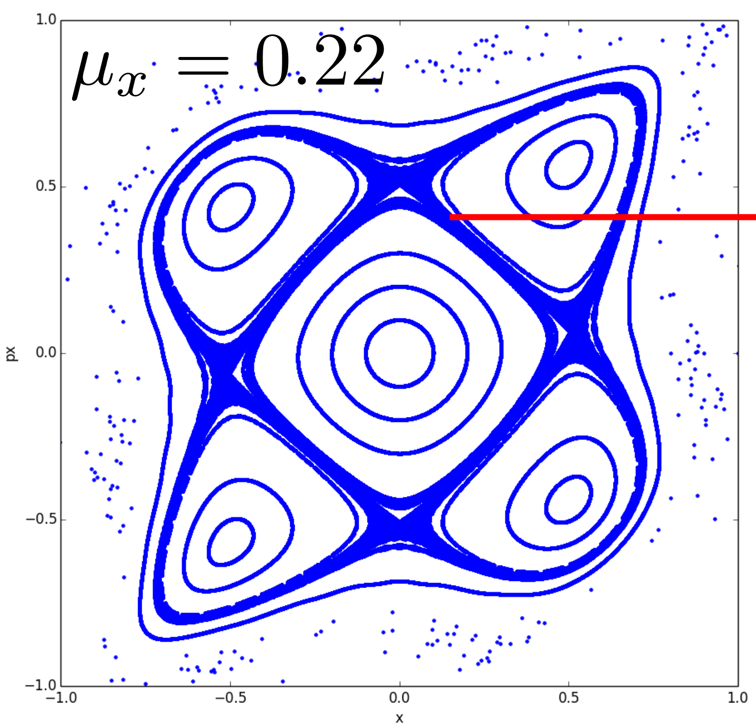
- When **perturbation** becomes **higher**, motion around the **separatrix** becomes **chaotic** (producing **tongues** or **splitting** of the separatrix)
- **Unstable** fixed points are indeed the **source of chaos** when a perturbation is added

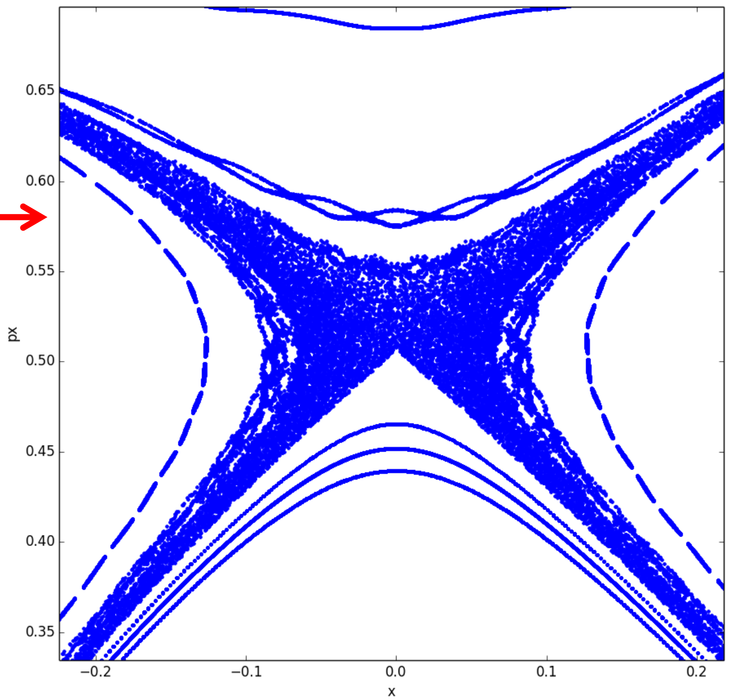




- **Poincare-Birkhoff** theorem states that under **perturbation** of a **resonance** only an **even number** of **fixed points** survives (**half stable** and the other **half unstable**)
- **Themselves** get **destroyed** when perturbation gets **higher**, etc. (**self-similar** fixed points)
- Resonance **islands grow** and **resonances** can **overlap** allowing diffusion of particles



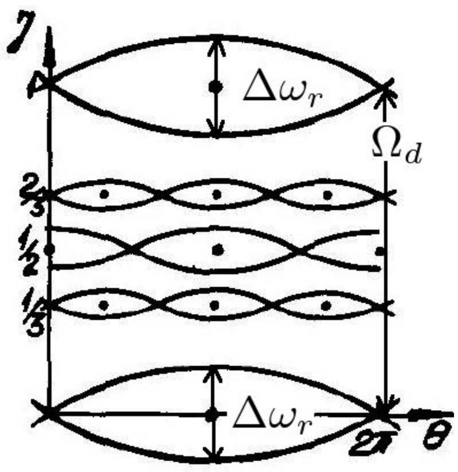




Resonance overlap criterion

- When perturbation grows, the resonance island width grows
- **Chirikov** (1960, 1979) proposed a **criterion** for the **overlap** of two **neighboring resonances** and the onset of orbit diffusion
- The **distance** between two resonances is $\delta\hat{J}_{1\ n,n'} = \frac{2\left(\frac{1}{n_1+n_2} - \frac{1}{n'_1+n'_2}\right)}{\left|\left.\frac{\partial^2 \bar{H}_0(\hat{\mathbf{J}})}{\partial \hat{J}_1^{\,2}}\right|_{\hat{J}_1=\hat{J}_{10}}\right|}$
- The **simple overlap criterion** is

$$\Delta\hat{J}_{n\ max} + \Delta\hat{J}_{n'\ max} \geq \delta\hat{J}_{n,n'}$$



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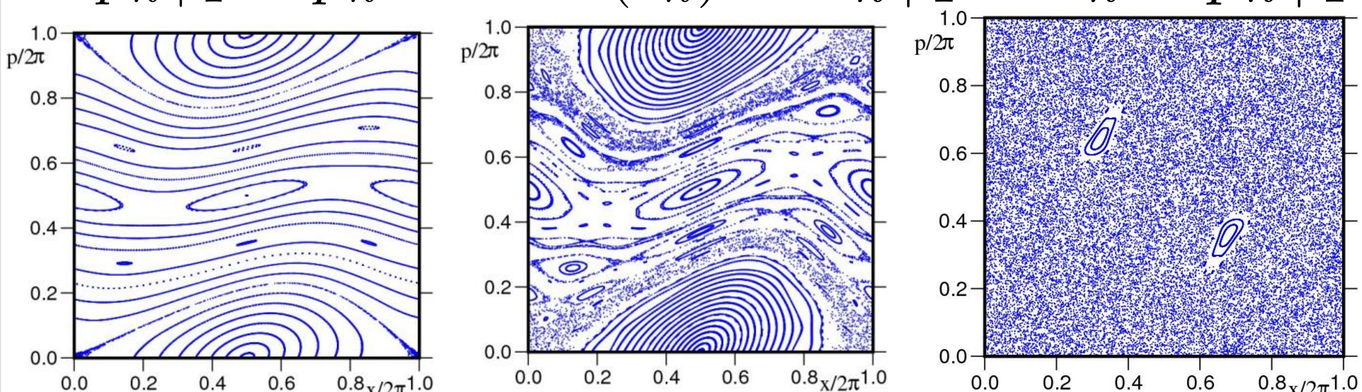
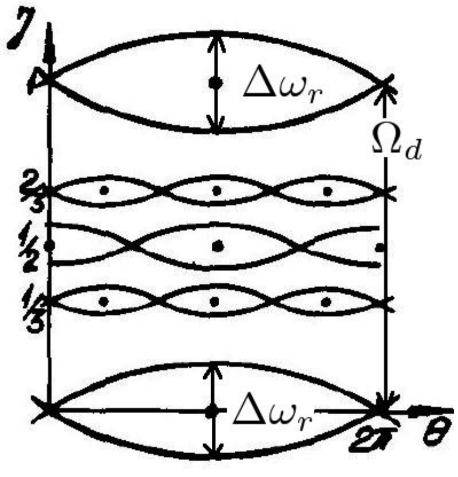
- The **simple overlap criterion** is
  $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \delta \hat{J}_{n,n'}$
- Considering the **width** of **chaotic layer** and **secondary islands**, the "two thirds" rule apply $\Delta \hat{J}_{n\ max} + \Delta \hat{J}_{n'\ max} \geq \frac{2}{3}\delta \hat{J}_{n,n'}$
- Example: **Chirikov's standard map**

$$p_{n+1} = p_n + K\sin(\theta_n) \qquad \theta_{n+1} = \theta_n + p_{n+1}$$





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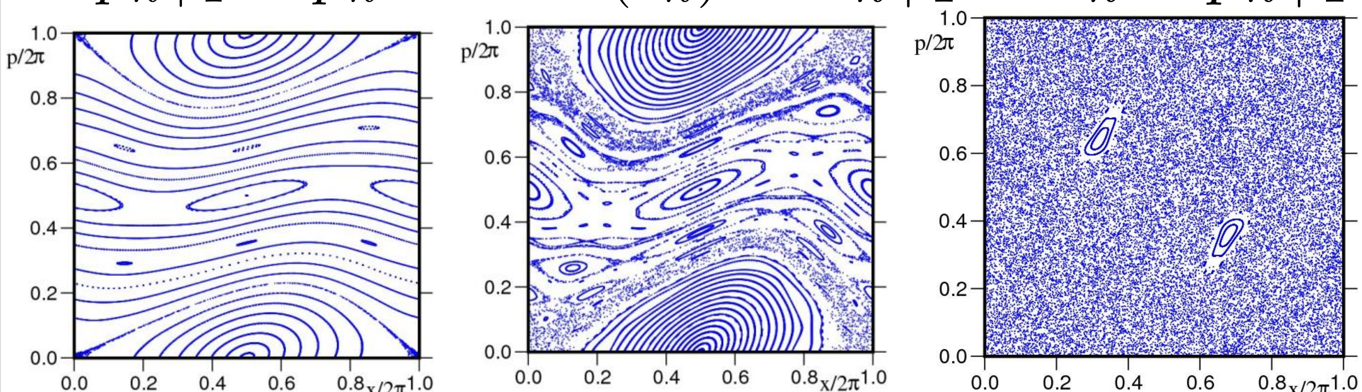
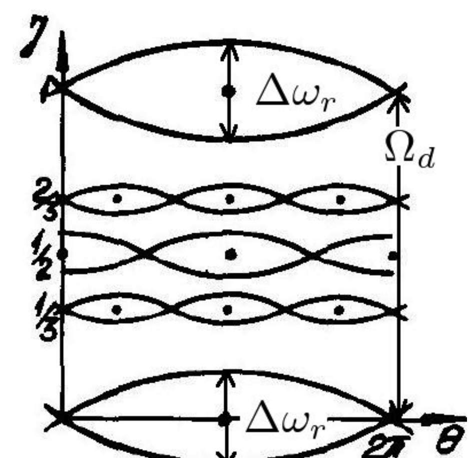
- The main limitation is the **geometrical nature** of the criterion (**difficulty** to be **extended** for **> 2 degrees of freedom**)

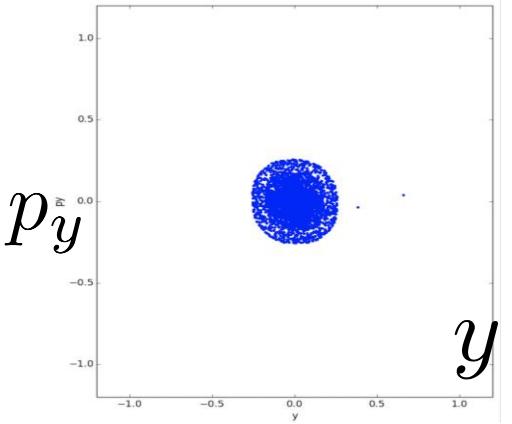
$$p_{n+1} = p_n + K \sin(\theta_n) \qquad \theta_{n+1} = \theta_n + p_{n+1}$$



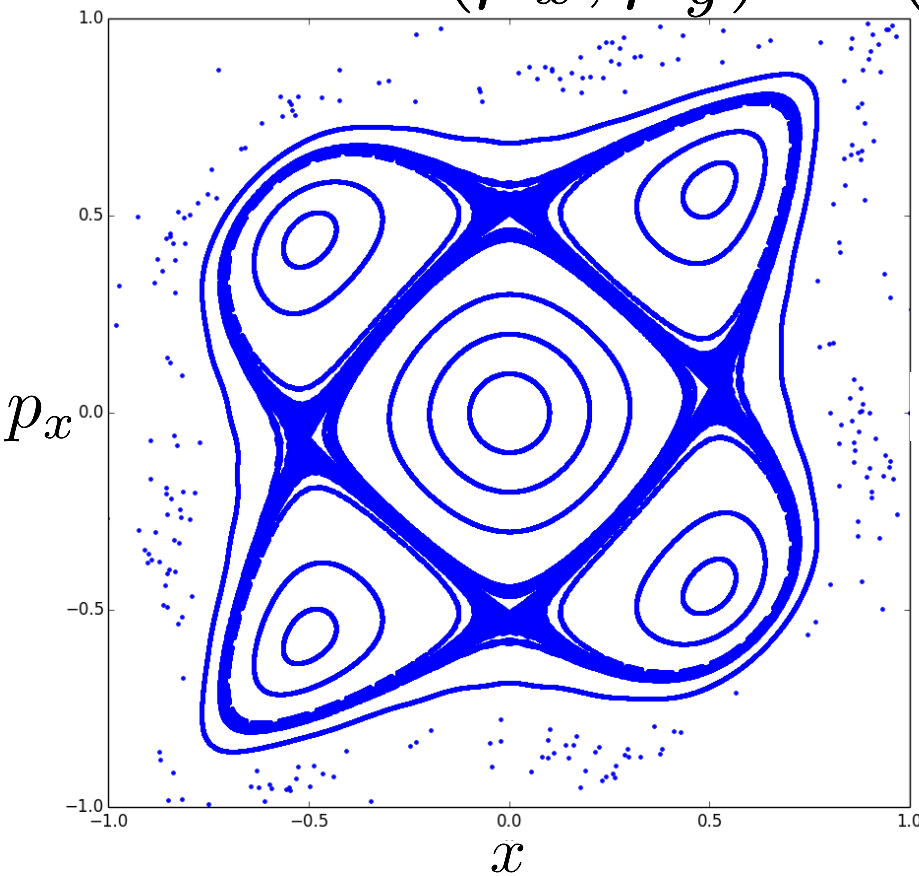


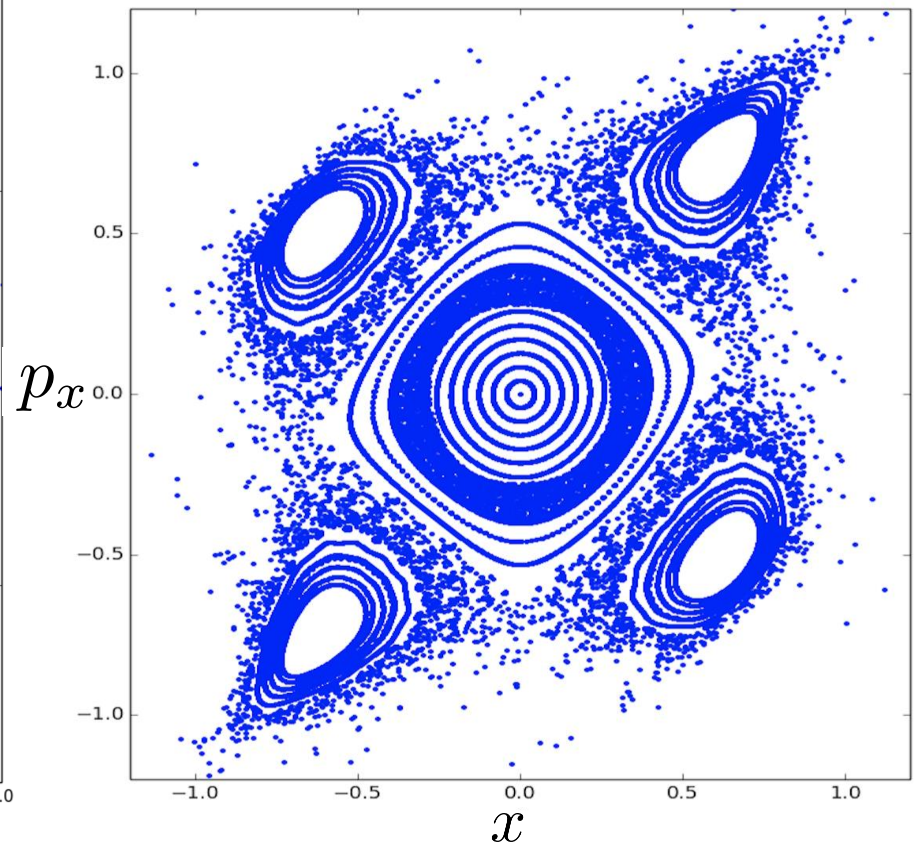
Increasing dimensions

- For $(y_0, p_{y0}) \neq (0,0)$, i.e. by adding another degree of freedom **chaotic motion** is **enhanced**



$$(\mu_x, \mu_y) = (0.22, 0.24)$$

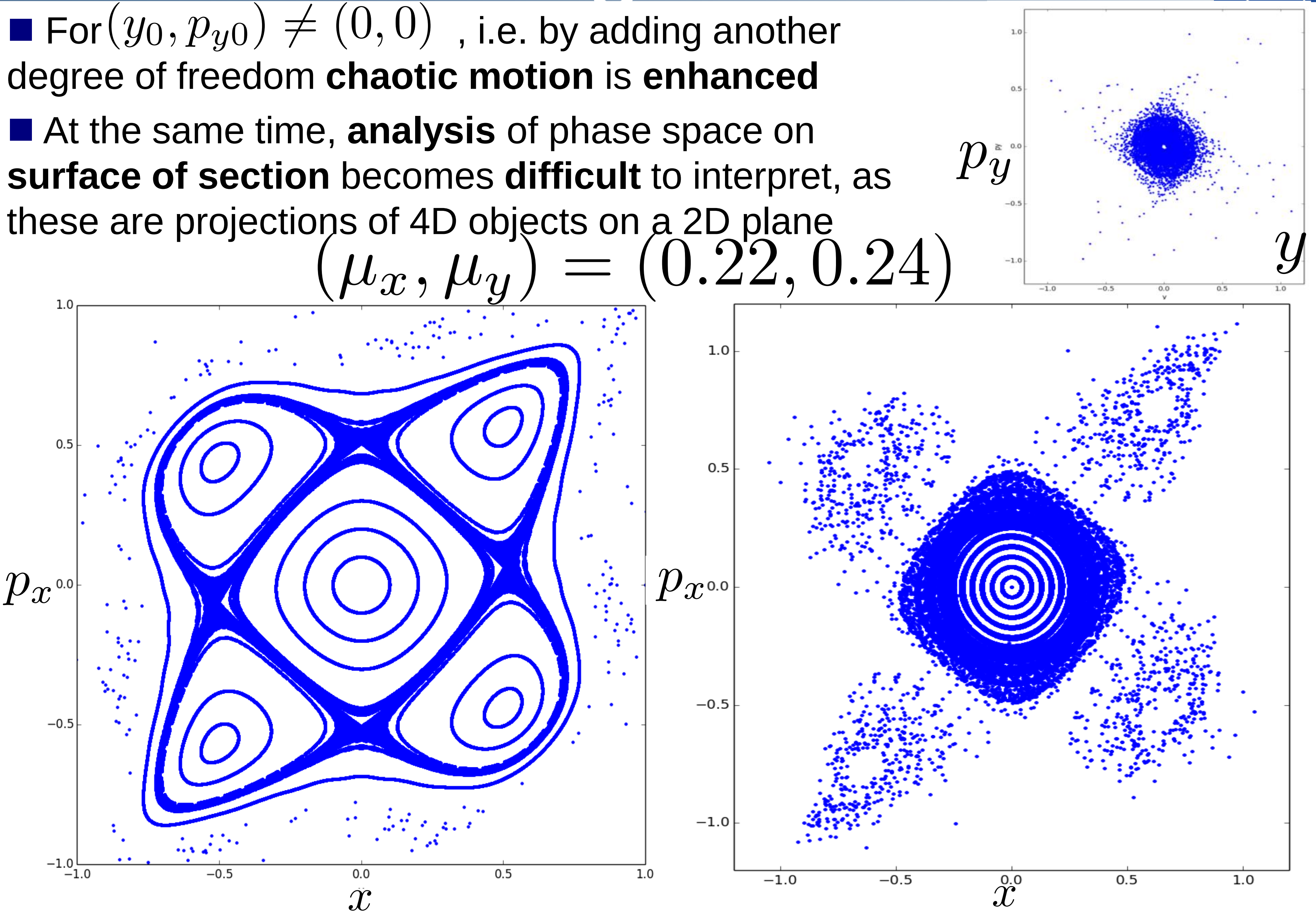




Increasing dimensions

- For $(y_0, p_{y0}) \neq (0,0)$, i.e. by adding another degree of freedom **chaotic motion** is **enhanced**
- At the same time, **analysis** of phase space on **surface of section** becomes **difficult** to interpret, as these are projections of 4D objects on a 2D plane

$$(\mu_x, \mu_y) = (0.22, 0.24)$$



Chaos detection methods

- Computing/measuring **dynamic aperture** (DA) or particle survival

  A. Chao et al., PRL 61, 24, 2752, 1988;
  F. Willeke, PAC95, 24, 109, 1989.

- Computation of Lyapunov exponents

  F. Schmidt, F. Willeke and F. Zimmermann, PA, 35, 249, 1991;
  M. Giovannozi, W. Scandale and E. Todesco, PA 56, 195, 1997

- Variance of unperturbed action (a la Chirikov)

  B. Chirikov, J. Ford and F. Vivaldi, AIP CP-57, 323, 1979
  J. Tennyson, SSC-155, 1988;
  J. Irwin, SSC-233, 1989

- Fokker-Planck diffusion coefficient in actions

  T. Sen and J.A. Elisson, PRL 77, 1051, 1996

- **Frequency map analysis**

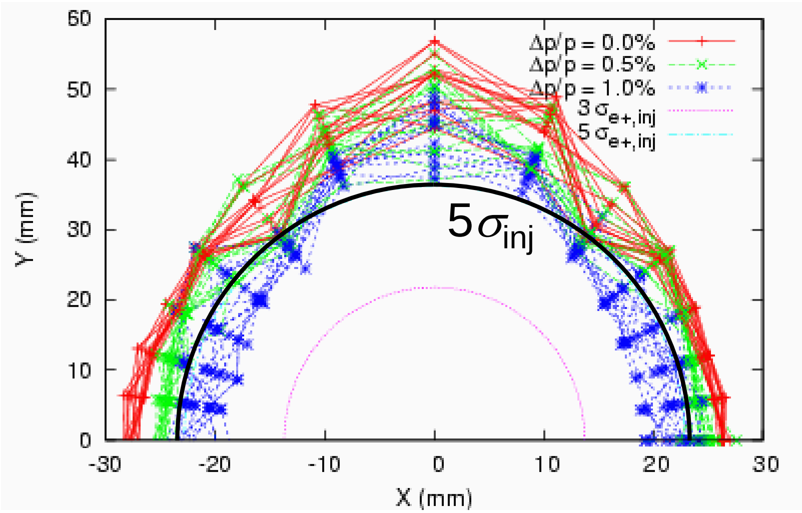
Dynamic aperture

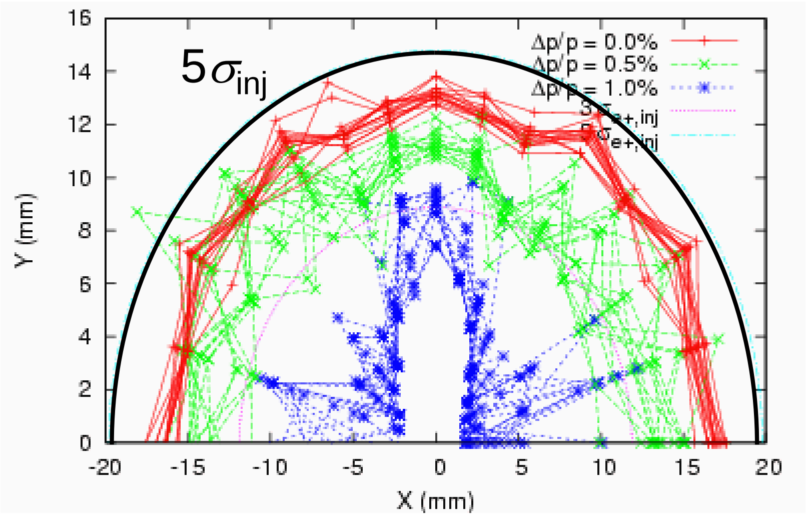
Dynamic Aperture

- The most direct way to evaluate the non-linear dynamics performance of a ring is the computation of **Dynamic Aperture**
- Particle motion due to multi-pole errors is generally non-bounded, so chaotic particles can **escape to infinity**
- This is not true for all non-linearities (e.g. the beam-beam force)
- Need a **symplectic** tracking code to follow particle trajectories (a lot of initial conditions) for a **number of turns** (depending on the given problem) until the particles start getting lost. This **boundary** defines the **Dynamic aperture**
- As multi-pole errors may not be completely known, one has to track through **several machine models** built by **random distribution** of these errors
- One could start with 4D (only transverse) tracking but certainly needs to simulate 5D (constant energy deviation) and finally 6D (synchrotron motion included)

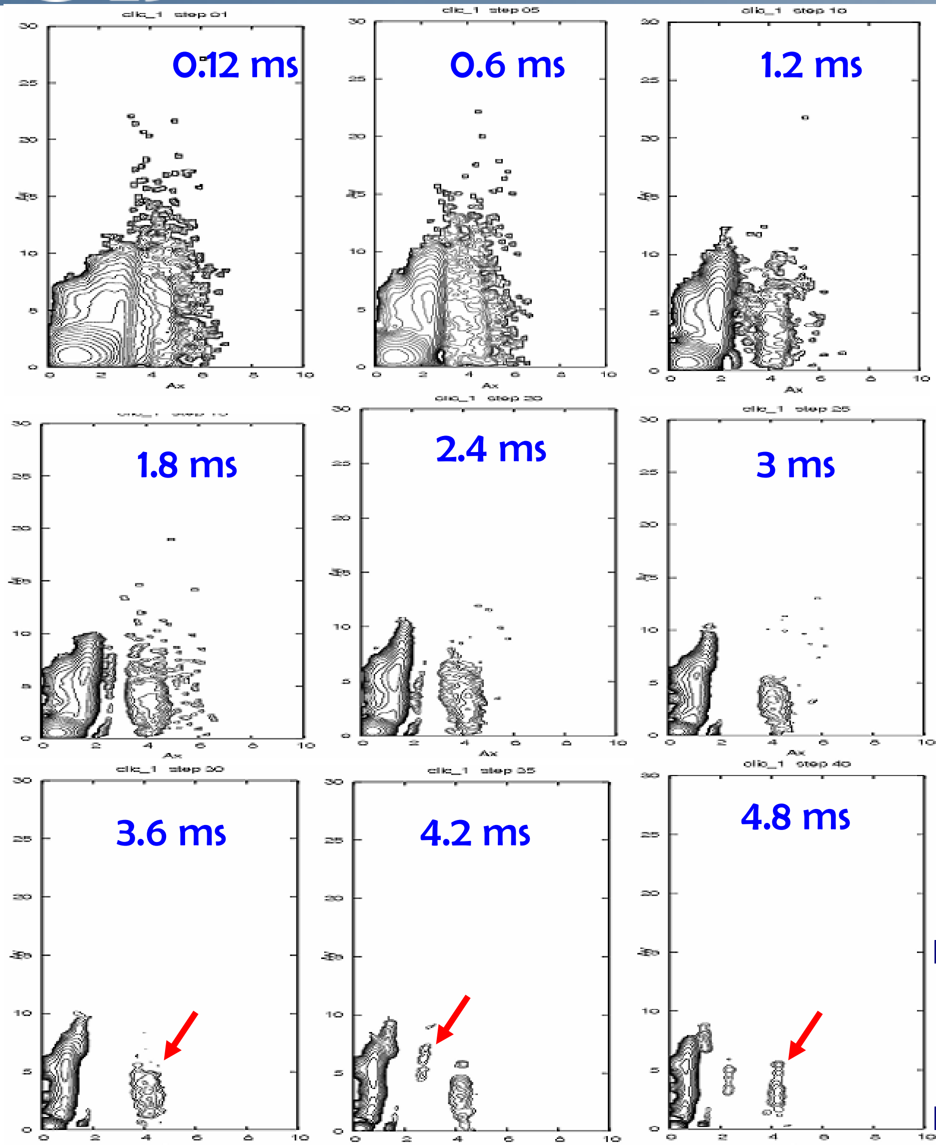
Dynamic Aperture plots

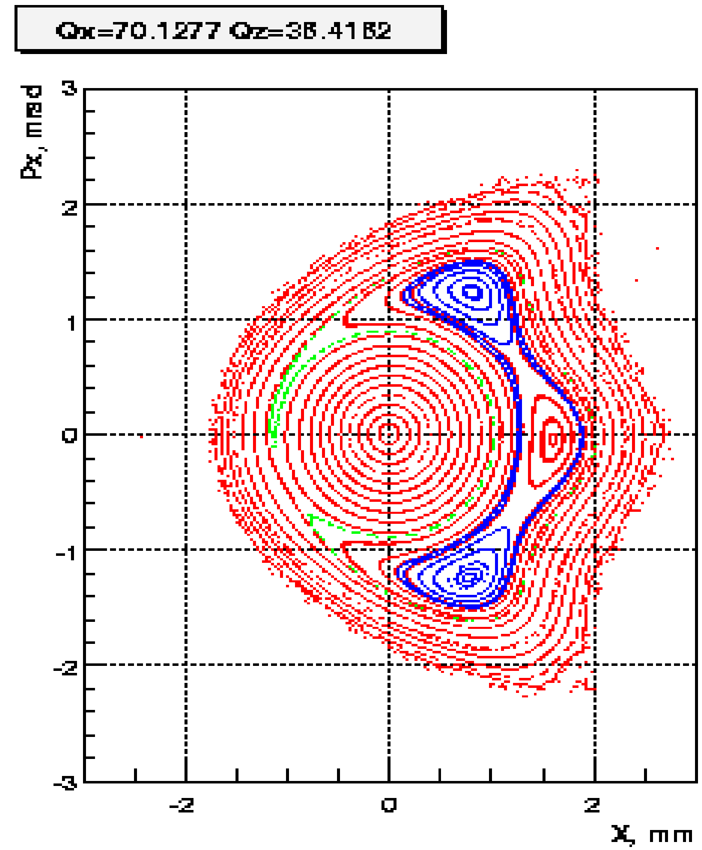
- Dynamic aperture plots show the maximum initial values of stable trajectories in x-y coordinate space at a particular point in the lattice, for a range of energy errors.
  - The beam size can be shown on the same plot.
  - Generally, the goal is to allow some significant margin in the design - the measured dynamic aperture is often smaller than the predicted dynamic aperture.





Dynamic aperture including damping

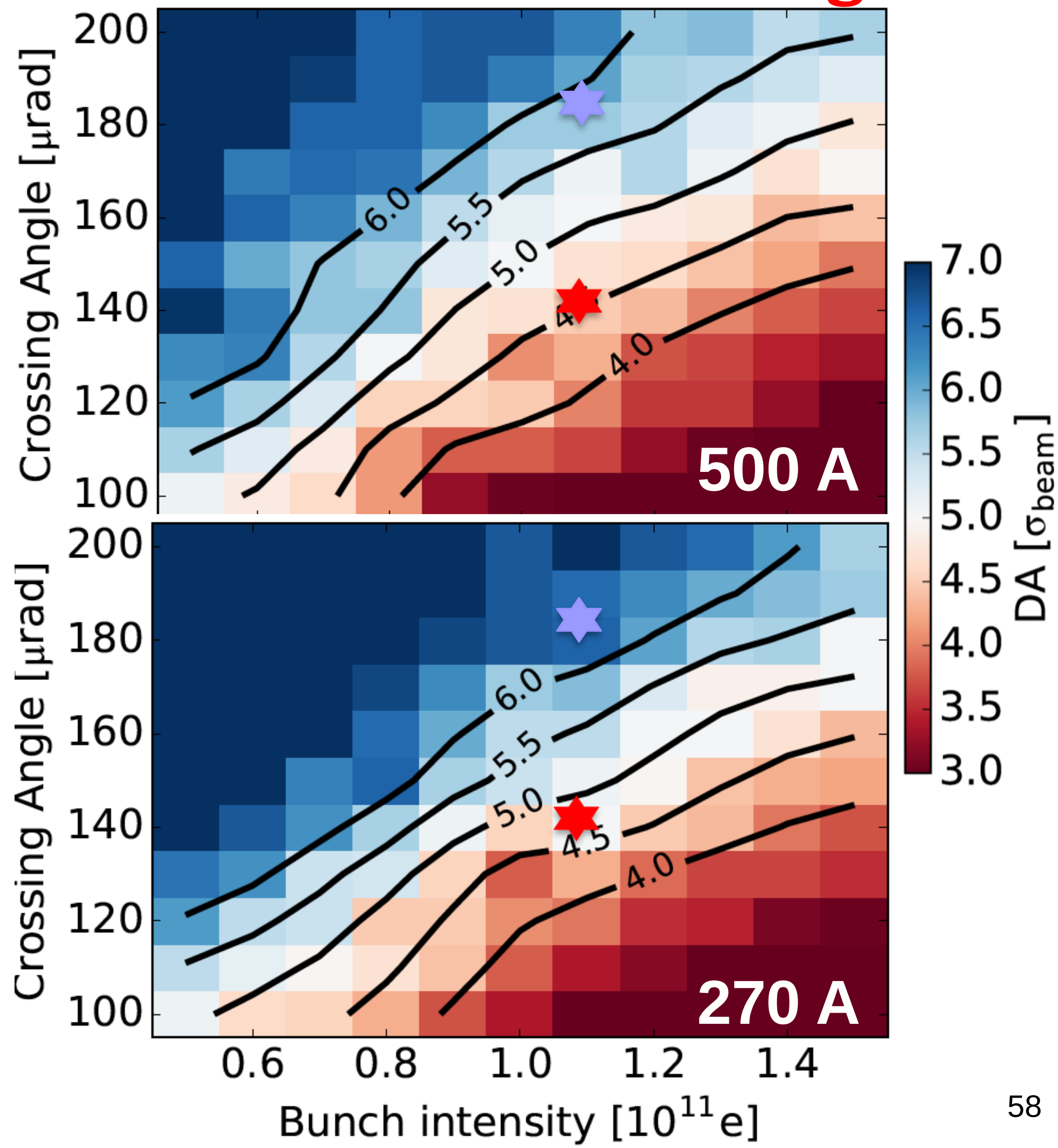




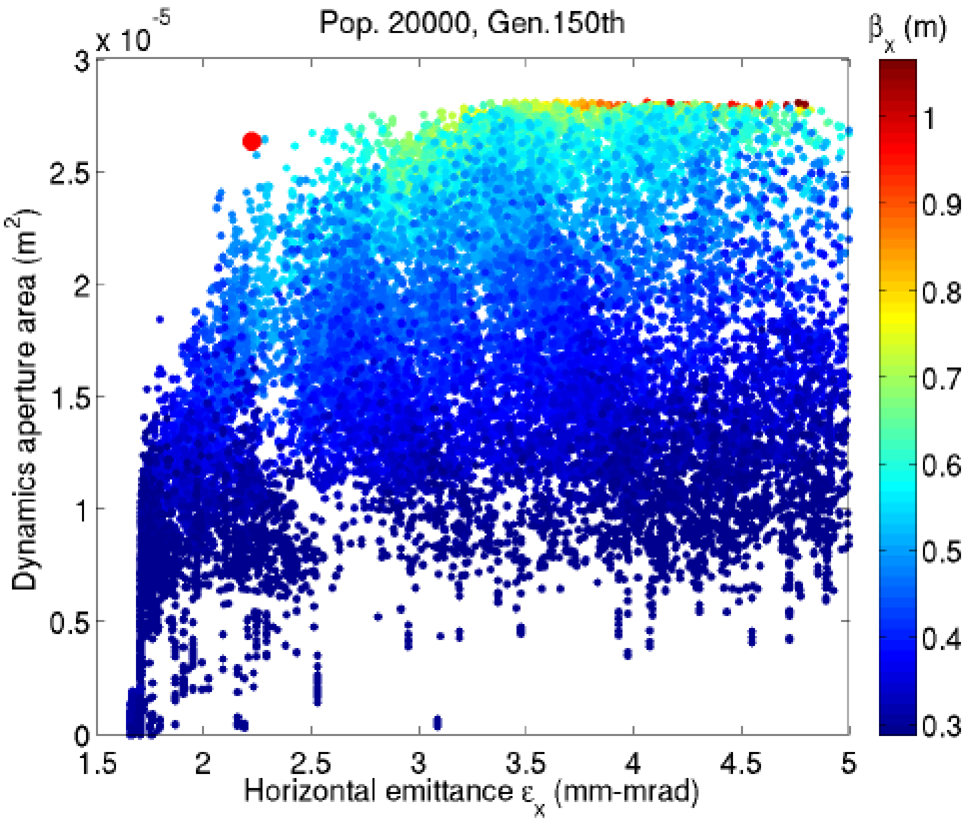
- Including radiation damping and excitation shows that 0.7% of the particles are lost during the damping
- Certain particles seem to damp away from the beam core, on resonance islands

D.Pellegrini

- **Min. Dynamic Aperture (DA)** with **intensity** vs **crossing angle**, for **nominal optics** (β^***= 40 cm**) and **BCMS** beam (**2.5 μm** emittance), 15 units of chromaticity
- For **1.1x10^{11} p**
  - At **$\theta_c/2$ = 185 μrad** (**~12 σ** separation), DA around **6 σ** **(good lifetime observed)**
  - At **$\theta_c/2$ = 140 μrad** (**~9 σ** separation), DA below **5 σ** **(reduced lifetime observed)**
  - **Improvement** for low octupoles, low chromaticity and WP optimisation (observed in **operation**)

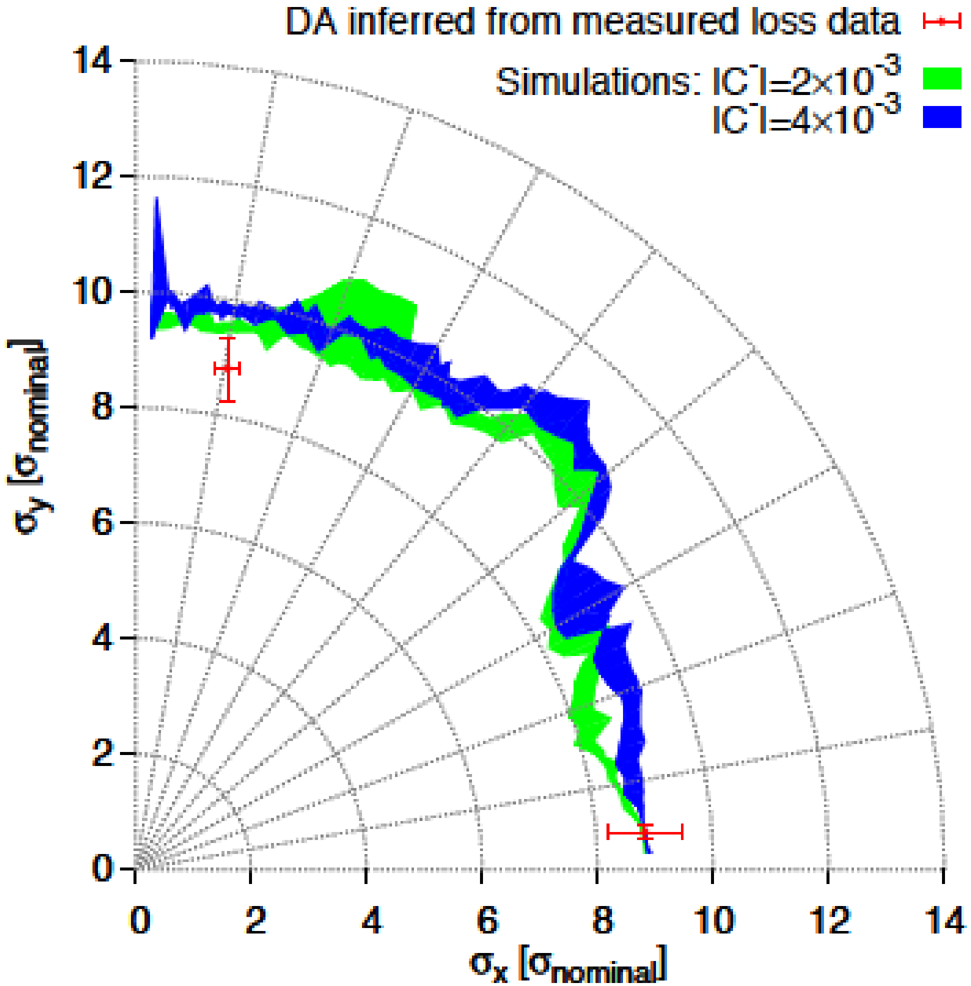


- MOGA –Multi Objective Genetic Algorithms are being recently used to optimise linear but also non-linear dynamics of electron low emittance storage rings
- Use knobs quadrupole strengths, chromaticity sextupoles and correctors with some constraints
- Target ultra-low horizontal emittance, increased lifetime and high dynamic aperture



Measuring Dynamic Aperture

- During LHC design phase, DA target was 2x higher than collimator position, due to statistical fluctuation, finite mesh, linear imperfections, short tracking time, multi-pole time dependence, ripple and a 20% safety margin
- Better knowledge of the model led to good agreement between measurements and simulations for actual LHC
- Necessity to build an accurate magnetic model (from beam based measurements)

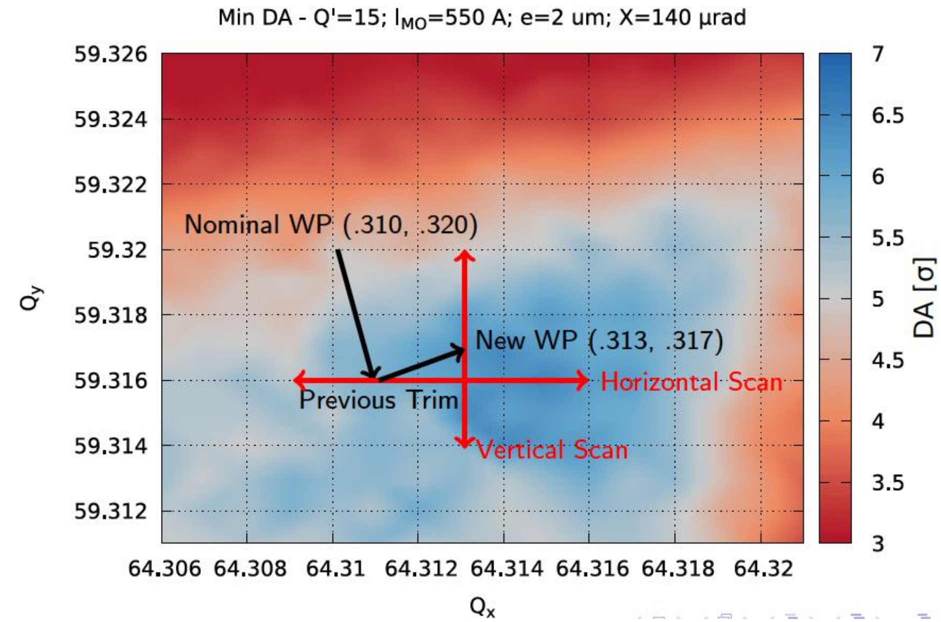


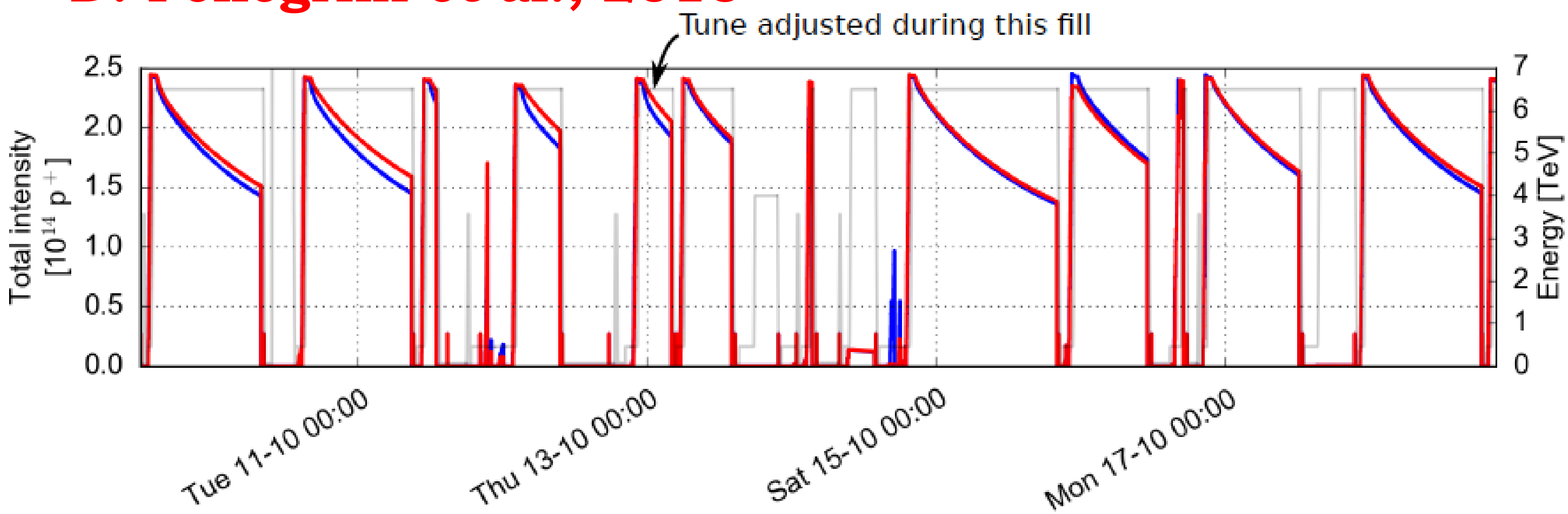
E.Mclean, PhD thesis, 2014

DA guiding machine performance

- B1 suffering from lower lifetime in the LHC
- DA simulations predicted the required adjustment
- Fine-tune scan performed and applied in operation, solving B1 lifetime problem

D. Pellegrini et al., 2016





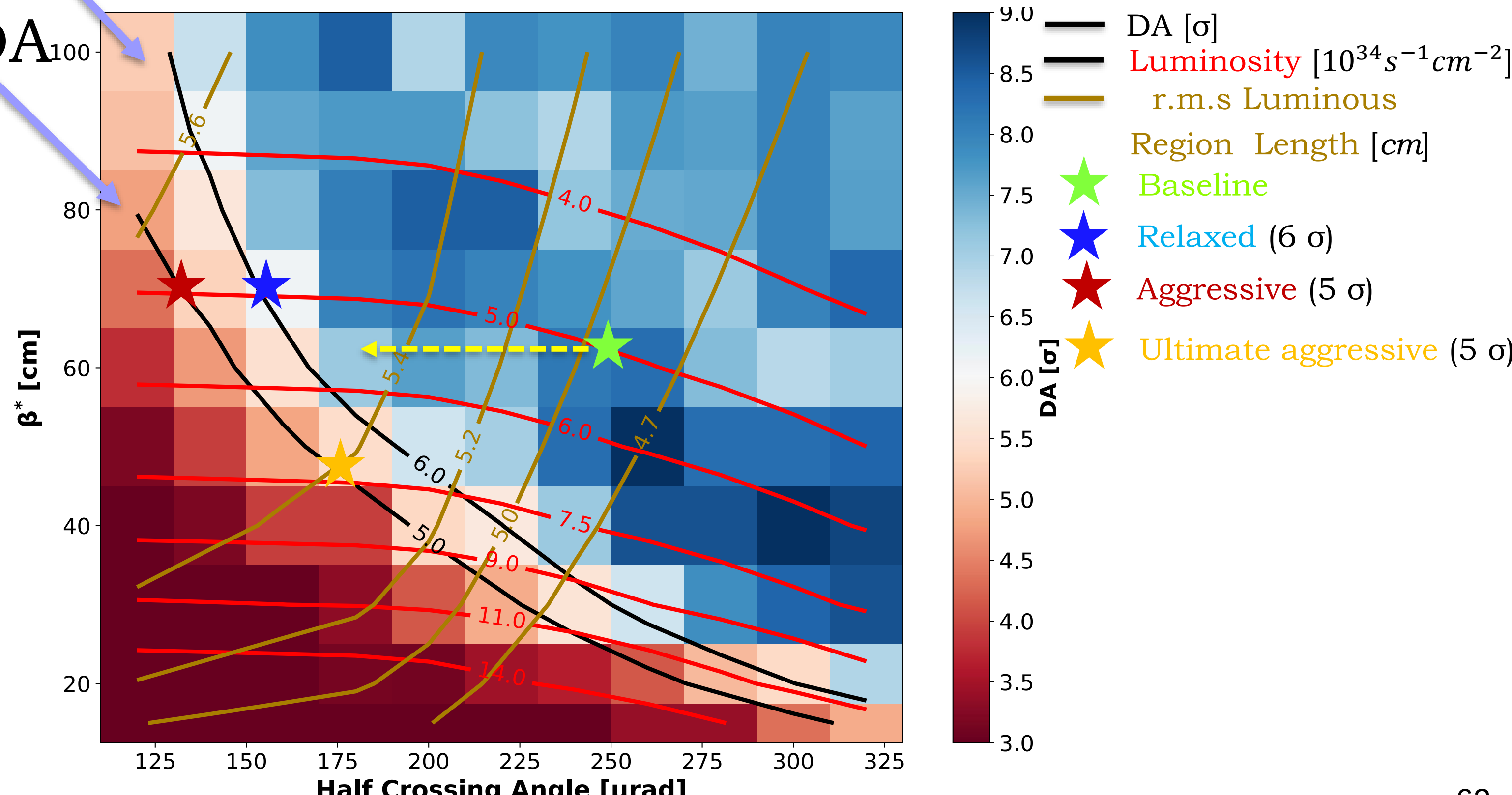
HL-LHC operational scenario

- Reduction of **crossing angle** at **constant luminosity**, reduces **pileup density** (by elongating the luminous region) and **triplet irradiation**

YP, N. Karastathis and D. Pellegrini et al., 2018

Relaxed DA

Aggressive DA



Lyapunov exponent

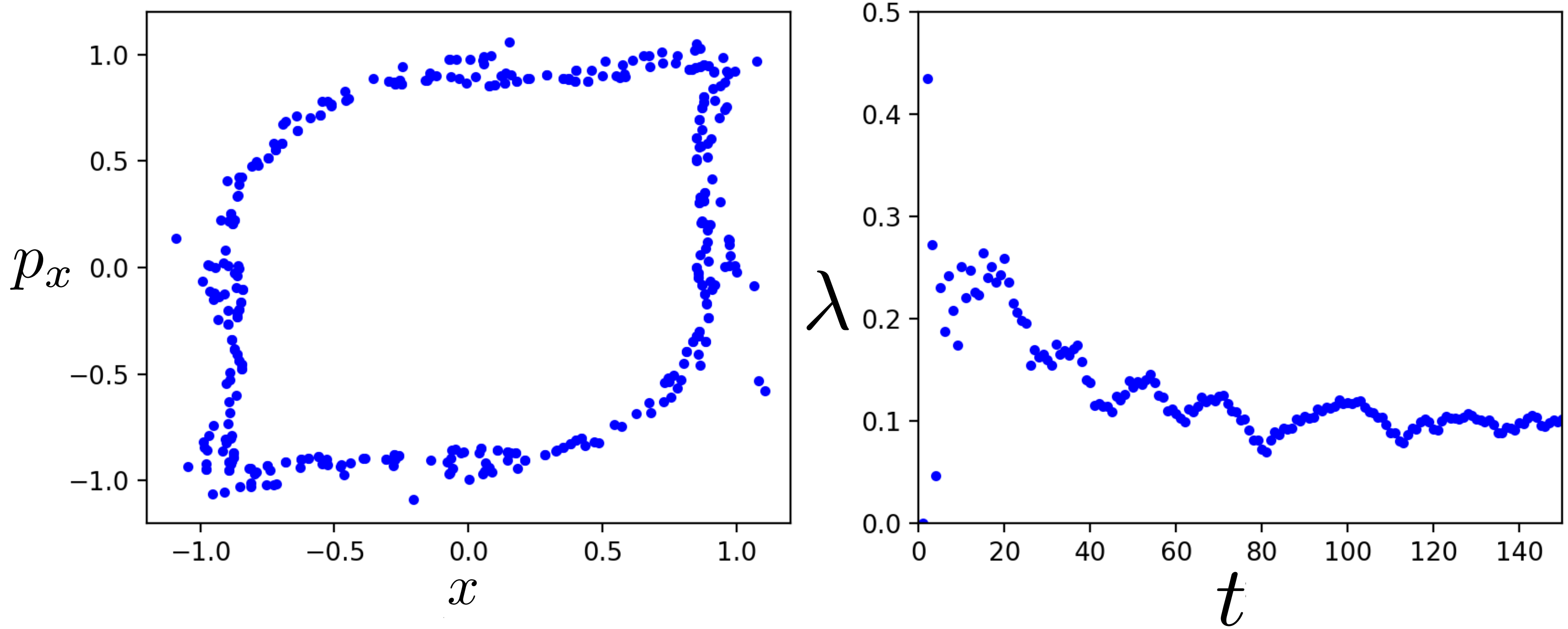
Lyapunov exponent

- Chaotic motion implies **sensitivity** to **initial condition**
- Two infinitesimally close **chaotic trajectories** in phase space with initial difference $\delta\mathbf{Z}_0$ will end-up diverging with rate

$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t}|\delta\mathbf{Z}_0|$$

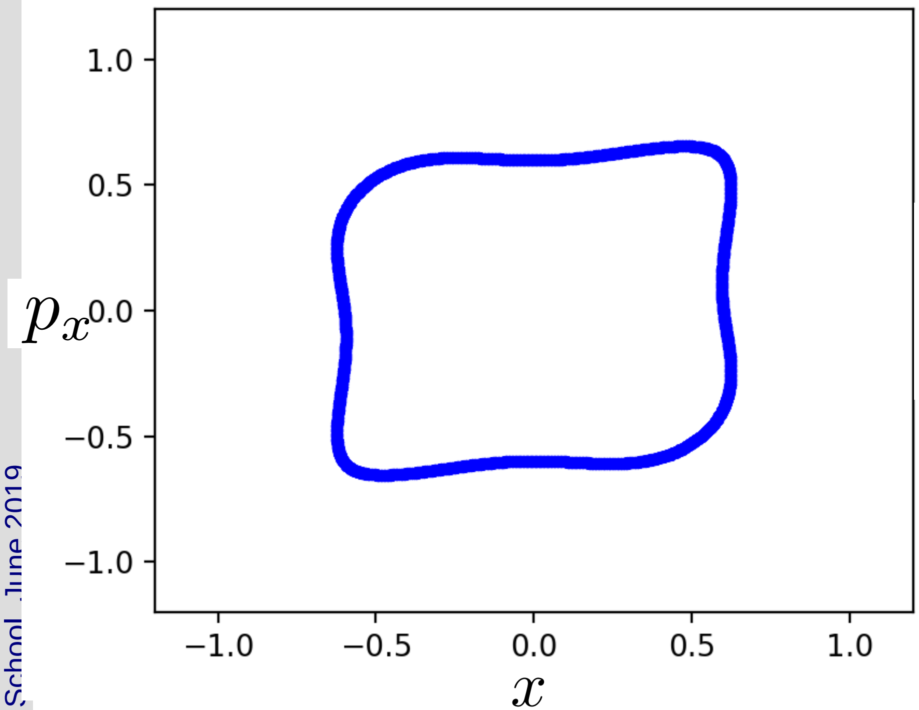
with λ the maximum **Lyapunov exponent**
- There is as many exponents as the phase space dimensions (Lyapunov spectrum)
- The largest one is the **Maximal Lyapunov exponent** (MLE) is defined as

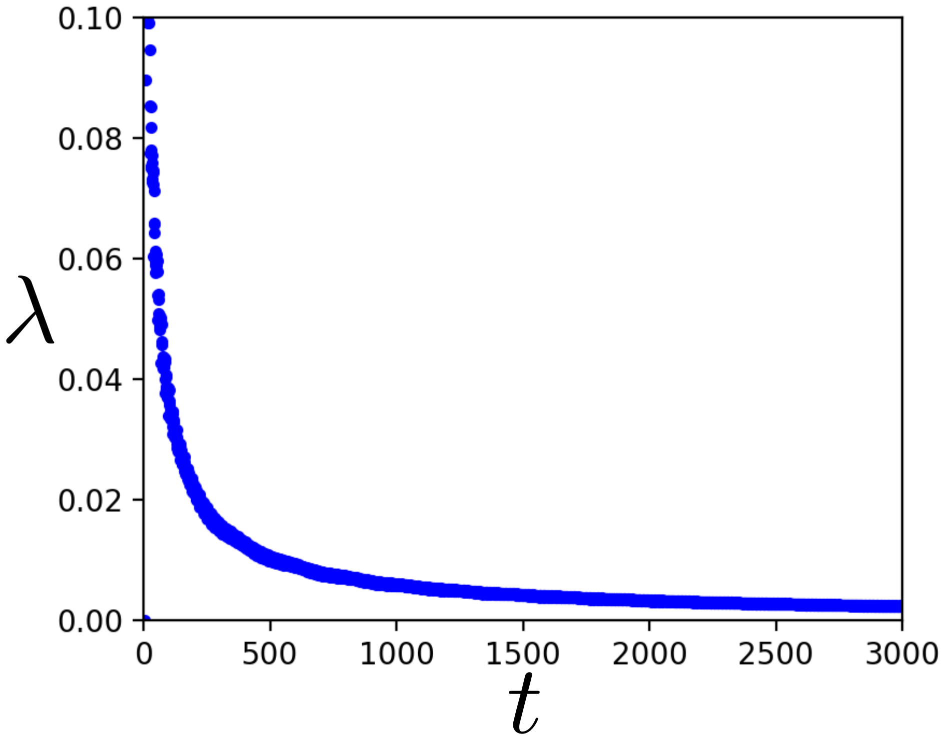
$$\lambda = \lim_{t\to\infty} \lim_{\delta\mathbf{Z}_0\to 0} \frac{1}{t} \ln \frac{|\delta\mathbf{Z}(t)|}{|\delta\mathbf{Z}_0|}$$



- Maximum Lyapounov exponent converges towards a **positive value** for a chaotic orbit

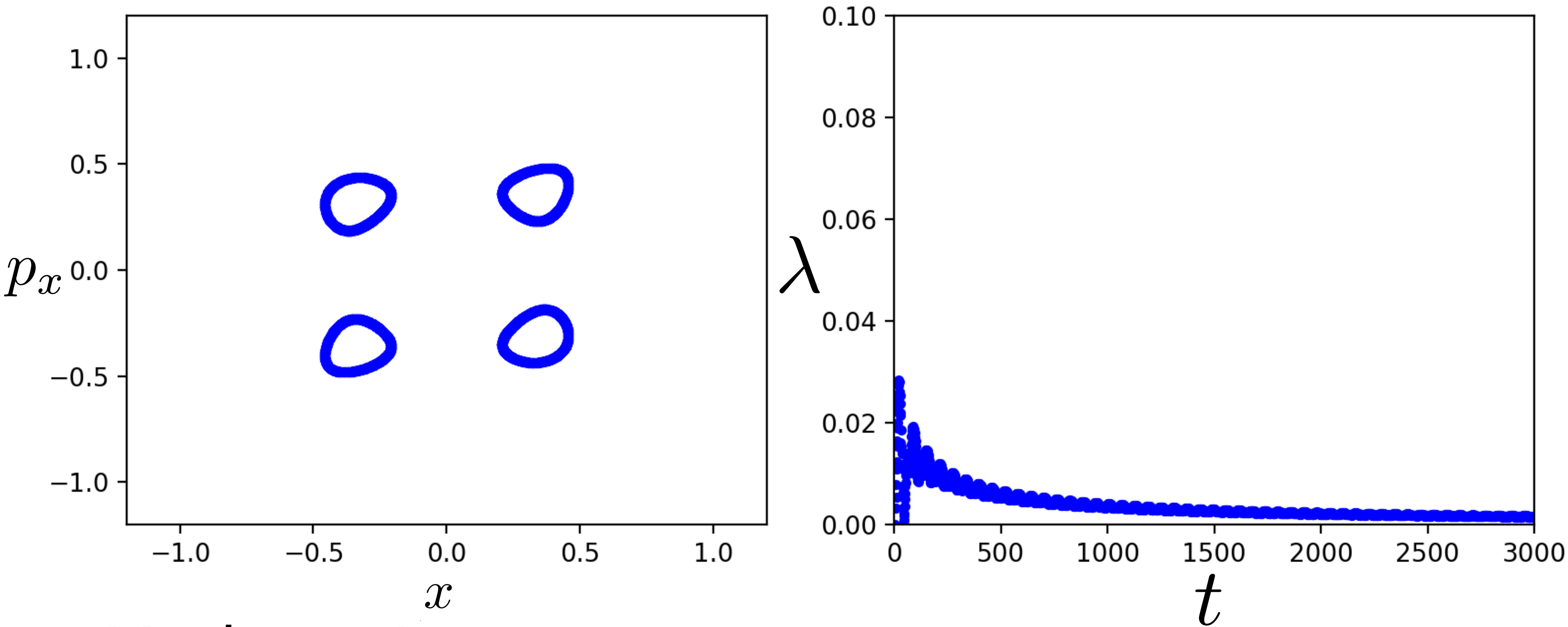
$$\lambda = \lim_{t\to\infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$





- Maximum Lyapounov exponent converges towards **zero** for a chaotic orbit

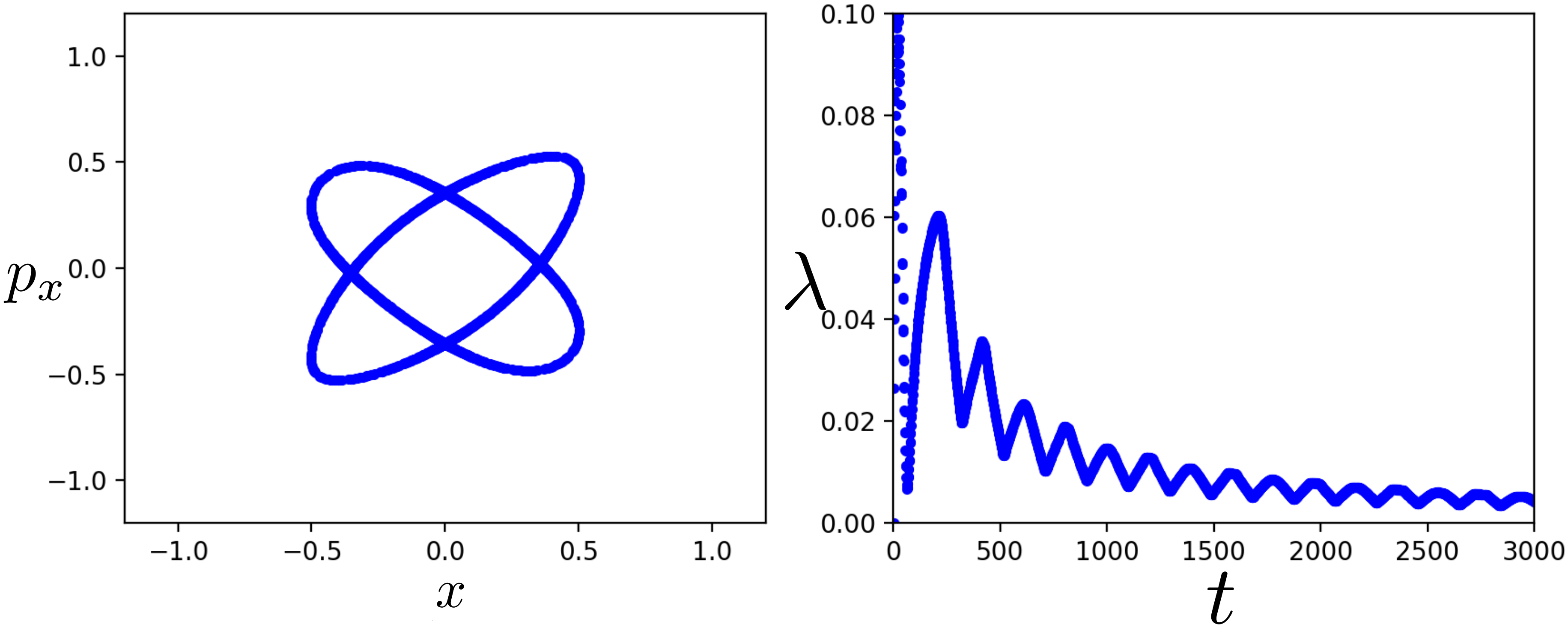
$$\lambda = \lim_{t\to\infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$



- Maximum Lyapounov exponent converges more slowly towards **zero** for a resonant orbit

$$\lambda = \lim_{t\to\infty} \lim_{\delta\mathbf{Z}_0\to 0} \frac{1}{t} \ln \frac{|\delta\mathbf{Z}(t)|}{|\delta\mathbf{Z}_0|}$$

Lyapunov exponent: regular orbit



- Maximum Lyapounov exponent converges more slowly towards **zero** for a resonant orbit, in particular close to the separatrix

$$\lambda = \lim_{t\to\infty} \lim_{\delta \mathbf{Z}_0 \to 0} \frac{1}{t} \ln \frac{|\delta \mathbf{Z}(t)|}{|\delta \mathbf{Z}_0|}$$

Frequency Map Analysis

Frequency map analysis

- Frequency Map Analysis (FMA) is a numerical method which springs from the studies of J. Laskar (Paris Observatory) putting in evidence the chaotic motion in the Solar Systems
- FMA was successively applied to several dynamical systems
  - Stability of Earth Obliquity and climate stabilization (Laskar, Robutel, 1993)
  - 4D maps (Laskar 1993)
  - Galactic Dynamics (Y.P and Laskar, 1996 and 1998)
  - Accelerator beam dynamics: lepton and hadron rings (Dumas, Laskar, 1993, Laskar, Robin, 1996, Y.P, 1999, Nadolski and Laskar 2001)

- Consider an integrable Hamiltonian system of the usual form

$$H(\boldsymbol{J},\boldsymbol{\varphi},\theta) = H_0(\mathbf{J})$$

- Hamilton's equations give

$$\dot{\phi}_j = \frac{\partial H_0(\mathbf{J})}{\partial J_j} = \omega_j(\mathbf{J}) \Rightarrow \phi_j = \omega_j(\mathbf{J})t + \phi_{j0}$$

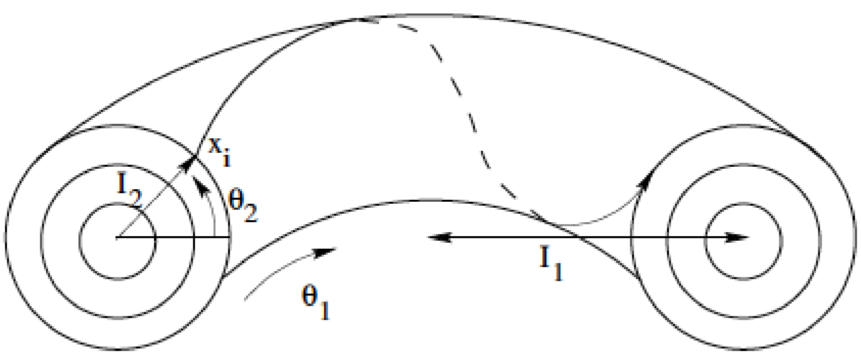
$$\dot{J}_j = -\frac{\partial H_0(\mathbf{J})}{\partial \phi_j} = 0 \Rightarrow J_j = \text{const.}$$

- The actions define the surface of an invariant torus
- In complex coordinates the motion is described by

$$\zeta_j(t) = J_j(0)e^{i\omega_j t} = z_{j0}e^{i\omega_j t}$$

- For a **non-degenerate** system $\det\left|\frac{\partial \omega(J)}{\partial J}\right| = \det\left|\frac{\partial^2 H_0(J)}{\partial J^2}\right| \neq 0$ there is a one-to-one correspondence between the actions and the frequency, a frequency map can be defined parameterizing the tori in the frequency space

$$F: \quad (\mathbf{I}) \longrightarrow (\omega)$$



- If a transformation is made to some new variables

$$\zeta_j = I_j e^{i\theta_j t} = z_j + \epsilon G_j(\mathbf{z}) = z_j + \epsilon \sum_{\mathbf{m}} c_{\mathbf{m}} z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}$$

- The system is still integrable but the tori are distorted
- The motion is then described by

$$\zeta_j(t) = z_{j0} e^{i\omega_j t} + \sum_{\mathbf{m}} a_{\mathbf{m}} e^{i\,(\mathbf{m}\cdot\omega)\,t}$$

i.e. a quasi-periodic function of time, with

$$a_{\mathbf{m}} = \epsilon\, c_{\mathbf{m}} z_{10}^{m_1} z_{20}^{m_2} \dots z_{n0}^{m_n} \text{ and } \mathbf{m}\cdot\omega = m_1\omega_1 + m_2\omega_2 + \dots + m_n\omega_n$$

- For a non-integrable Hamiltonian, $H(\mathbf{I},\theta) = H_0(\mathbf{I}) + \epsilon H'(\mathbf{I},\theta)$ and especially if the perturbation is small, most tori persist (**KAM** theory)
- In that case, the motion is still quasi-periodic and a frequency map can be built
- The **regularity** (or not) of the map reveals stable (or chaotic) motion

Building the frequency map

- When a quasi-periodic function $f(t) = q(t) + ip(t)$ in the complex domain is given numerically, it is possible to recover a quasi-periodic approximation

$$f'(t) = \sum_{k=1}^{N} a'_k e^{i\omega'_k t}$$

in a very precise way over a finite time span $[-T, T]$ several orders of magnitude more precisely than simple Fourier techniques

- This approximation is provided by the Numerical Analysis of Fundamental Frequencies – **NAFF** algorithm
- The frequencies ω'_k and complex amplitudes a'_k are computed through an iterative scheme.

The NAFF algorithm

- The first frequency ω'_1 is found by the location of the maximum of

$$\phi(\sigma) = \langle f(t), e^{i\sigma t} \rangle = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\sigma t} \chi(t) dt$$

where $\chi(t)$ is a weight function

- In most of the cases the Hanning window filter is used $\chi_1(t) = 1 + \cos(\pi t / T)$
- Once the first term $e^{i\omega'_1 t}$ is found, its complex amplitude a'_1 is obtained and the process is restarted on the remaining part of the function

$$f_1(t) = f(t) - a'_1 e^{i\omega'_1 t}$$

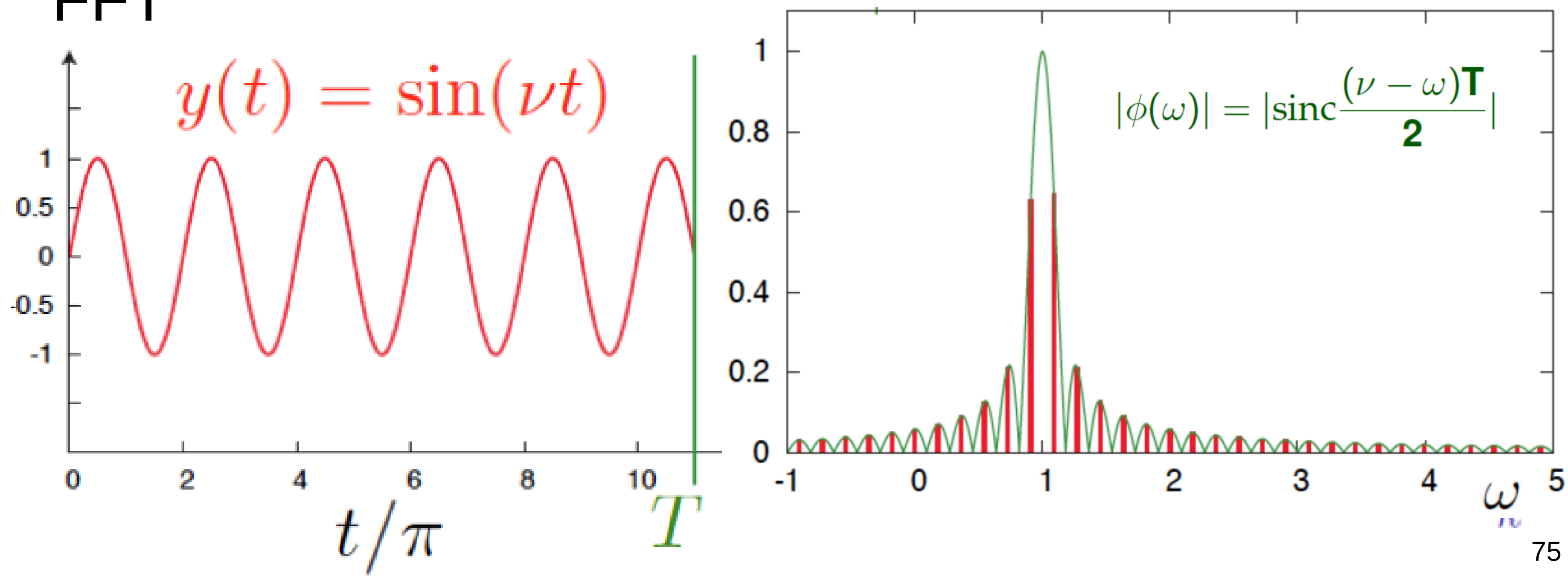
- The procedure is continued for the number of desired terms, or until a required precision is reached

Frequency determination

- The accuracy of a simple FFT even for a simple sinusoidal signal is not better than $|\nu - \nu_T| = \frac{1}{T}$
- Calculating the Fourier integral explicitly

$$\phi(\omega) = < f(t),\ e^{i\omega t} > = \frac{1}{T}\int_0^T f(t)\ e^{-i\omega t}\ dt$$

shows that the maximum lies in between the main peaks of the FFT

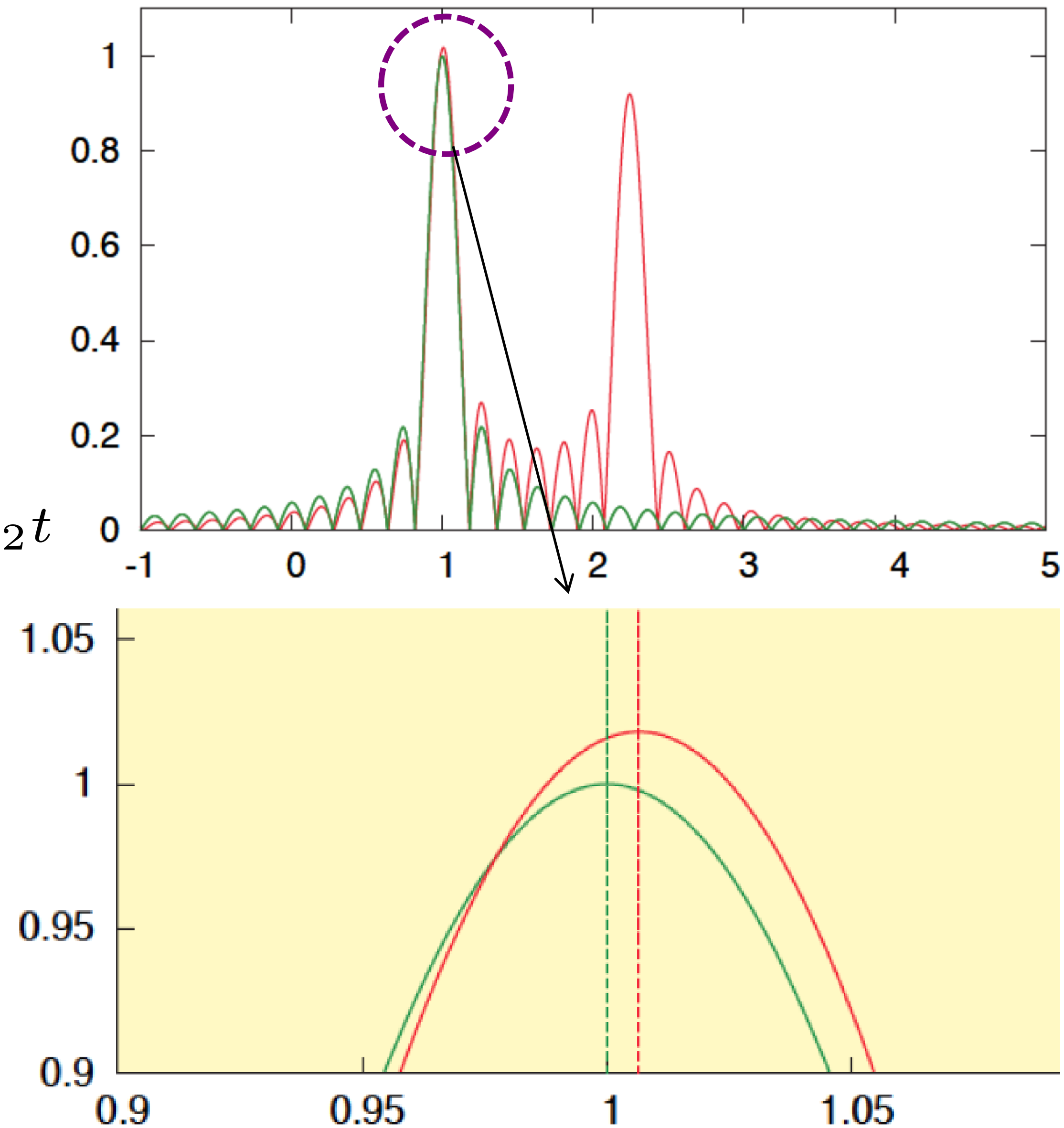


Frequency determination

- A more complicated signal with two frequencies

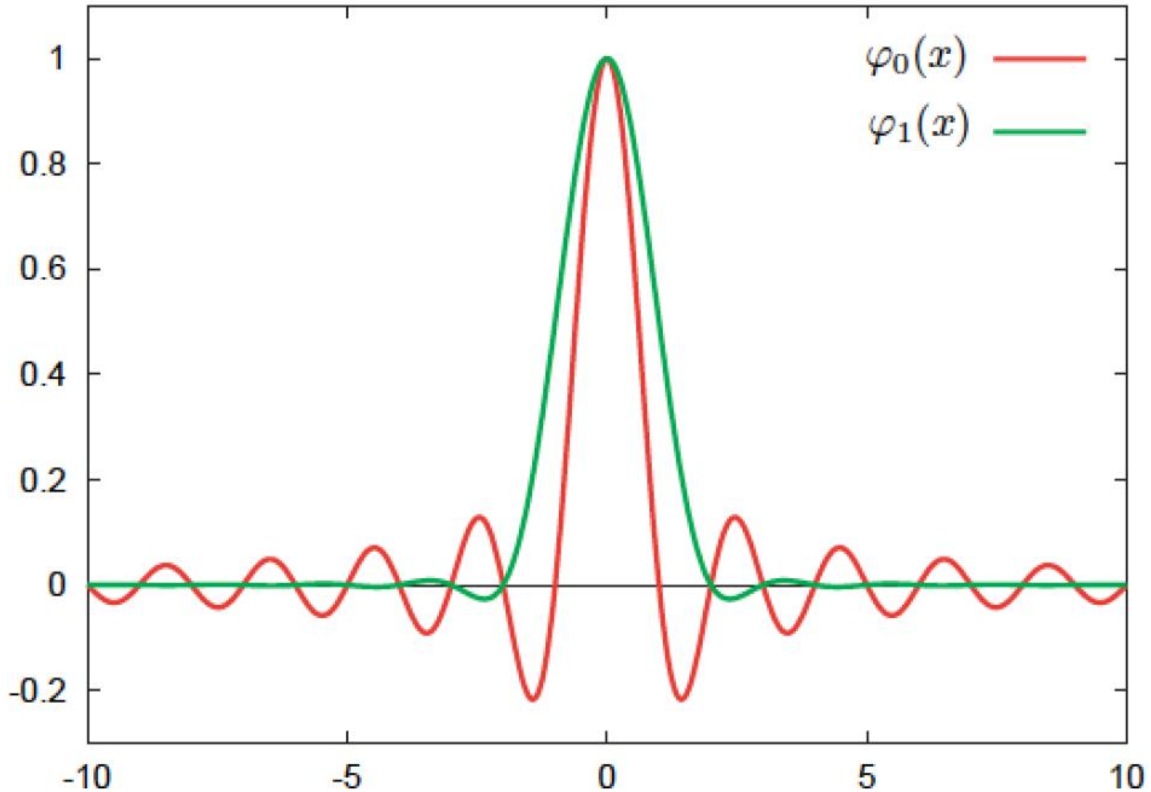
$$f(t) = a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$$

shifts slightly the maximum with respect to its real location



Window function

- A window function like the Hanning filter $\chi_1(t) = 1 + \cos(\pi t/T)$ kills side-lobs and allows a very accurate determination of the frequency



Precision of NAFF

- For a general window function of order p

$$\chi_p(t) = \frac{2^p (p!)^2}{(2p)!}(1+\cos \pi t)^p$$

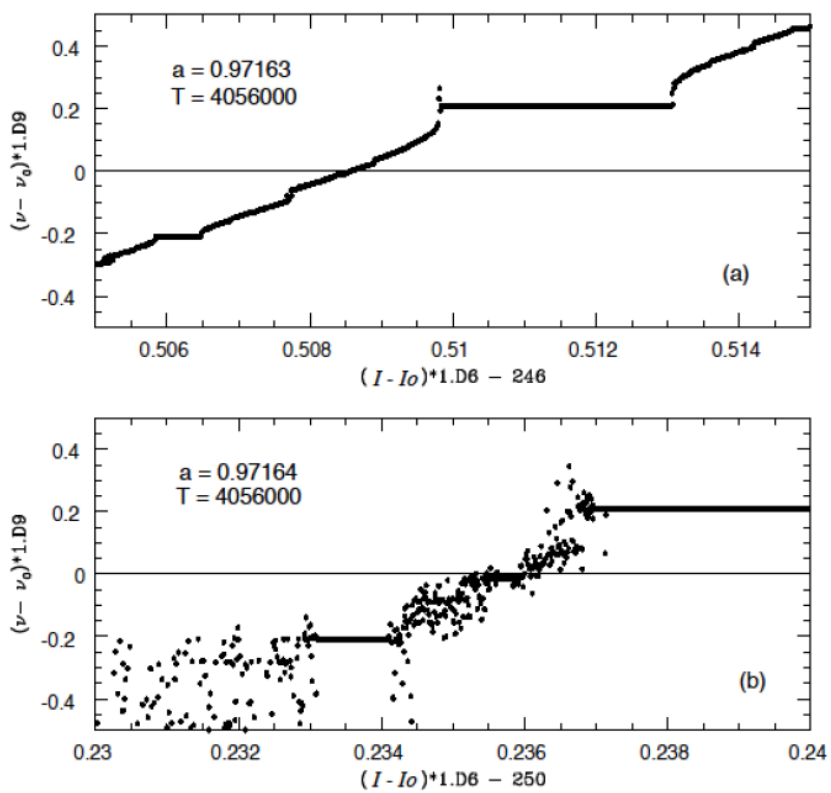
Laskar (1996) proved a theorem stating that the solution provided by the NAFF algorithm converges asymptotically towards the real KAM quasi-periodic solution with precision

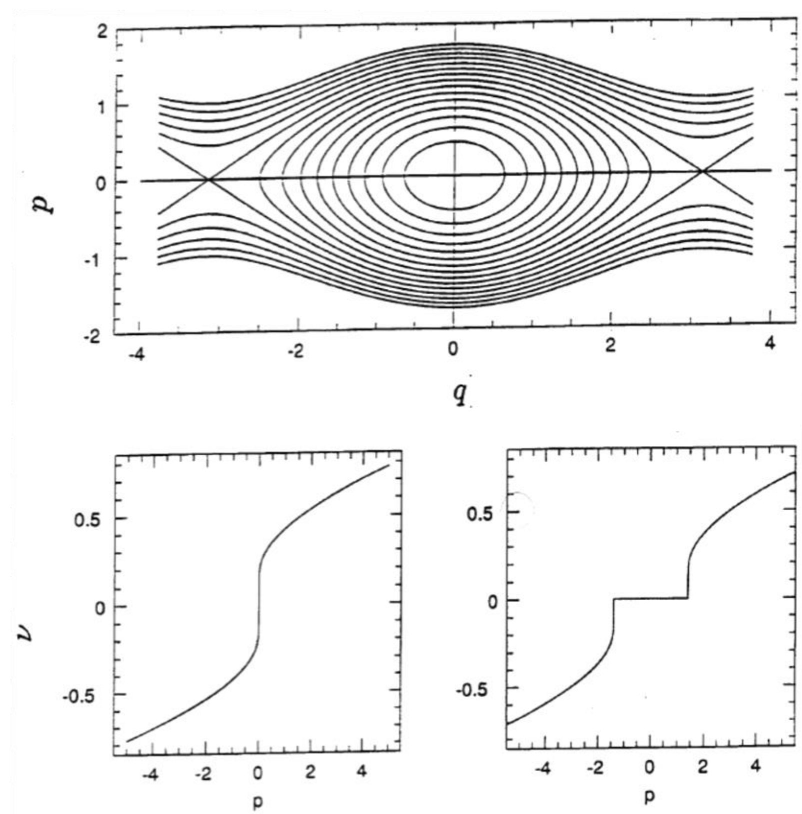
$$\nu_1 - \nu_1^T \propto \frac{1}{T^{2p+2}}$$

- In particular, for no filter (i.e. $p = 0$) the precision is $\frac{1}{T^2}$, whereas for the Hanning filter ($p = 1$), the precision is of the order of $\frac{1}{T^4}$

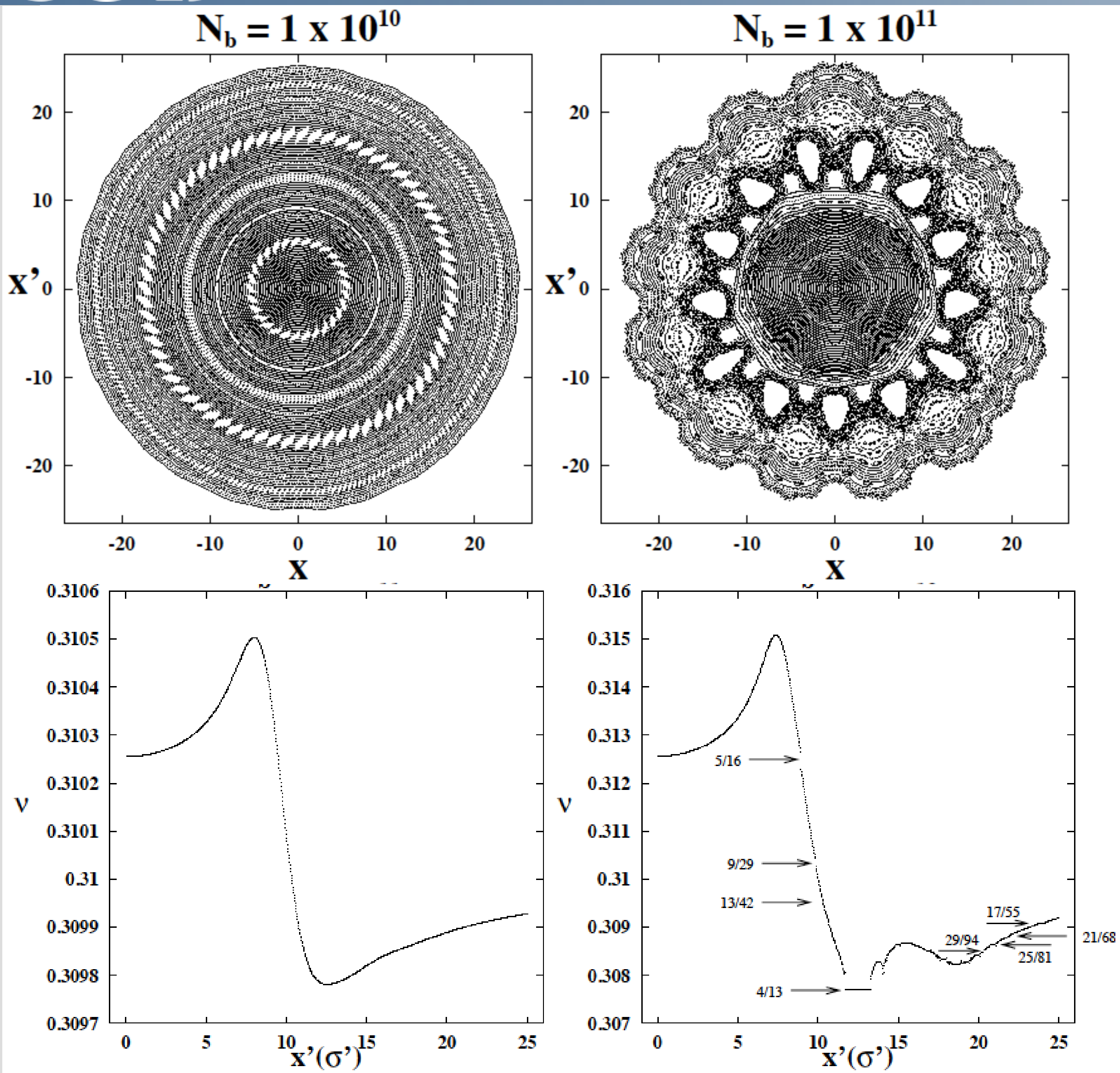
Aspects of the frequency map

- In the vicinity of a resonance the system behaves like a **pendulum**
- Passing through the **elliptic point** for a fixed angle, a **fixed frequency** (or rotation number) is observed
- Passing through the **hyperbolic point**, a **frequency jump** is observed





Example: Frequency map for BBLR

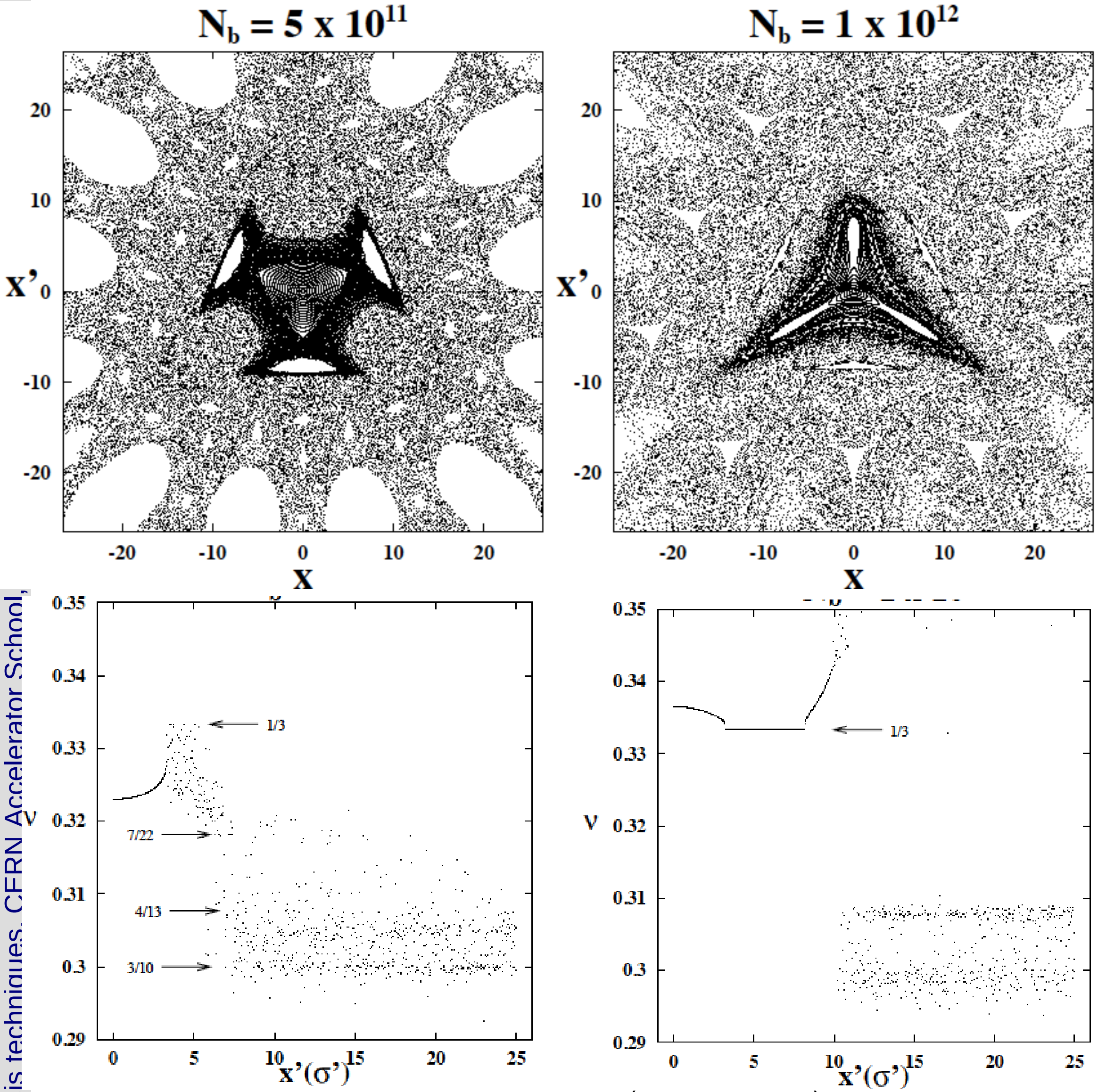


- Simple Beam-beam long range (BBLR) kick and a rotation

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} \cos\mu & \beta^*\sin\mu \\ -\sin\mu/\beta^* & \cos\mu \end{pmatrix} \begin{pmatrix} x + f(x') \\ x' \end{pmatrix}_0$$

$$f(x') = K\left[\frac{1}{x'+\theta_c}\left(1-e^{-\frac{(x'+\theta_c)^2}{2{\sigma'_x}^2}}\right) - \frac{1}{\theta_c}\left(1-e^{-\frac{\theta_c^2}{2{\sigma'_x}^2}}\right)\right]$$

Example: Frequency map for BBLR

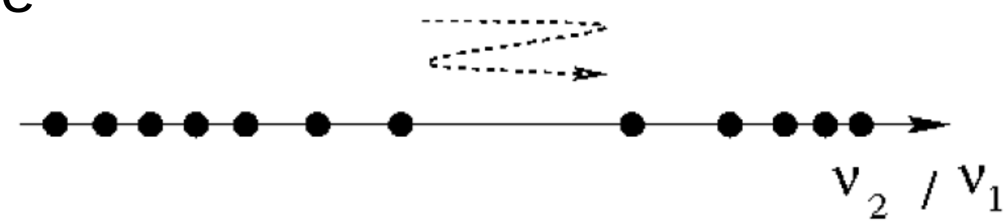


- Simple Beam-beam long range (BBLR) kick and a rotation

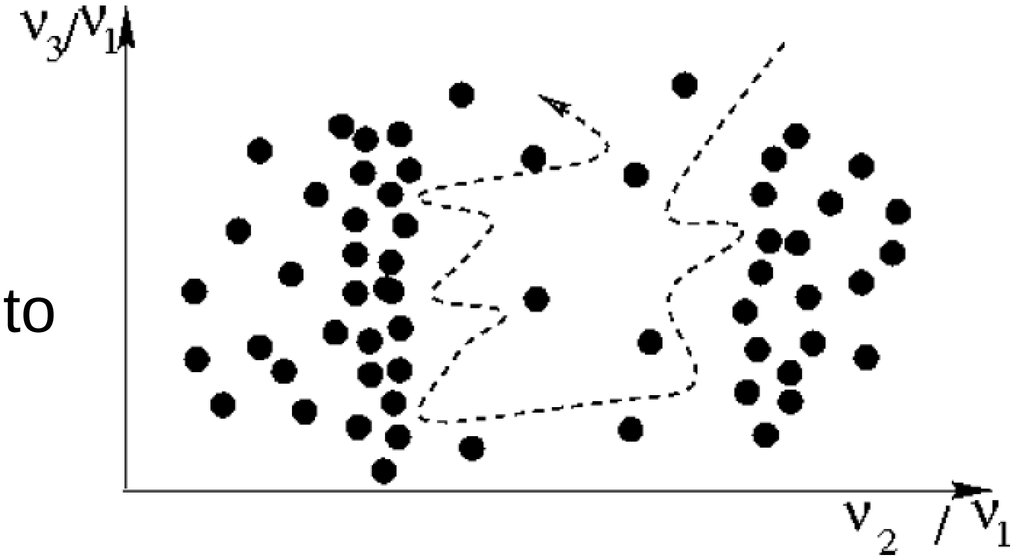
$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} \cos\mu & \beta^*\sin\mu \\ -\sin\mu/\beta^* & \cos\mu \end{pmatrix} \begin{pmatrix} x+f(x') \\ x' \end{pmatrix}_0 \qquad f(x') = K\left[\frac{1}{x'+\theta_c}\left(1-e^{-\frac{(x'+\theta_c)^2}{2\sigma_x'^2}}\right)-\frac{1}{\theta_c}\left(1-e^{-\frac{\theta_c^2}{2\sigma_x'^2}}\right)\right]$$

Diffusion in frequency space

- For a **2 degrees of freedom** Hamiltonian system, the **frequency space** is a **line**, the tori are dots on this lines, and the **chaotic zones** are **confined** by the existing KAM tori
- For a system with 3 or more degrees of freedom, KAM tori are still represented by dots but do not prevent chaotic trajectories to diffuse

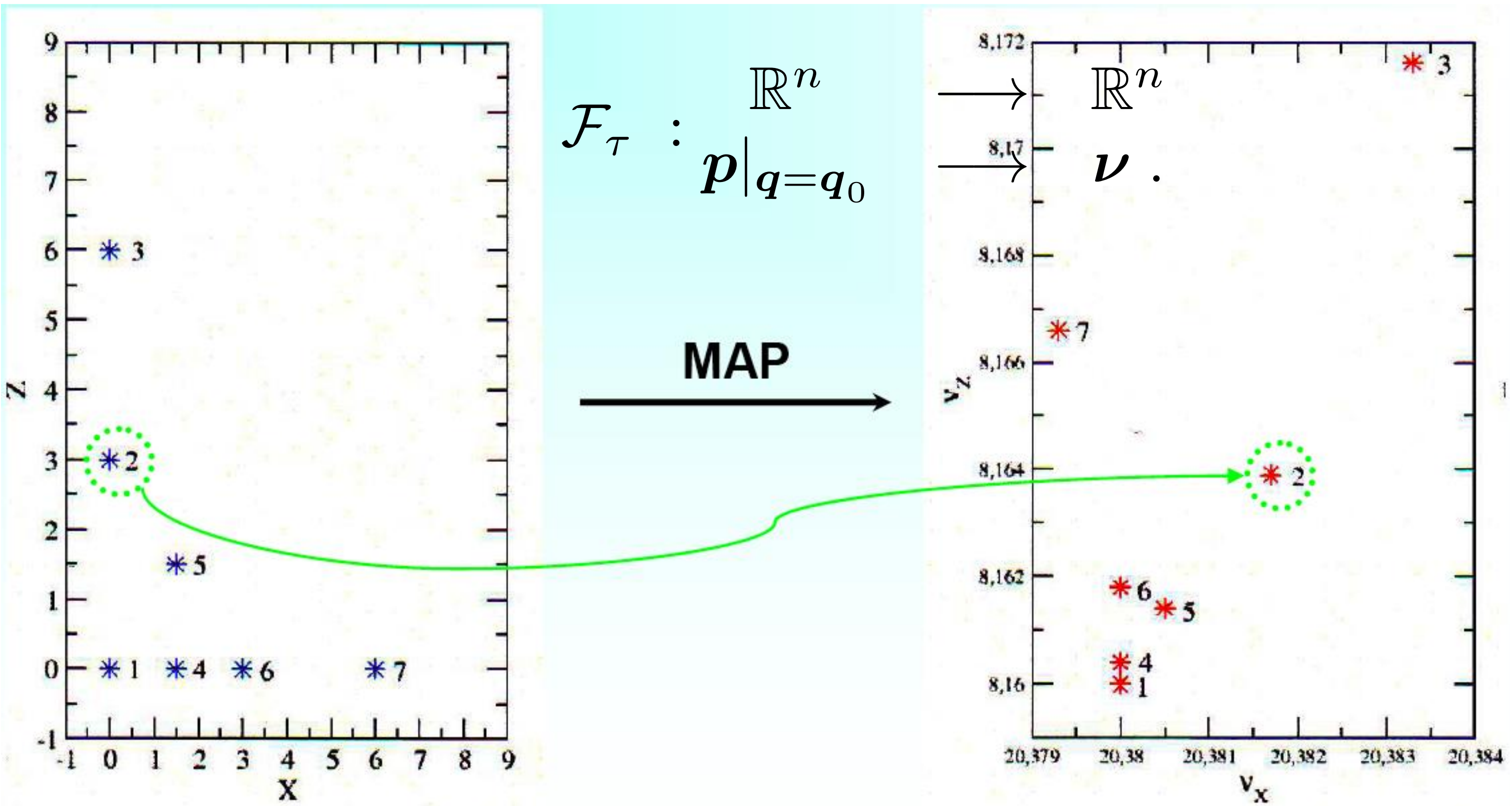


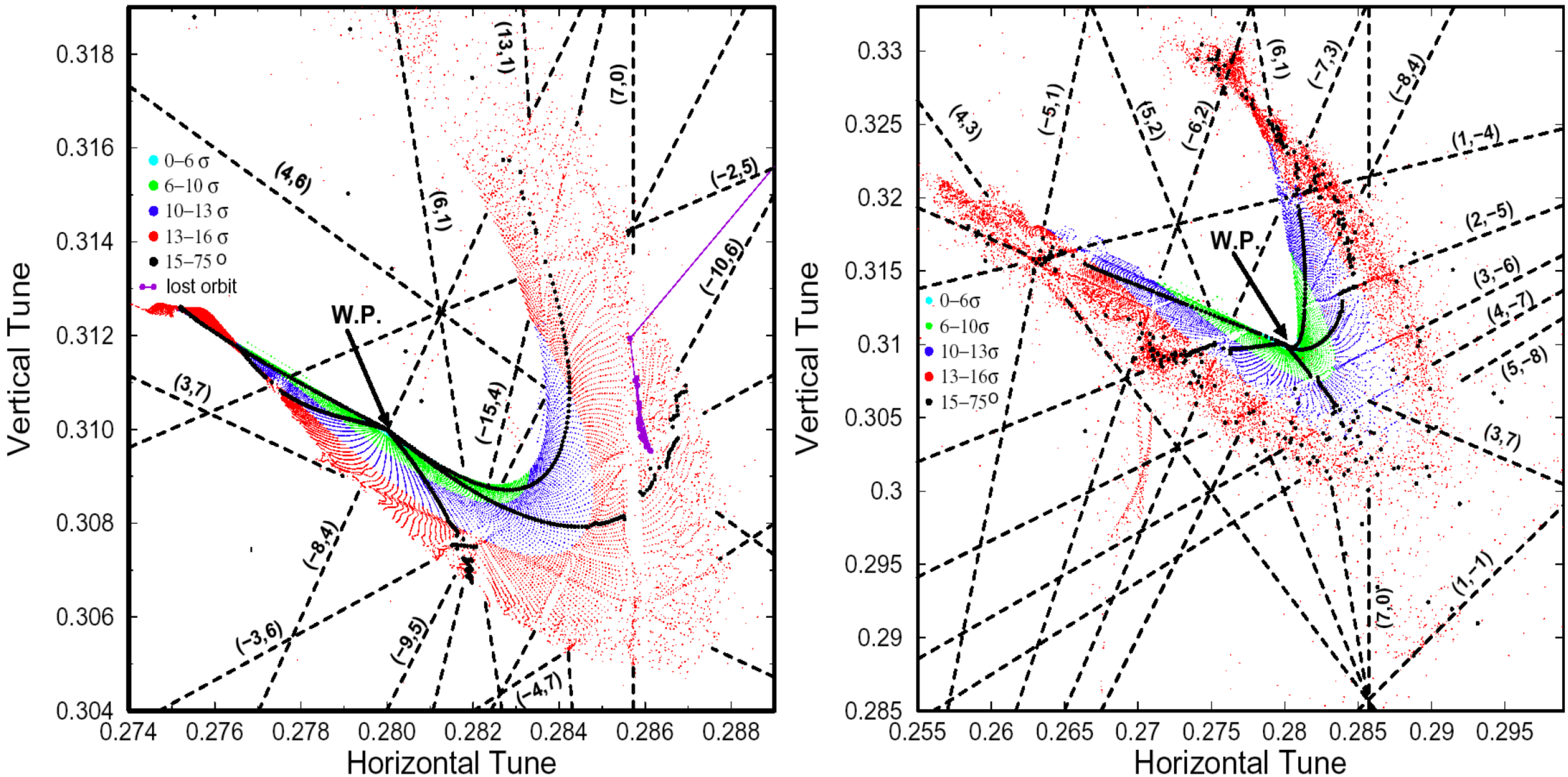
- This topological possibility of particles diffusing is called **Arnold diffusion**
- This diffusion is supposed to be extremely small in their vicinity, as tori act as effective barriers (**Nechoroshev theory**)



Building the frequency map

- Choose coordinates (x_i, y_i) with p_x and p_y=0
- Numerically integrate the phase trajectories through the lattice for sufficient number of turns
- Compute through NAFF Q_x and Q_y after sufficient number of turns
- Plot them in the tune diagram





- Frequency maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)

Diffusion Maps

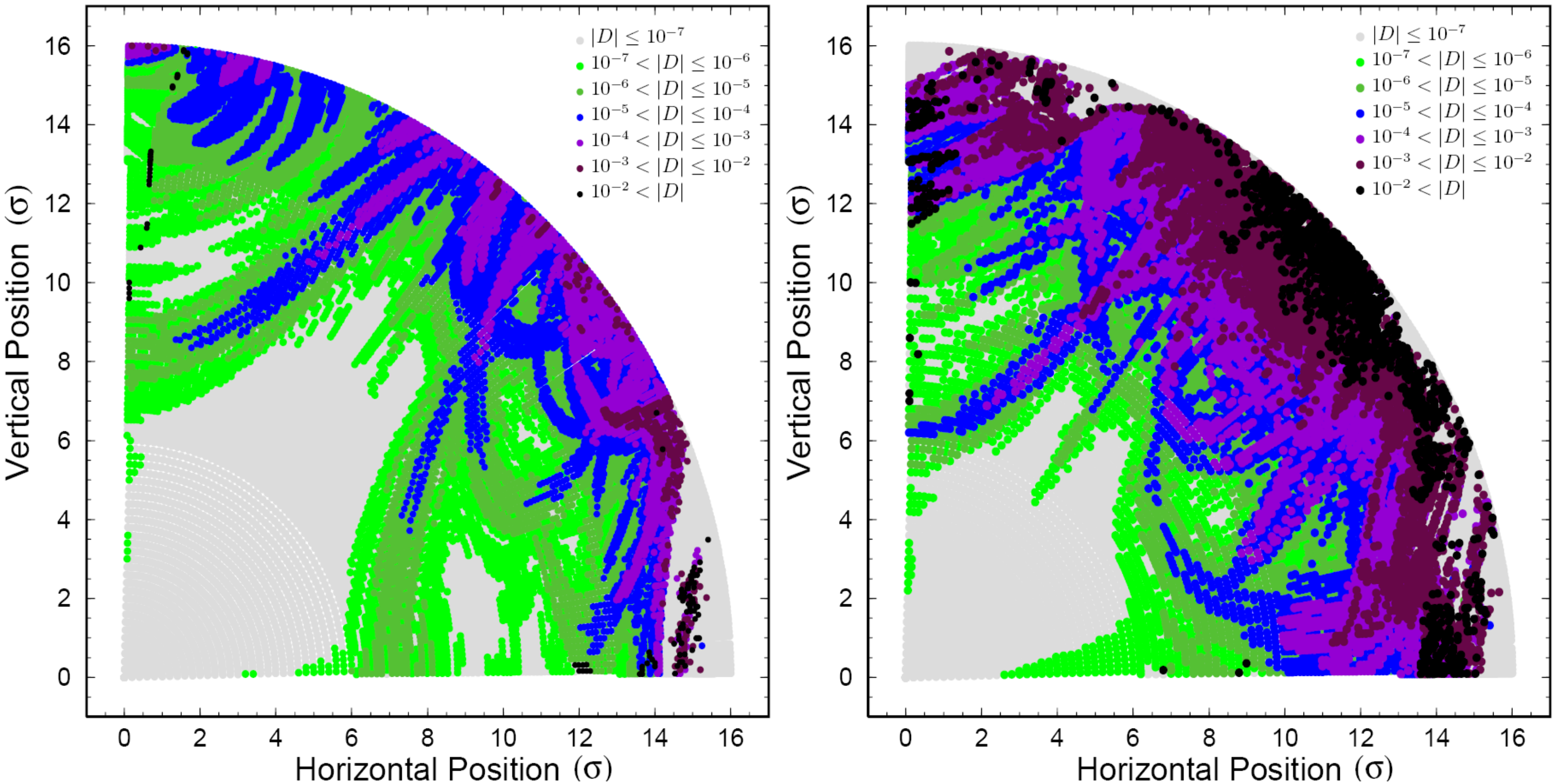
- Calculate frequencies for two equal and successive time spans and compute frequency diffusion vector:

$$\boldsymbol{D}\big|_{t=\tau} = \boldsymbol{\nu}\big|_{t\in(0,\tau/2]} - \boldsymbol{\nu}\big|_{t\in(\tau/2,\tau]}$$

- Plot the initial condition space color-coded with the norm of the diffusion vector
- Compute a diffusion quality factor by averaging all diffusion coefficients normalized with the initial conditions radius

$$D_{QF} = \left\langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \right\rangle_R$$

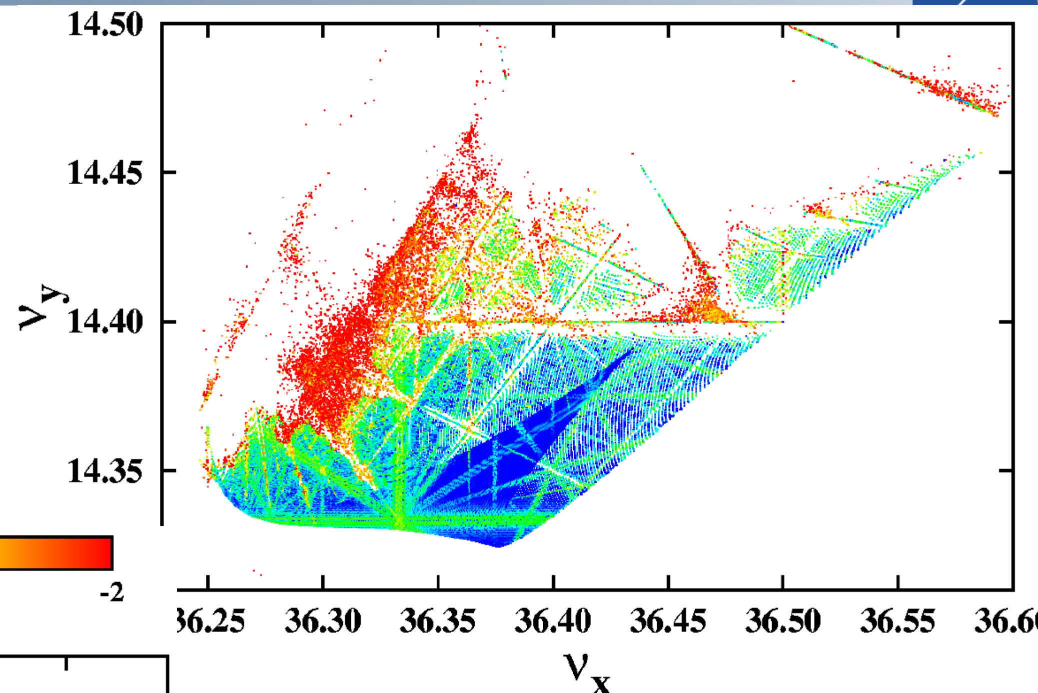
Example: Diffusion maps for the LHC

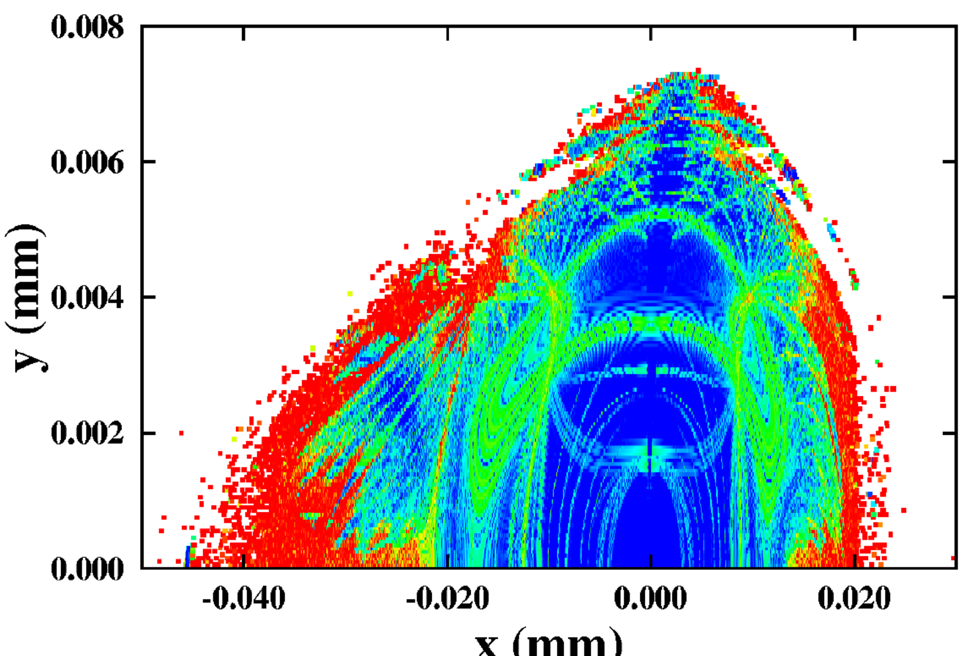


Diffusion maps for the target error table (left) and an increased random skew octupole error in the super-conducting dipoles (right)

Example: Frequency Map for the ESRF

- All dynamics represented in these two plots
- Regular motion represented by blue colors (close to zero amplitude particles or working point)

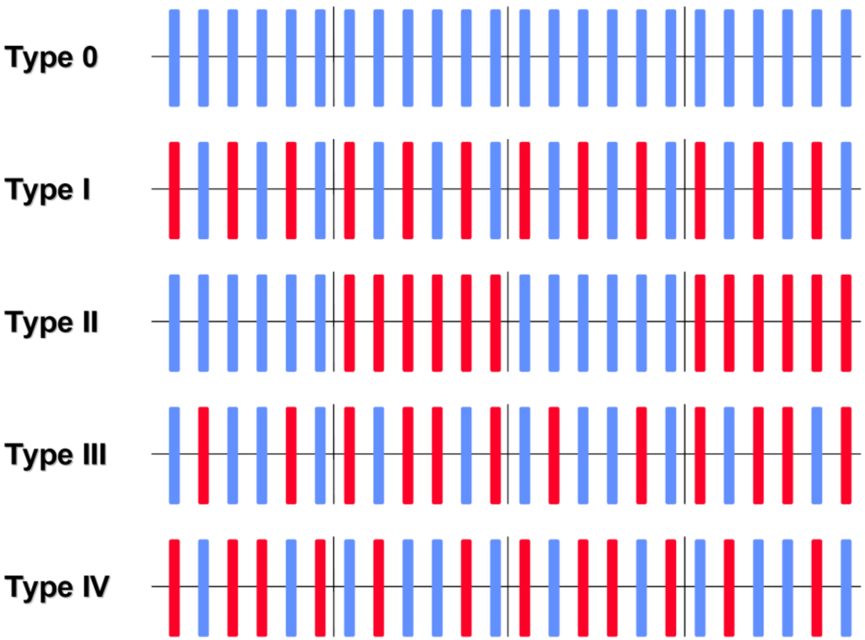


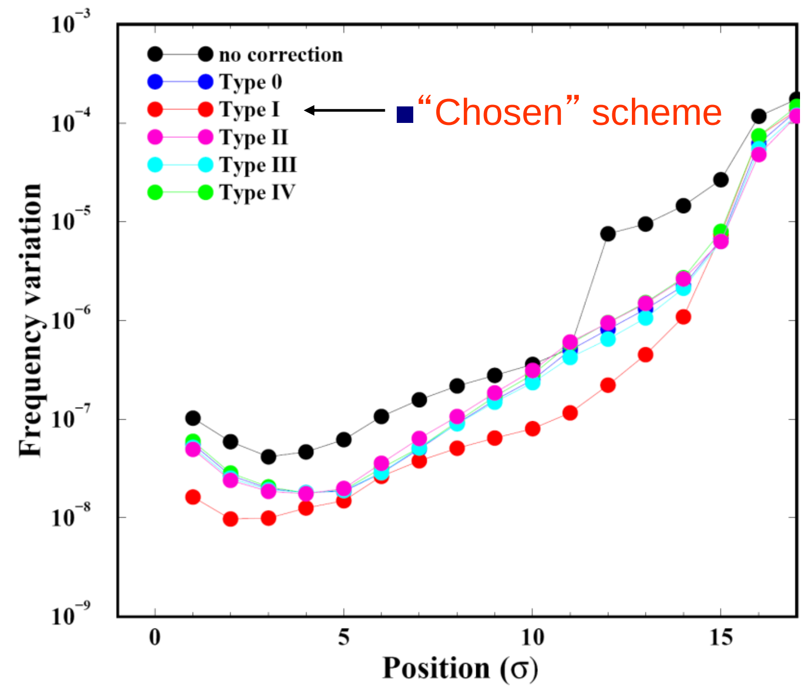


- Resonances appear as distorted lines in frequency space (or curves in initial condition space
- Chaotic motion is represented by red scattered particles and defines dynamic aperture of the machine

Numerical Applications

Correction schemes efficiency

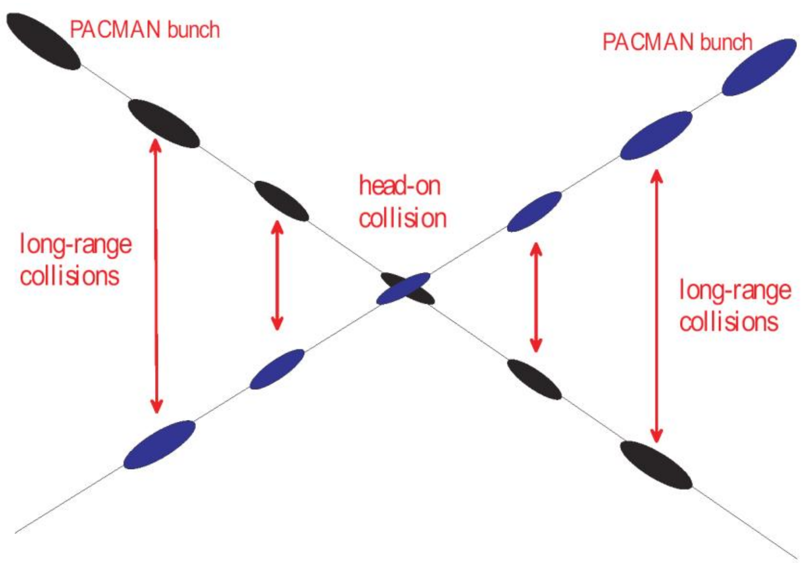




- Comparison of correction schemes for b_4 and b_5 errors in the LHC dipoles
- Frequency maps, resonance analysis, tune diffusion estimates, survival plots and short term tracking, proved that only half of the correctors are needed

Beam-Beam interaction

Variable	Symbol	Value
Beam energy	E	7 TeV
Particle species	$\cdots$	protons
Full crossing angle	θ_c	300 μrad
rms beam divergence	σ'_x	31.7 μrad
rms beam size	σ_x	15.9 μm
Normalized transv. rms emittance	$\gamma\varepsilon$	3.75 μm
IP beta function	β^*	0.5 m
Bunch charge	N_b	$(1\times10^{11}-2\times10^{12})$
Betatron tune	Q_0	0.31



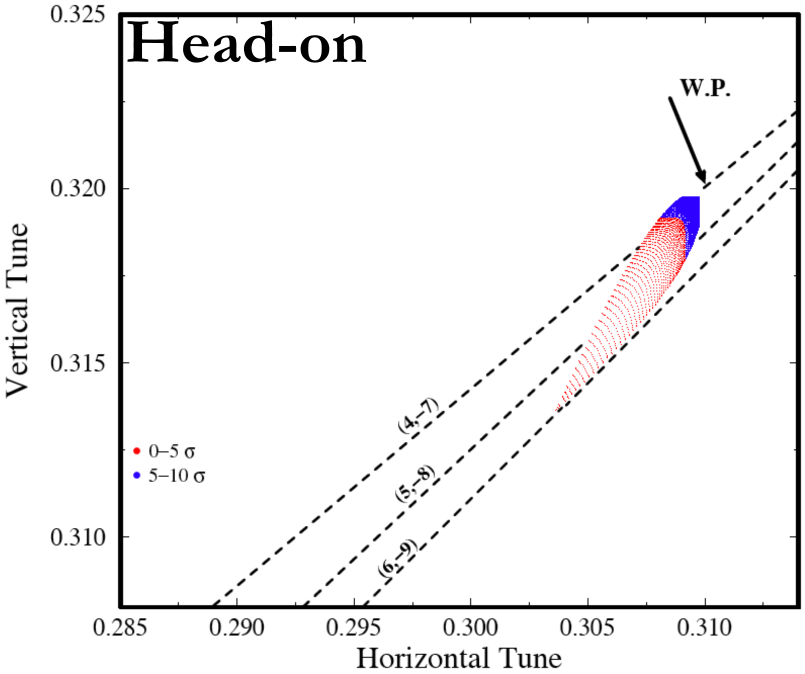
- Long range beam-beam interaction represented by a 4D kick-map

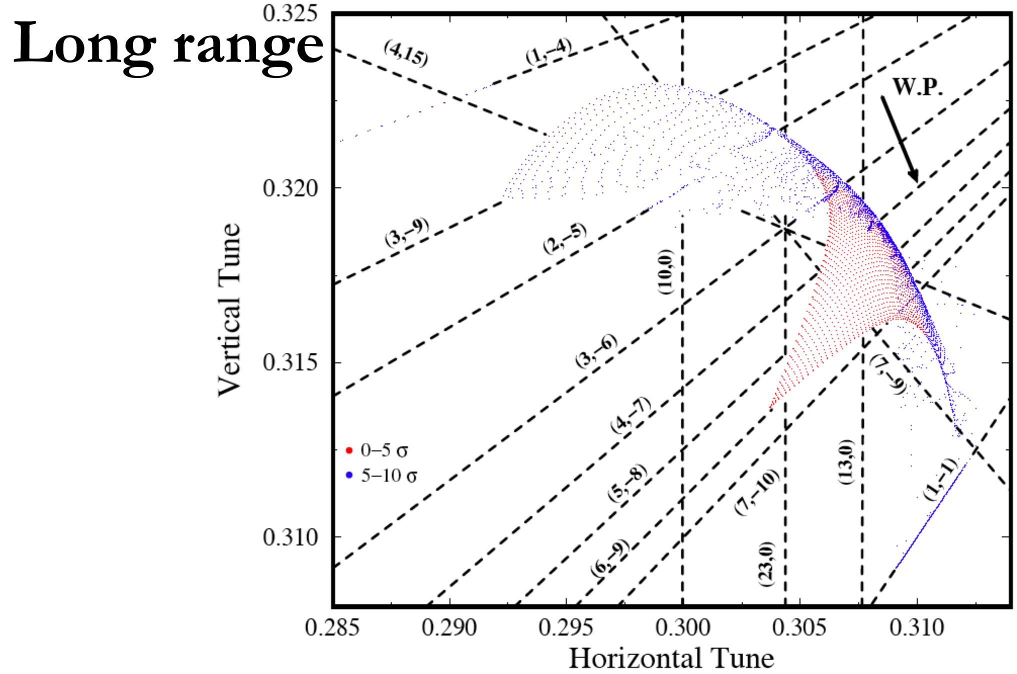
$$\Delta x = -\, n_{par}\frac{2r_pN_b}{\gamma}\left[\frac{x'+\theta_c}{\theta_t^2}\left(1-e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}}\right) - \frac{1}{\theta_c}\left(1-e^{-\frac{\theta_c^2}{2\theta_{x,y}^2}}\right)\right]$$

$$\Delta y = -\, n_{par}\frac{2r_pN_b}{\gamma}\frac{y'}{\theta_t^2}\left(1-e^{-\frac{\theta_t^2}{2\theta_{x,y}^2}}\right)$$

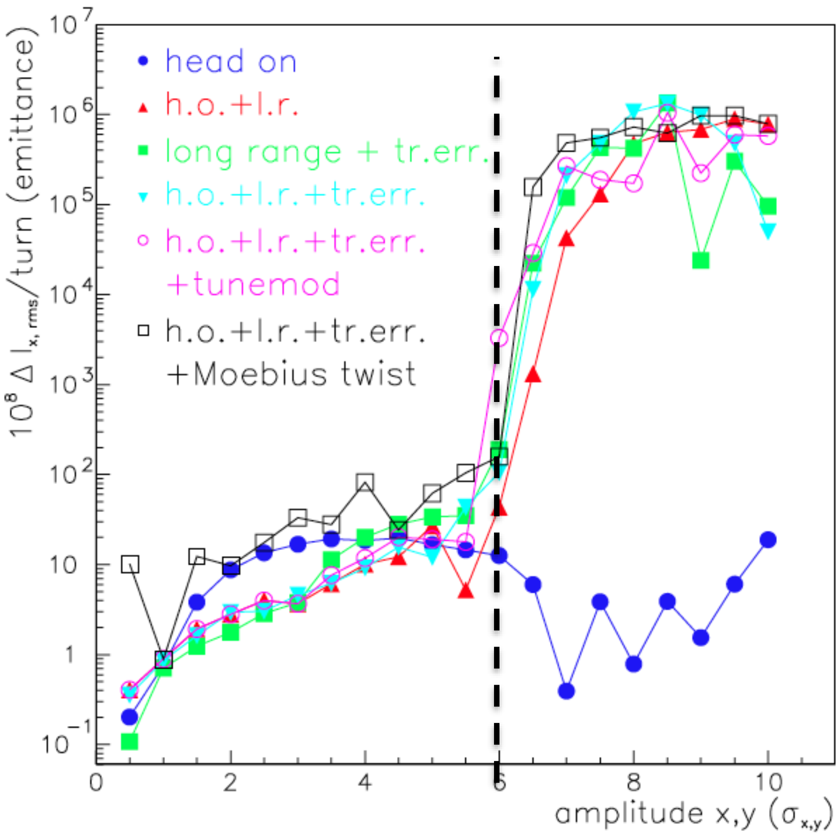
with $\theta_t \equiv \left((x'+\theta_c)^2 + {y'}^2\right)^{1/2}$

Head-on vs Long range interaction



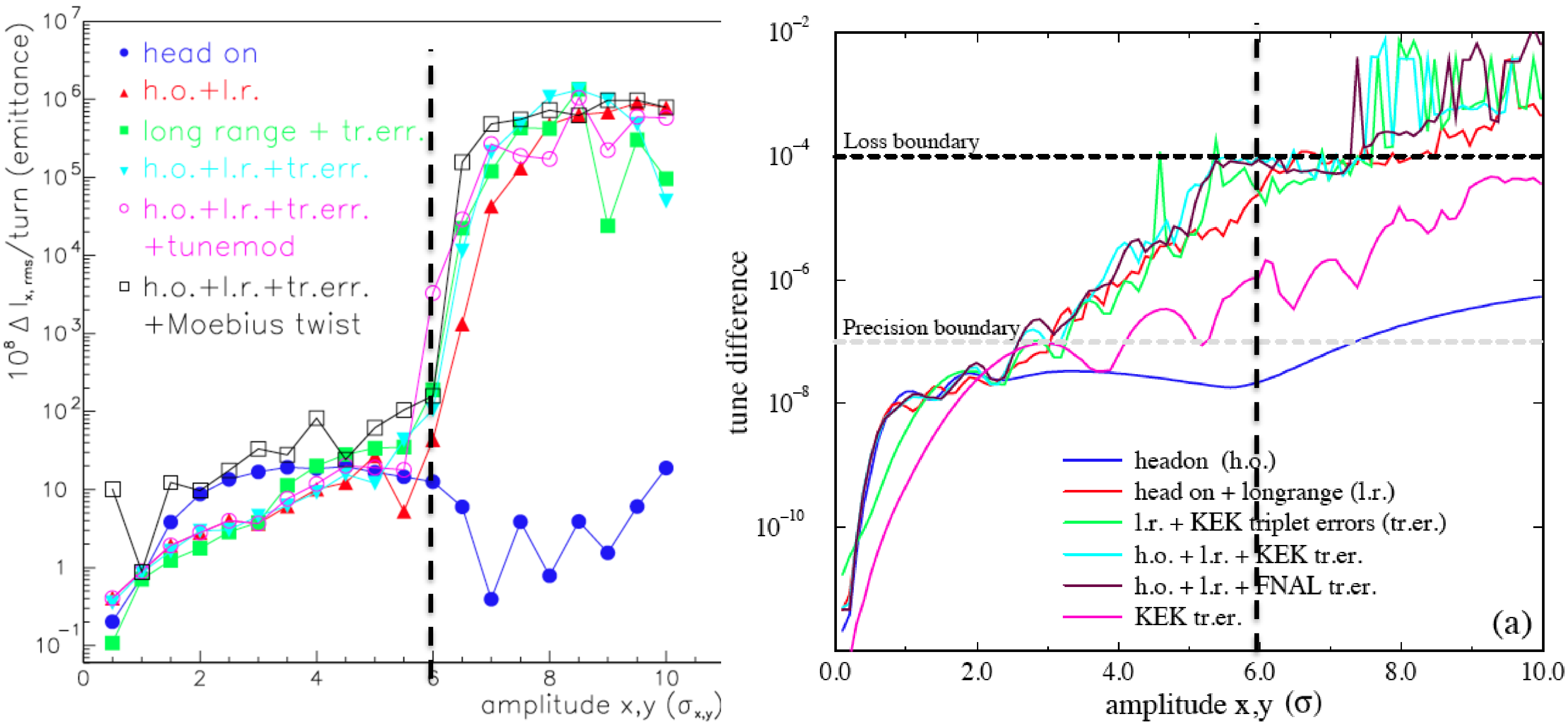


- Proved dominant effect of long range beam-beam effect
- Dynamic Aperture (around 6σ) located at the folding of the map (indefinite torsion)
- Experimental effort to compensate beam-beam long range effect with wires (1/r part of the force) or octupoles



- In the chaotic region of phase space, the action diffusion coefficient per turn can be estimated by averaging over the quasi-randomly varying betatron phase variable as

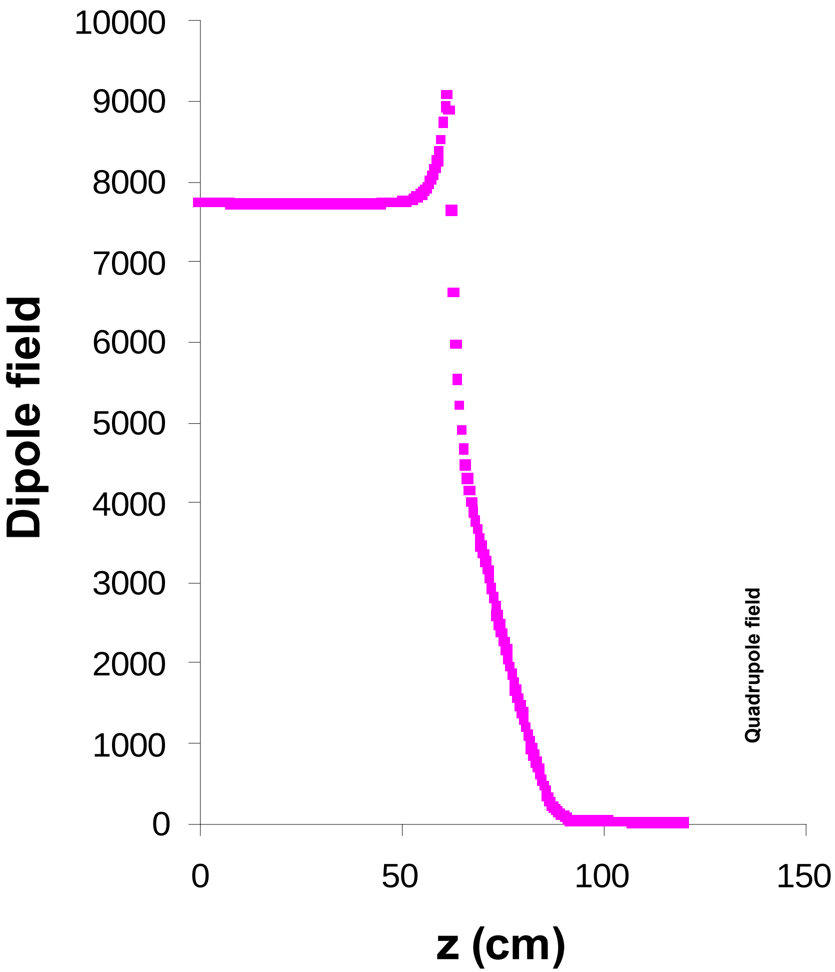
$$D(J) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, [\Delta J(\phi)]^2$$

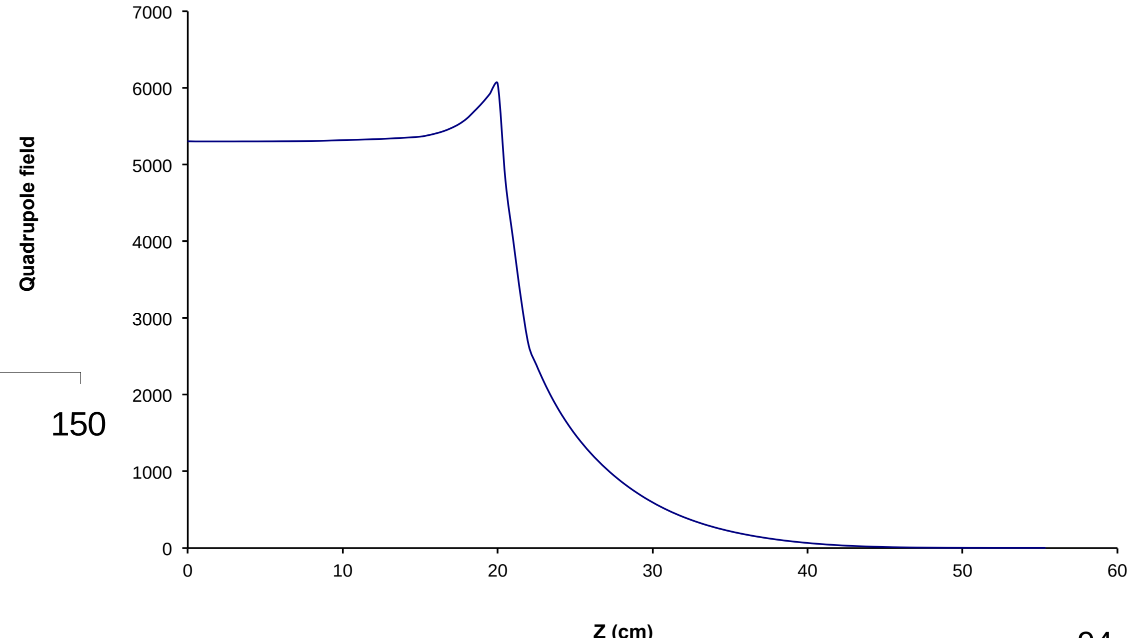


- Very good agreement of diffusive aperture boundary (action variance) with frequency variation (loss boundary corresponding to around 1 integer unit change in 10^7 turns)

Magnet fringe fields

- Up to now we considered only transverse fields
- Magnet fringe field is the longitudinal dependence of the field at the magnet edges
- Important when magnet aspect ratios and/or emittances are big





Quadrupole fringe field

General field expansion for a quadrupole magnet:

$$B_x = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n} y^{2m+1}}{(2n)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]}$$

$$B_y = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m}}{(2n+1)!(2m)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l]} \quad .$$

$$B_z = \sum_{m,n=0}^{\infty} \sum_{l=0}^{m} \frac{(-1)^m x^{2n+1} y^{2m+1}}{(2n+1)!(2m+1)!} \binom{m}{l} b_{2n+2m+1-2l}^{[2l+1]}$$

and to leading order

$$\begin{aligned} B_x &= y\left[b_1 - \frac{1}{12}(3x^2 + y^2) b_1^{[2]}\right] + O(5) \\ B_y &= x\left[b_1 - \frac{1}{12}(3y^2 + x^2) b_1^{[2]}\right] + O(5) \\ B_z &= xy b_1^{[1]} + O(4) \end{aligned}$$

The quadrupole fringe to leading order has an octupole-like effect

- From the hard-edge Hamiltonian

$$H_f = \frac{\pm Q}{12B\rho(1+\frac{\delta p}{p})}(y^3p_y - x^3p_x + 3x^2yp_y - 3y^2xp_x),$$

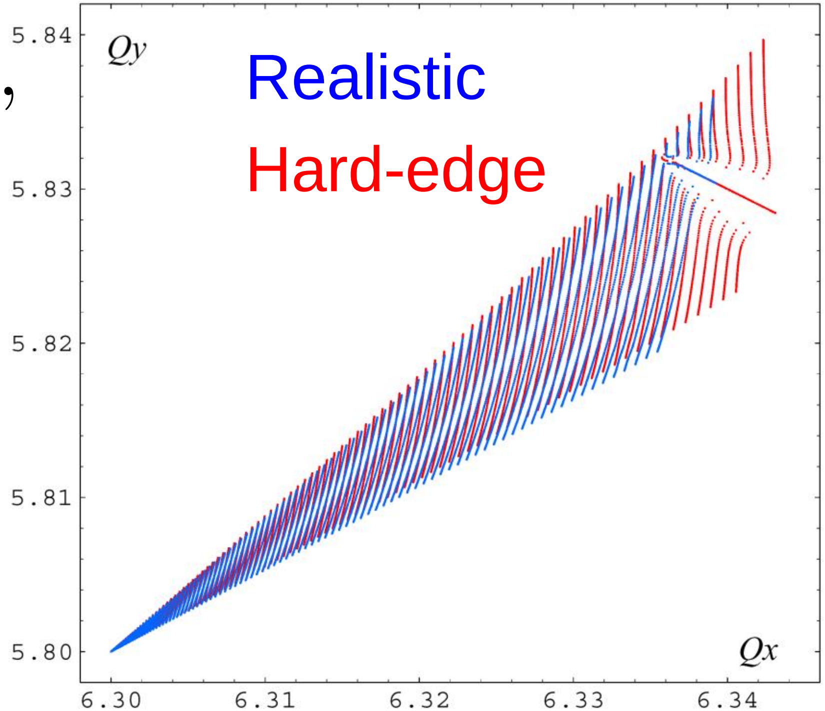
the first order shift of the frequencies with amplitude can be computed analytically

$$\begin{pmatrix}\delta\nu_x\\ \delta\nu_y\end{pmatrix} = \begin{pmatrix}a_{hh} & a_{hv}\\ a_{hv} & a_{vv}\end{pmatrix}\begin{pmatrix}2J_x\\ 2J_y\end{pmatrix},$$

with the ”anharmonicity” coefficients (torsion)

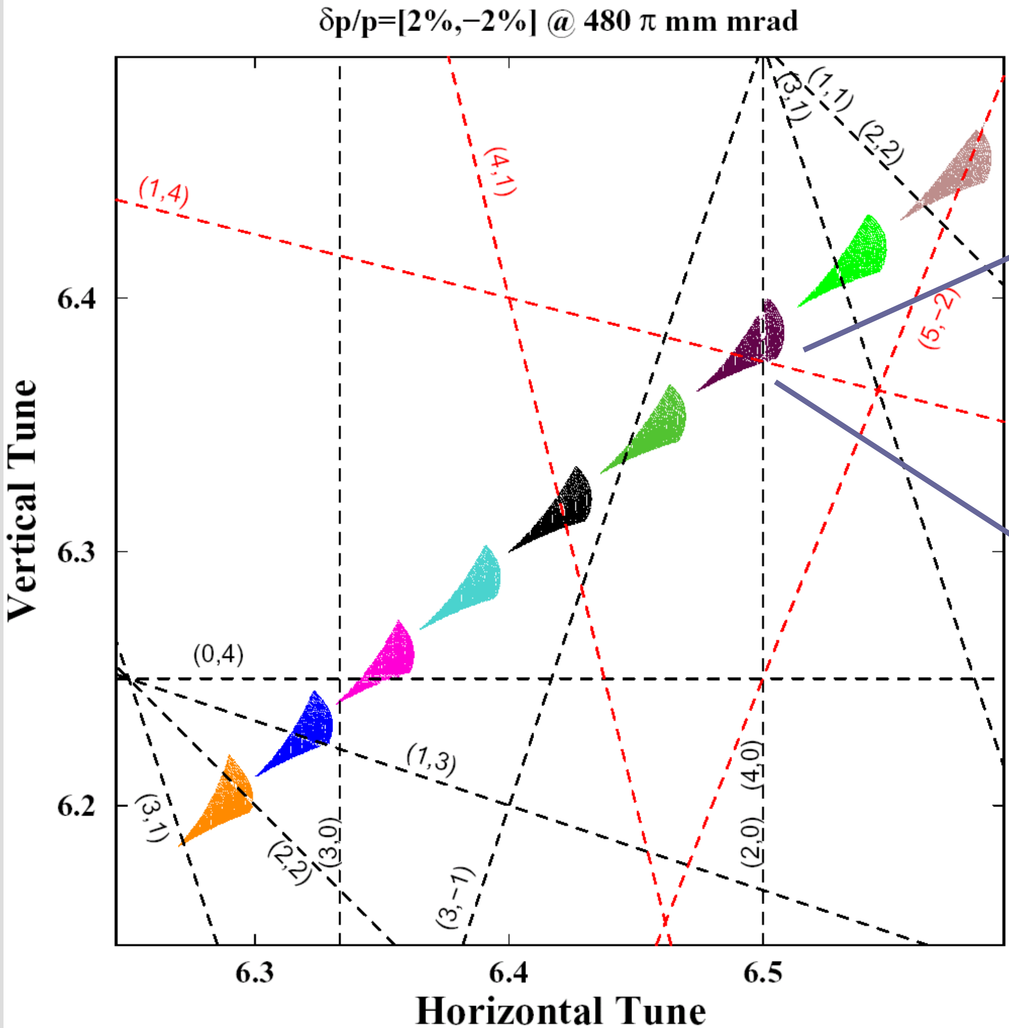
$$a_{hh} = \frac{-1}{16\pi B\rho}\sum_i \pm Q_i\beta_{xi}\alpha_{xi}$$
$$a_{hv} = \frac{1}{16\pi B\rho}\sum_i \pm Q_i(\beta_{xi}\alpha_{yi} - \beta_{yi}\alpha_{xi})$$
$$a_{vv} = \frac{1}{16\pi B\rho}\sum_i \pm Q_i\beta_{yi}\alpha_{yi}$$

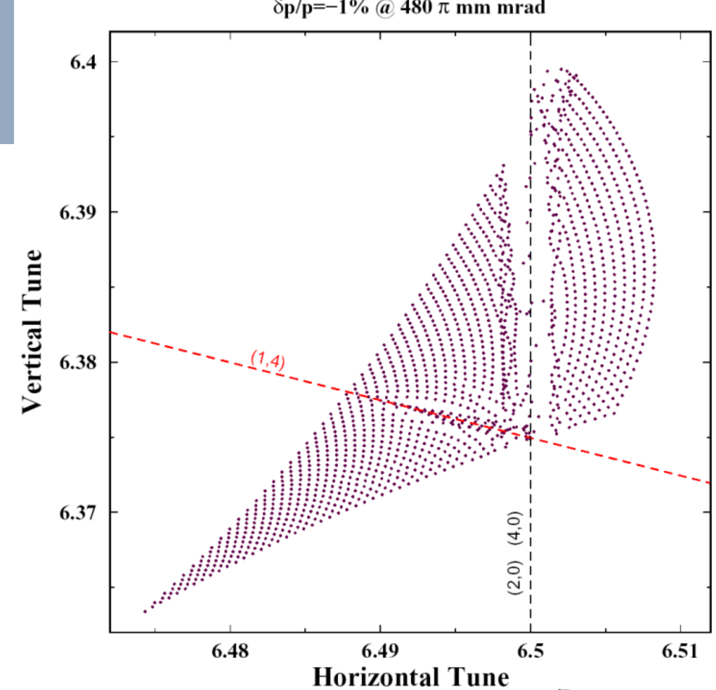
Tune footprint for the SNS based on hard-edge (red) and realistic (blue) quadrupole fringe-field

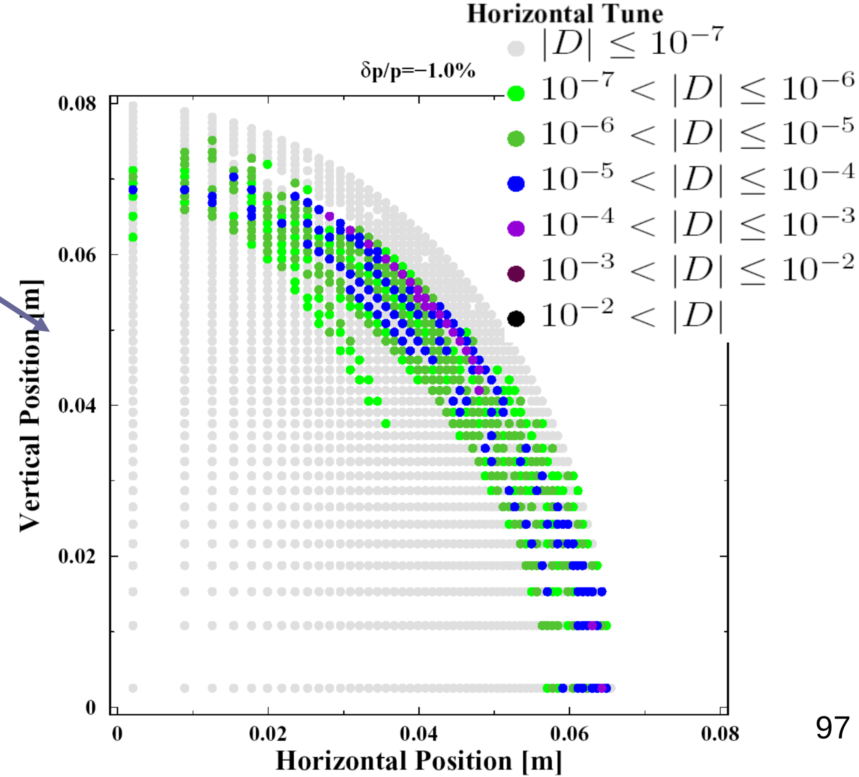


Off-momentum frequency maps

SNS Working Point $(Q_x,Q_y)=(6.4,6.3)$

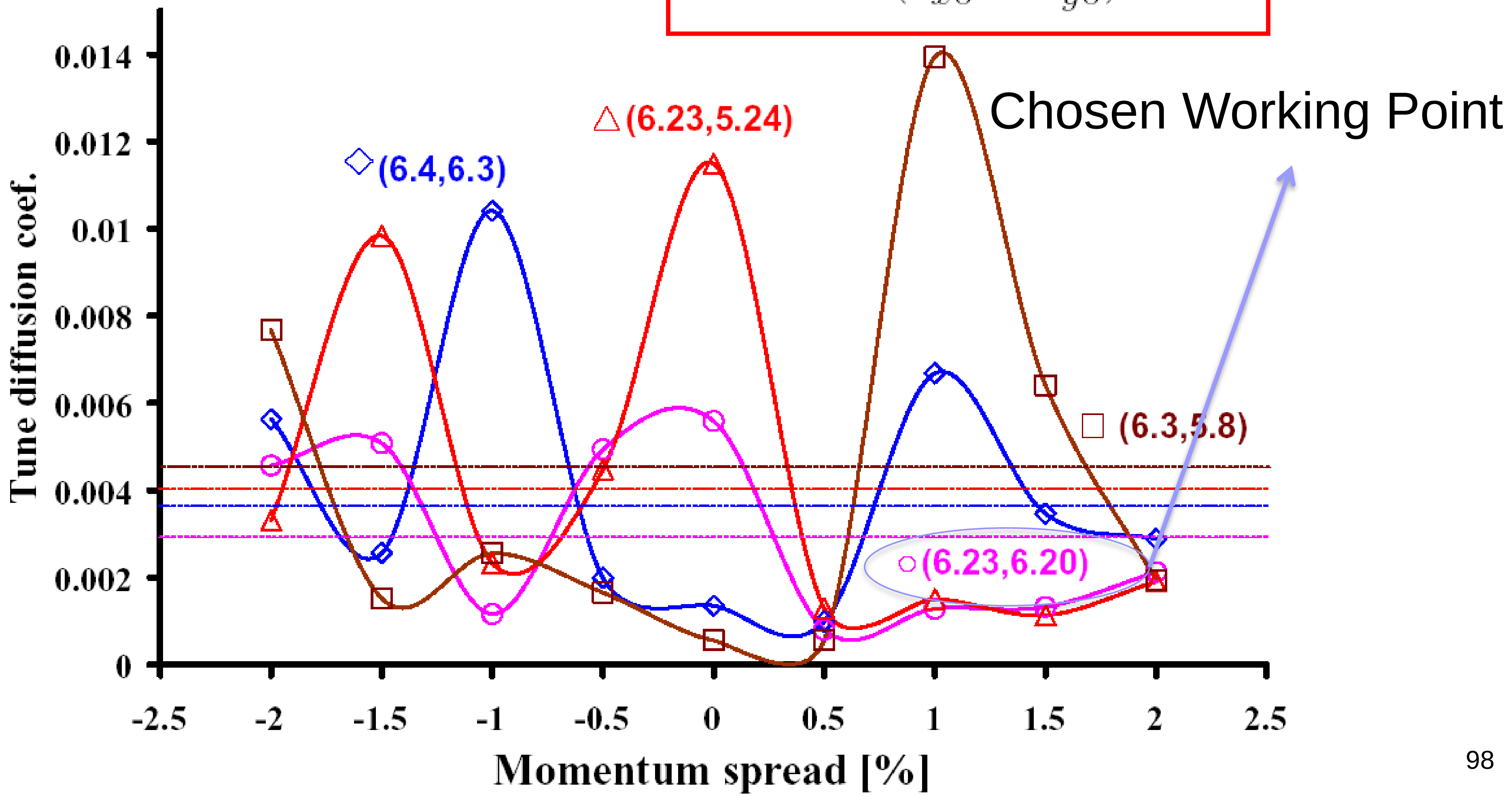






Tune Diffusion quality factor

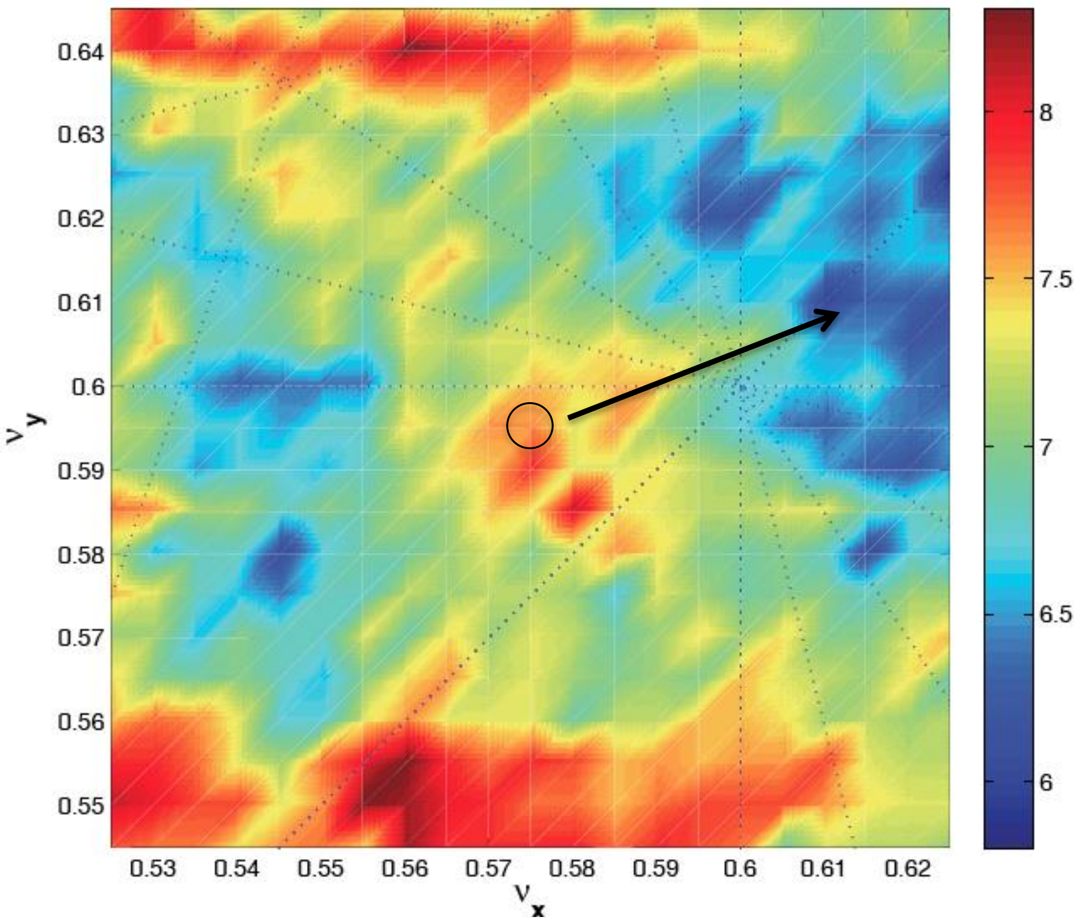
$$D_{QF} = \langle \frac{|\boldsymbol{D}|}{(I_{x0}^2 + I_{y0}^2)^{1/2}} \rangle_R$$



Global Working point choice

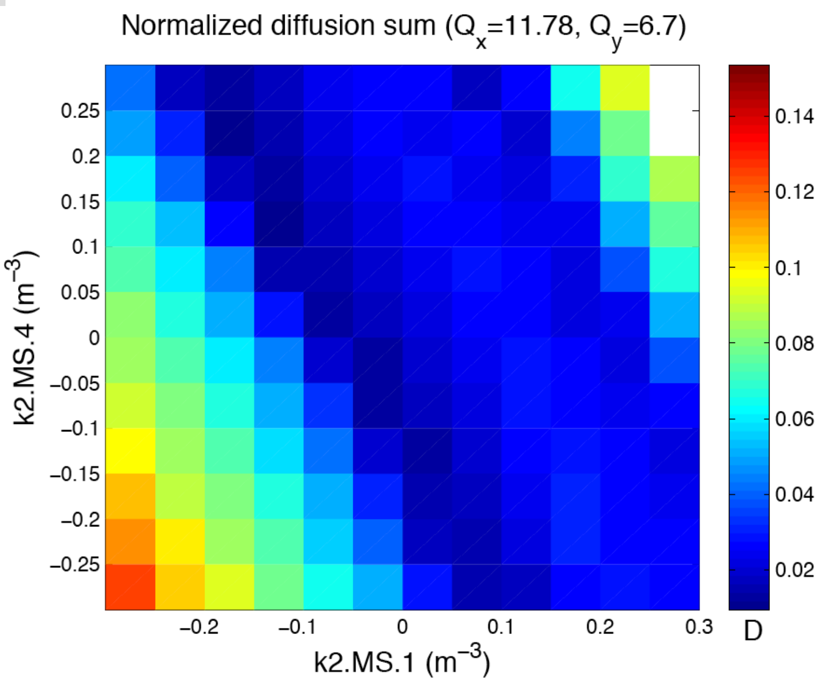
- Figure of merit for choosing best working point is sum of diffusion rates with a constant added for every lost particle
- Each point is produced after tracking 100 particles
- Nominal working point had to be moved towards "blue" area

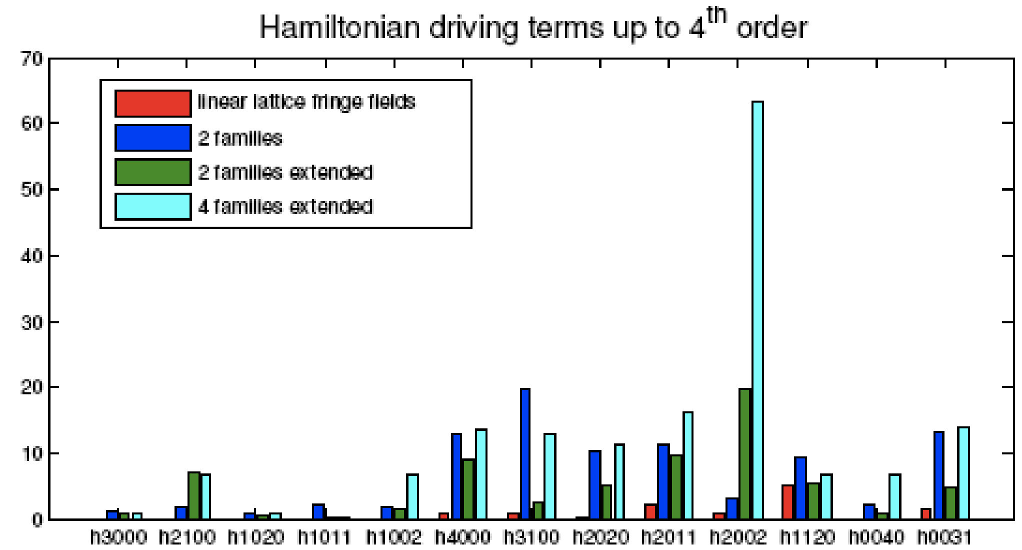
$$e^D = \sqrt{\frac{(\nu_{x,1} - \nu_{x,2})^2 + (\nu_{y,1} - \nu_{y,2})^2}{N/2}}$$



$$WPS = 0.1 N_{lost} + \sum e^D$$

Sextupole scheme optimization

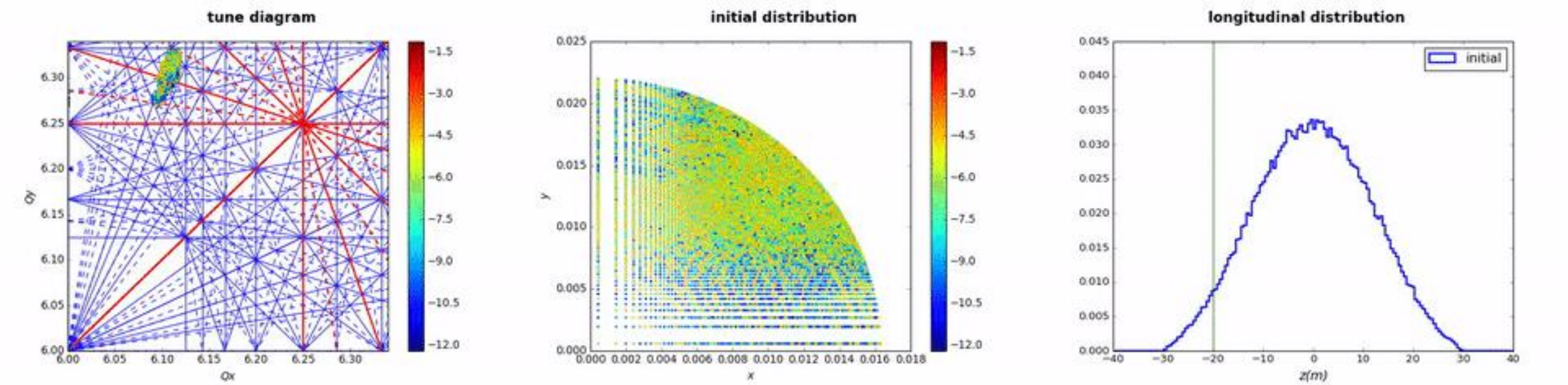




- Comparing different chromaticity sextupole correction schemes and working point optimization using normal form analysis, frequency maps and finally particle tracking
- Finding the adequate sextupole strengths through the tune diffusion coefficient

Frequency Map Analysis with modulation

Frequency maps with space-charge



F.Asvesta, et al., 2017

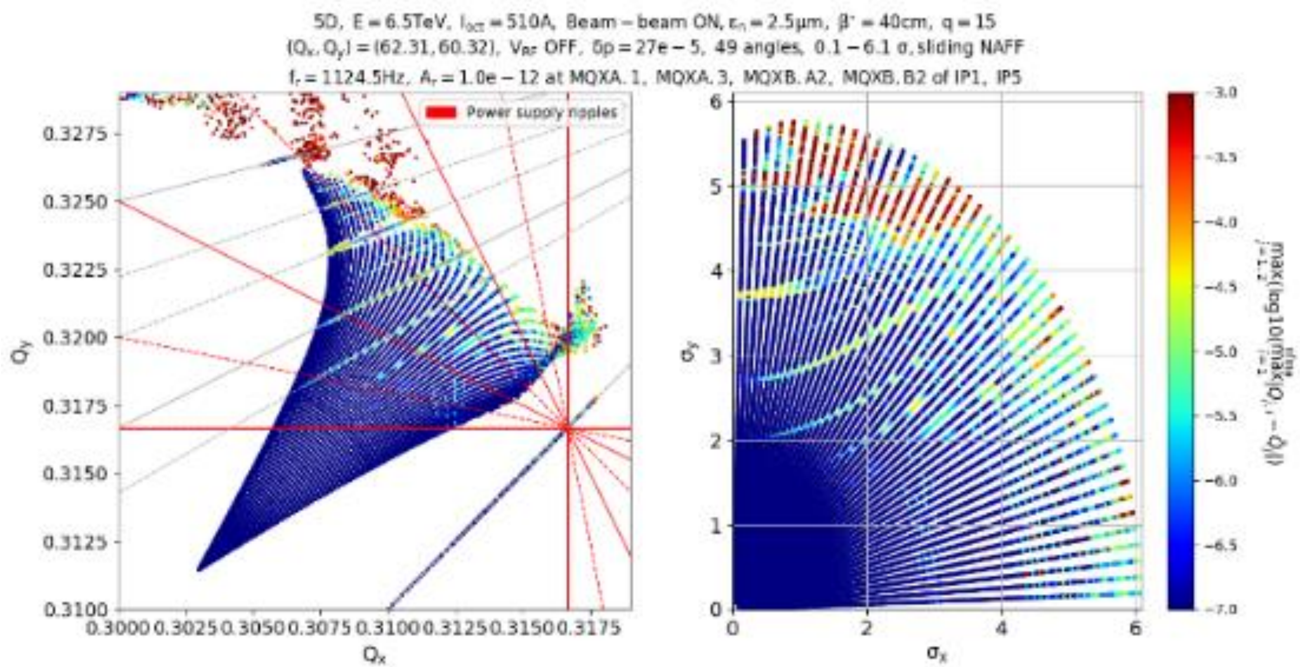
- Evolution of frequency map over different longitudinal position
- Tunes acquired over each longitudinal period
- Particles with similar longitudinal offset but different amplitudes experience the resonance in different manne
- Particles with different longitudinal offset may experienc different resonances

- Quadrupoles of the **inner triplet** right and left **of IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects
- **Resonance conditions**:

S. Kostoglou, et al., 2018

$$aQ_x + bQ_y + c\frac{f_{modulation}}{f_{revolution}} = k \text{ for } a, b, c, k \text{ integers}$$

-By increasing the modulation depth, sidebands start to appear in the FMAs



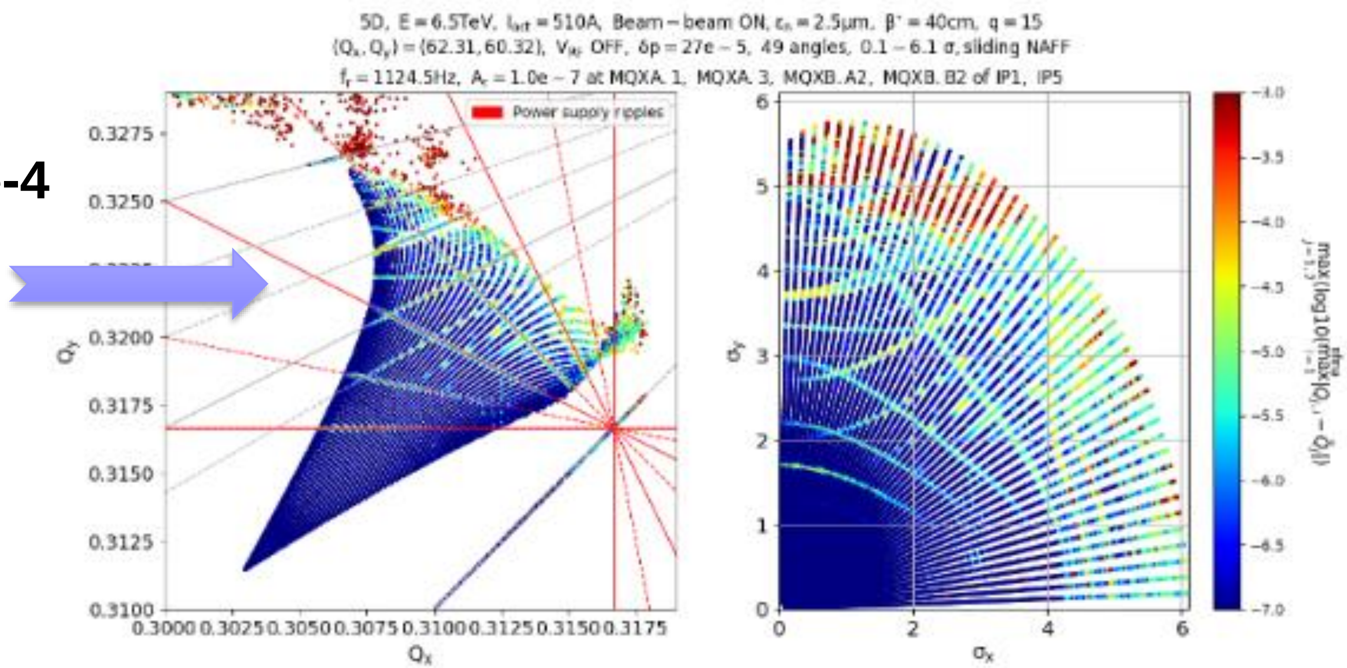
- Quadrupoles of the **inner triplet** right and left **of IP1 and IP5**, **large beta-functions** increase the sensitivity to non-linear effects
- **Resonance conditions**:

S. Kostoglou, et al., 2018

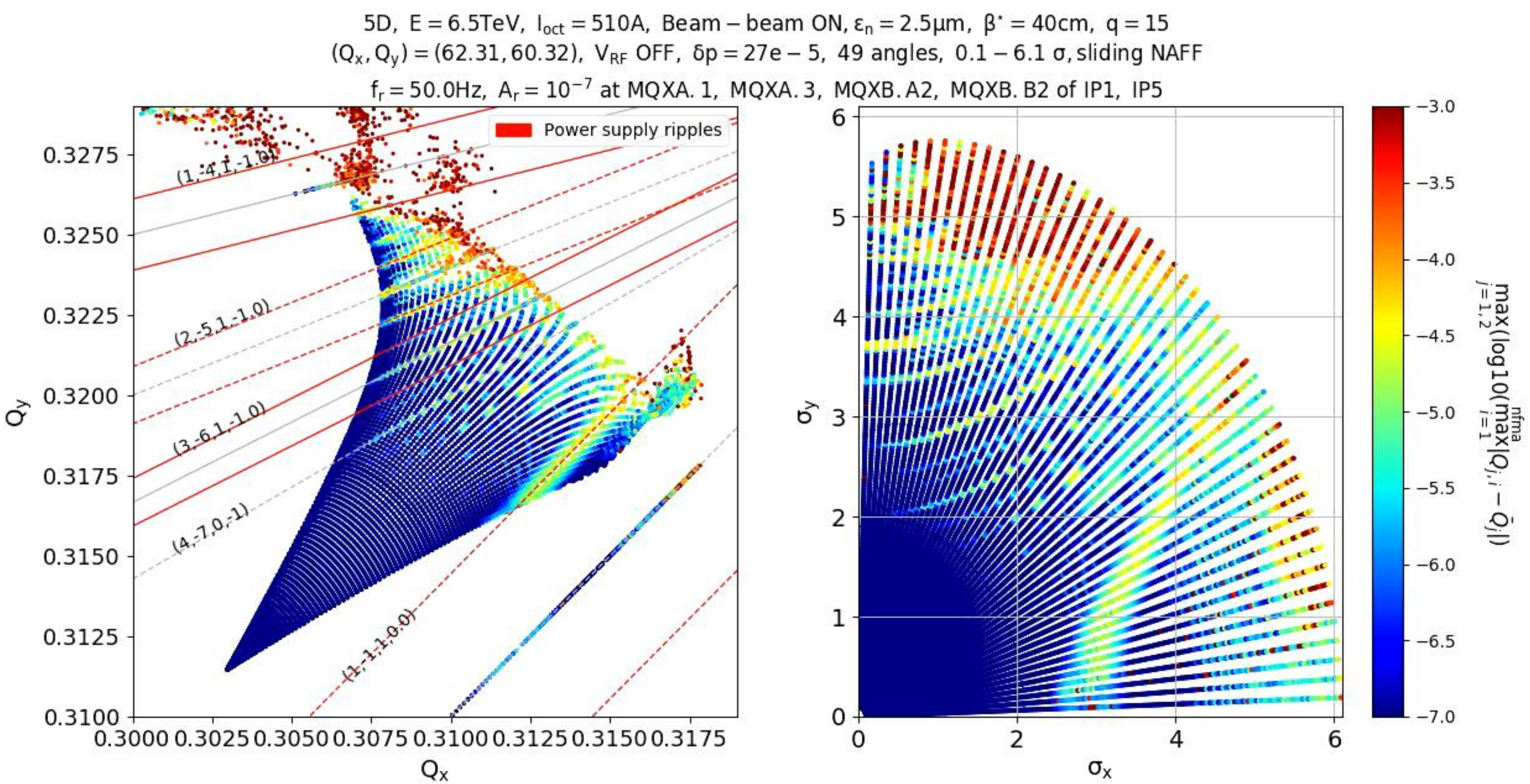
$$aQ_x + bQ_y + c\frac{f_{modulation}}{f_{revolution}} = k \text{ for } a, b, c, k \text{ integers}$$

-**By increasing the modulation depth, sidebands start to appear in the FMAs**

ΔQ=1e-4

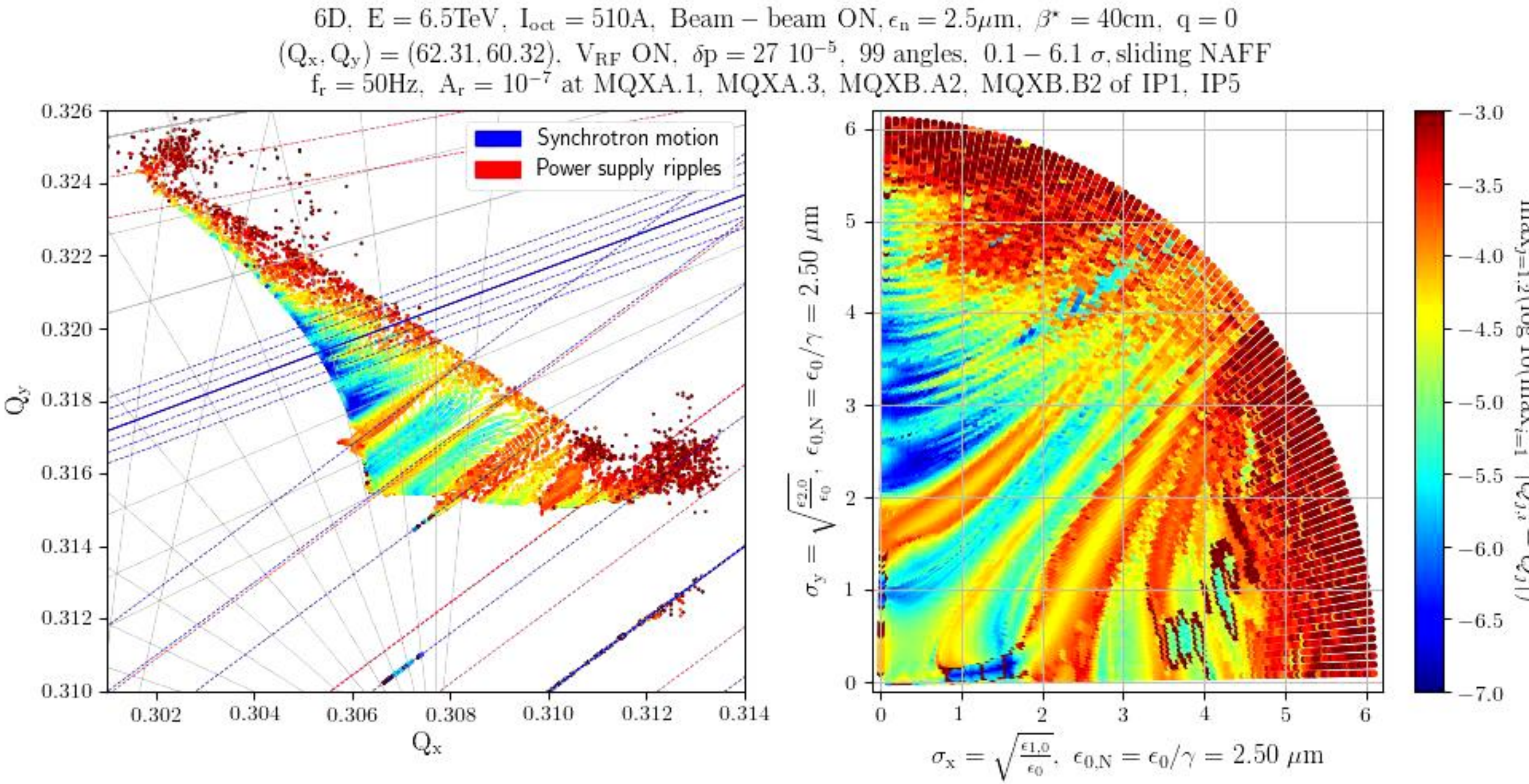


- Scan of different ripple frequencies **(50-900 Hz)**



S. Kostoglou, YP et al., 2018

6D FMAs with power supply ripples



S. Kostoglou, YP et al., 2018

Summary

- Appearance of **fixed points** (periodic orbits) determine **topology** of the phase space
- **Perturbation** of unstable (hyperbolic points) opens the path to chaotic motion
- Resonance can overlap enabling the rapid diffusion of orbits
- **Dynamic aperture** by brute force tracking (with symplectic numerical integrators) is the usual quality criterion for evaluating non-linear dynamics performance of a machine
- **Frequency Map Analysis** is a numerical tool that enables to study in a global way the dynamics, by identifying the excited **resonances** and the extent of **chaotic** regions
- It can be directly applied to **tracking** and **experimental** data
- A combination of these modern methods enable a thorough analysis of non-linear dynamics and lead to a robust design

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