Randomness Characterization through Bayesian Model Selection

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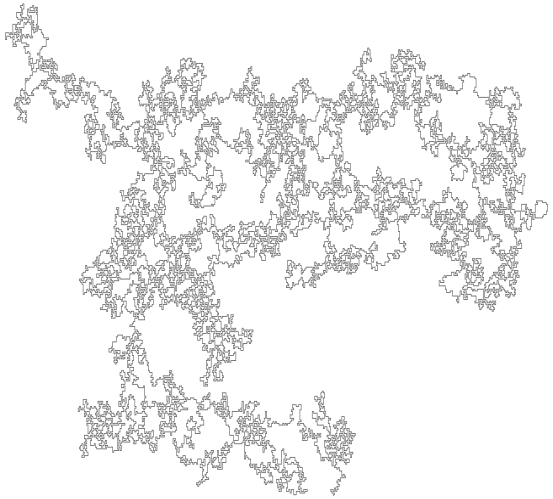
AISIS, UNAM. October 2019

How to tell if a number sequence is random?

$\hat{s} = HHTTTHTHT \ldots THTT$



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- Dynamical systems mappings
- Spin systems
- Correlated photons
- Particles decay

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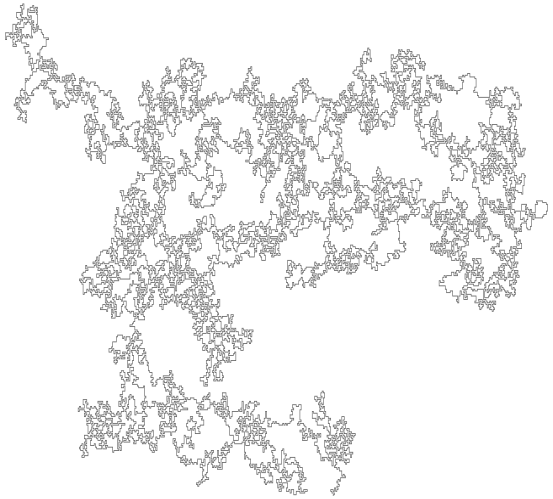

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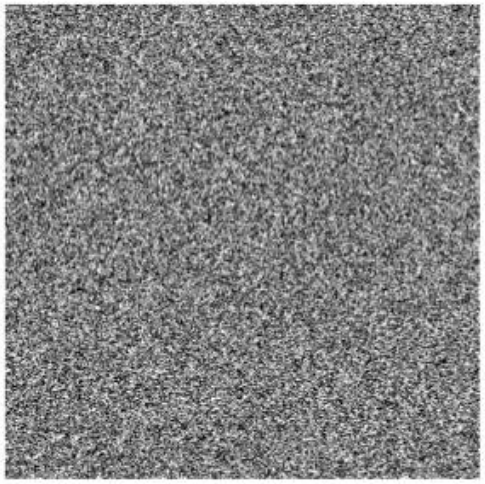
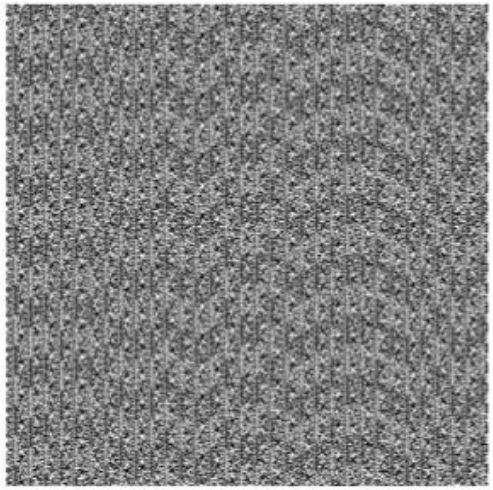
- Monte Carlo methods
- Cryptography
- Probabilistc algorithms

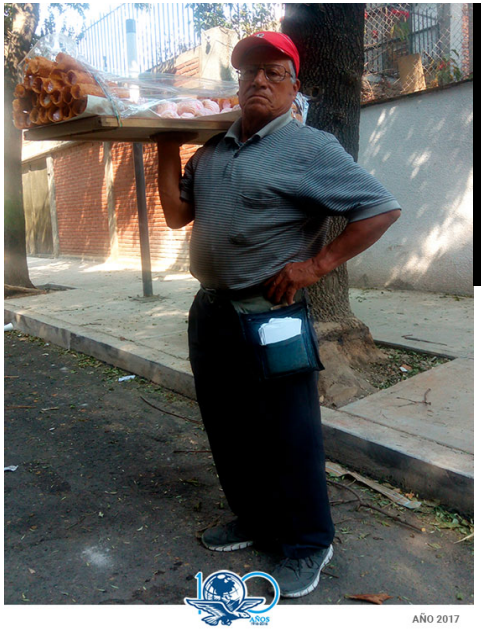
Pseudo-random bits



rand() in PHP on Windows



(Maximally) random sequences



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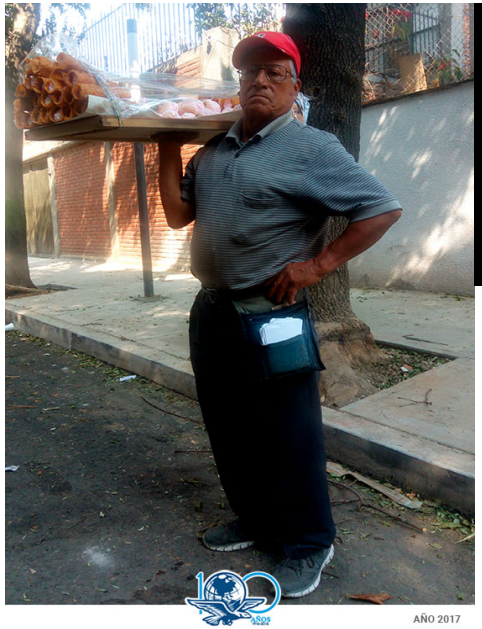

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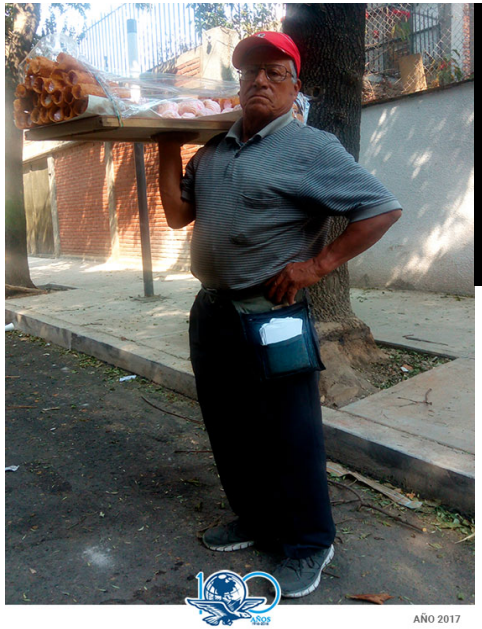

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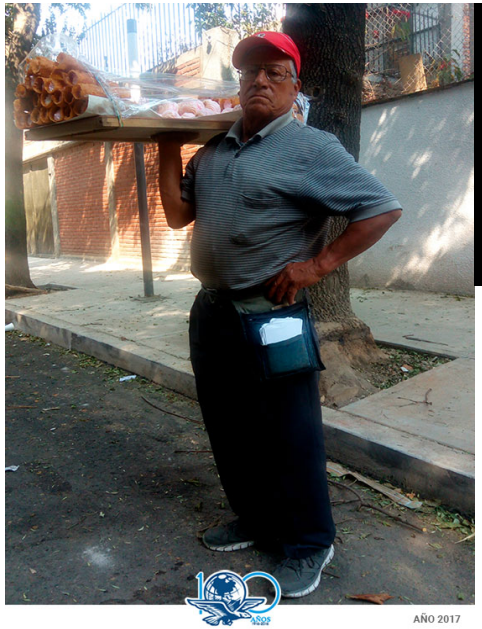

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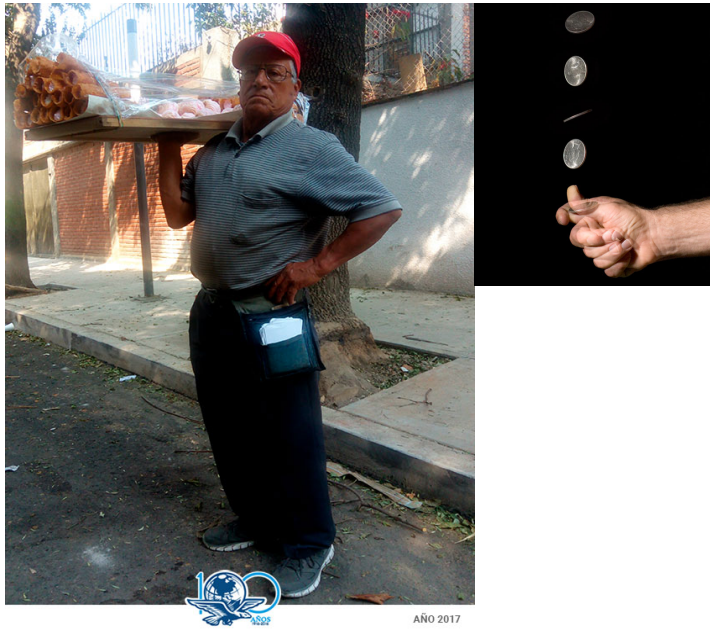

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$H[X] \sim$ Measure of randomness

$$H[X] = -\sum_x p_x \log_2 p_x$$

$$0 \leq H[X] \leq \log_2 |\mathbf{X}|$$

$$H_{\max} = \log_2 |\mathbf{X}| \iff p_x = \frac{1}{|\mathbf{X}|}$$

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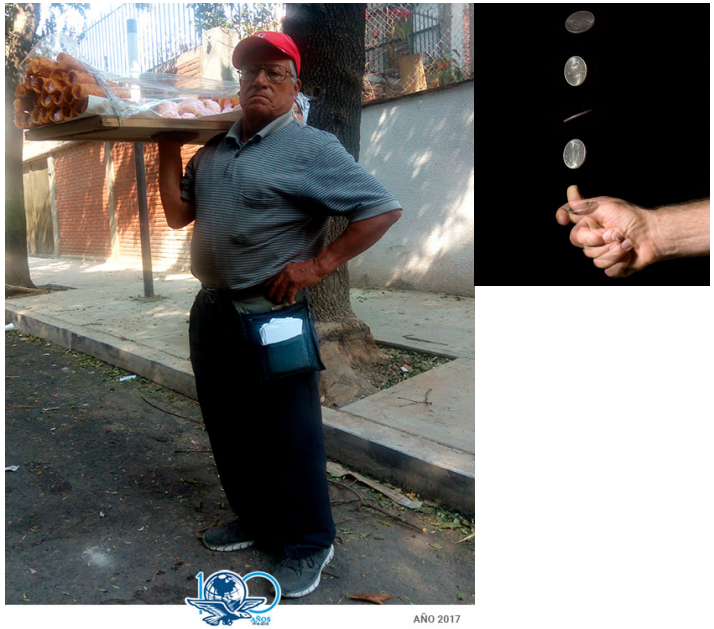

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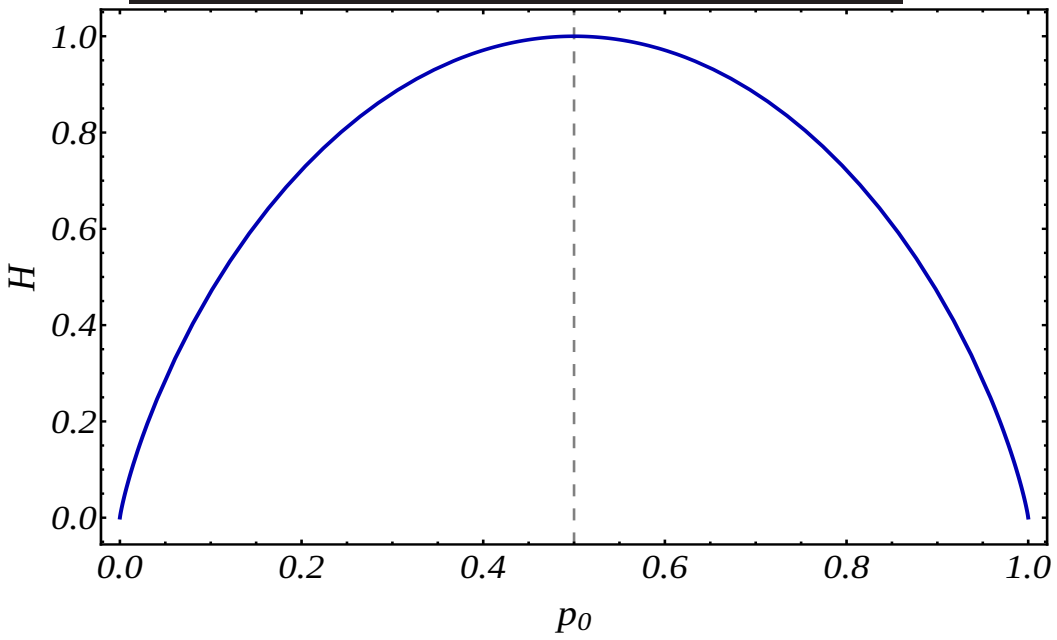

$$H[X] = -p_0 \log_2 p_0 - (1 - p_0) \log_2 (1 - p_0)$$

Pragmatic approach: NIST battery of tests

$$\hat{s} = \texttt{01001101...0010} \quad \Longrightarrow \begin{cases} \text{Same frequency of `0' and `1' } (k_0 \approx k_1) \\ \text{Longest string of consecutive 0's} \\ \text{Fourier transform } \sim \text{ white noise} \\ \vdots \end{cases}$$

Each property is analysed as an *hypothesis test* $\Longrightarrow$ obtain a p-value

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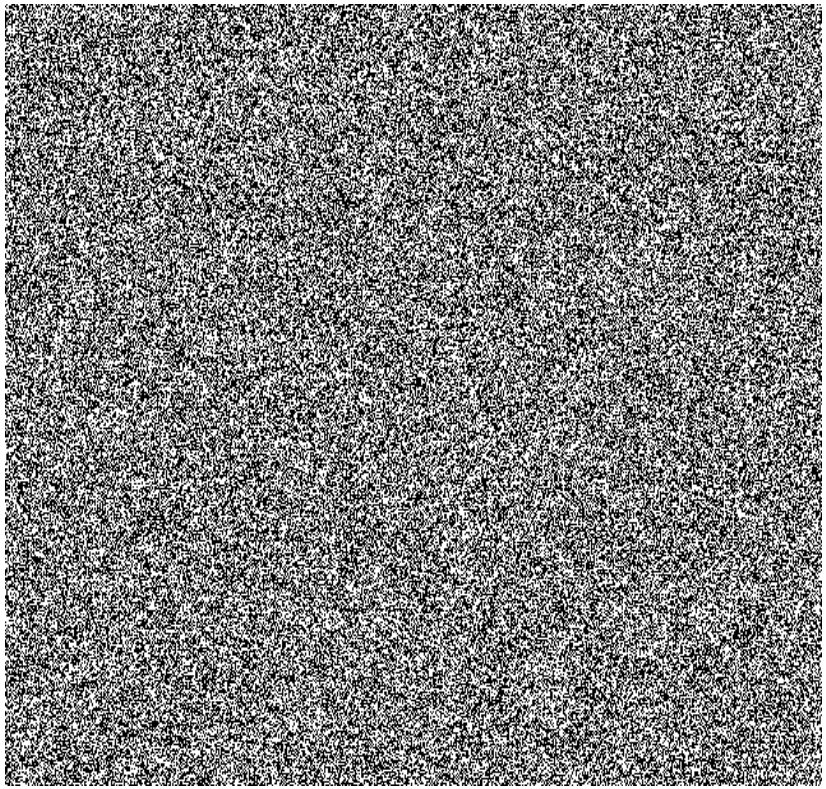
- **If** "random" $\Longrightarrow$ properties examined with the tests.
- Properties $\not\Longrightarrow$ randomness
- *Frequentist* approach based on p-values

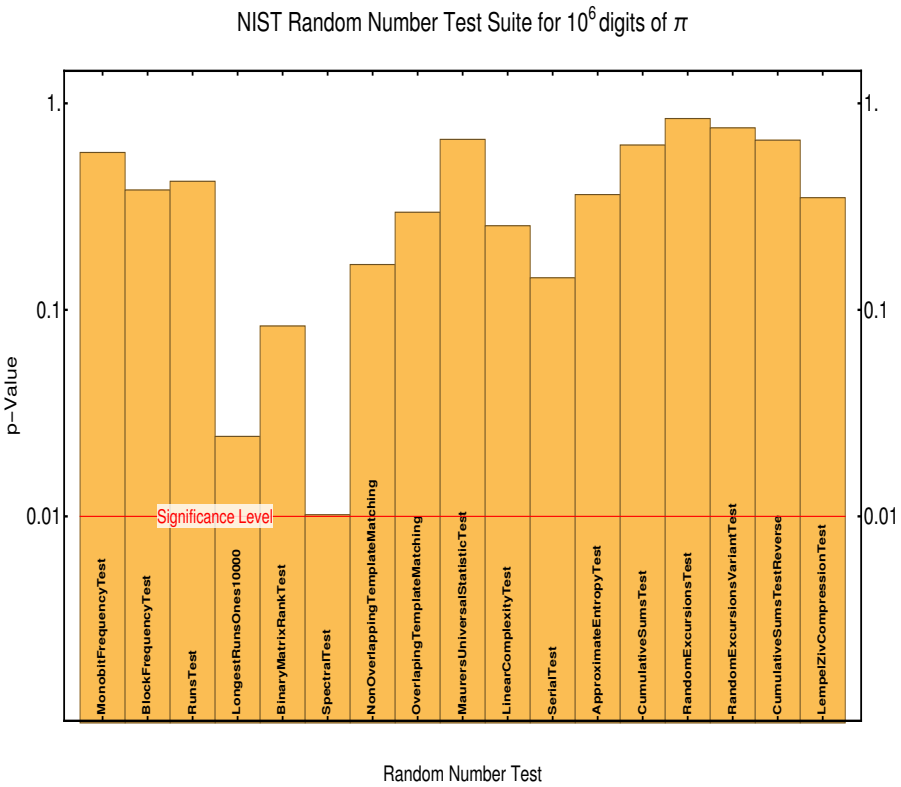
R. L. Wasserstein and N. A. Lazar, "The ASA's statement on p-values: context, process, and purpose", The American Statistician, 129–133 (2016)

M. Baker, "Statisticians issue warning on p-values", Nature **531**, 151 (2016)

Is π random?



(a) Binary representation of the first 302,500 digits of π.



(b) Results of the 15 NIST tests

Randomness as "incompressibility"

Algorithmic Information Theory

- $\hat{s}$ is random **iff** the "shortest" *algorithm* to generate it is `print`($\hat{s}$).
- AIT (Chaitin, Kolmogorov, Somolonoff): a *mathematically formal* theory that identifies (computationally) random $\sim$ incompressible.
- There is NO *general* algorithm capable of assessing whether *any* sequence is random.
- ... *but* Borel's Normality criterion

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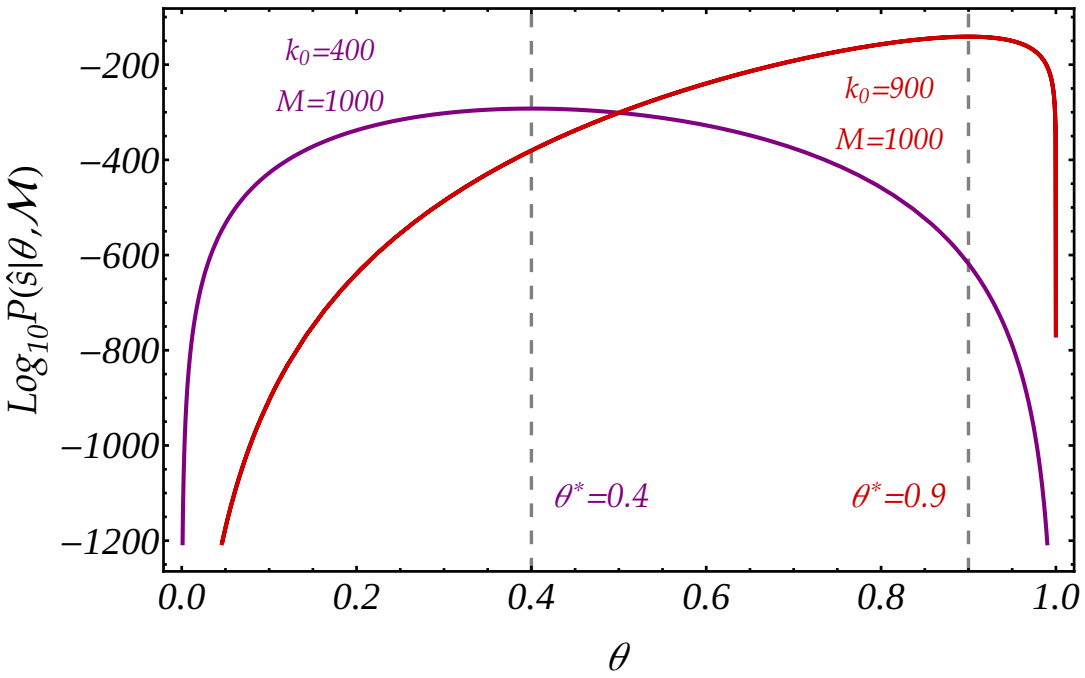
$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}, \qquad \pi = 2\cdot\frac{2}{\sqrt{2}}\cdot\frac{2}{\sqrt{2+\sqrt{2}}}\cdot\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots$$

Primer on statistical inference for bit sequences

$|\hat{s}| = M$ bits, with k_0 and k_1 being the frequencies of `0`'s and '`1`'s; $k_0 + k_1 = M$.

$$\mathcal{M}: \quad p_0 = \theta; \quad p_1 = 1 - \theta \qquad \Longrightarrow \qquad P(\hat{s}|\theta, \mathcal{M}) = \theta^{k_0}(1-\theta)^{k_1} \ .$$



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$$\theta^* = k_0/M$$

Fair "coin" (**RNG**)
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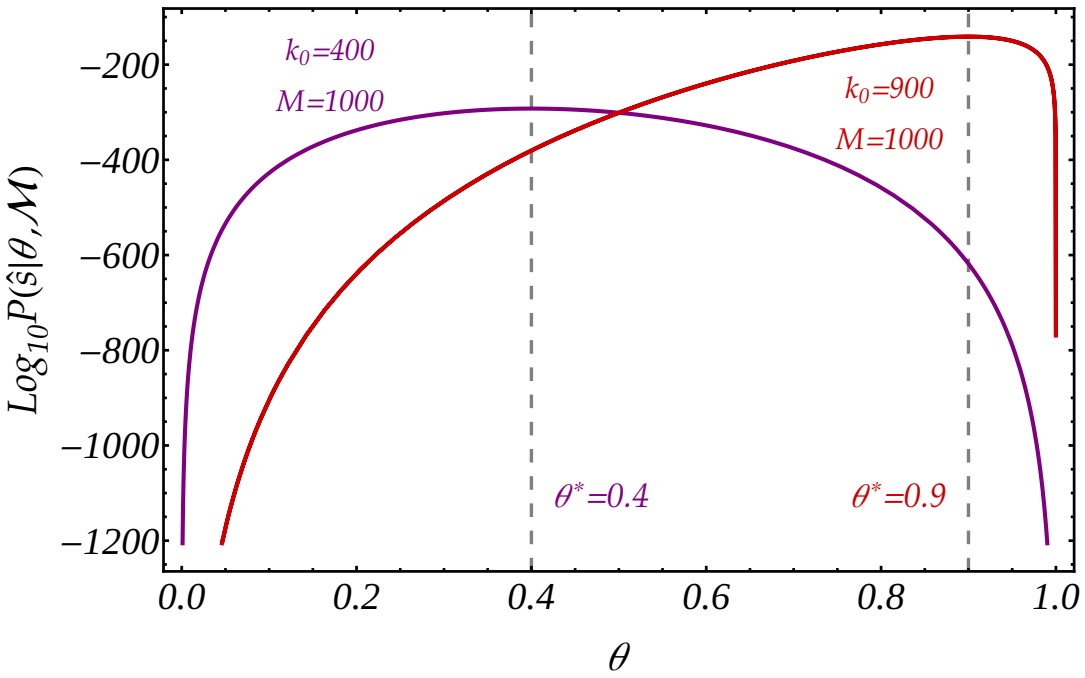

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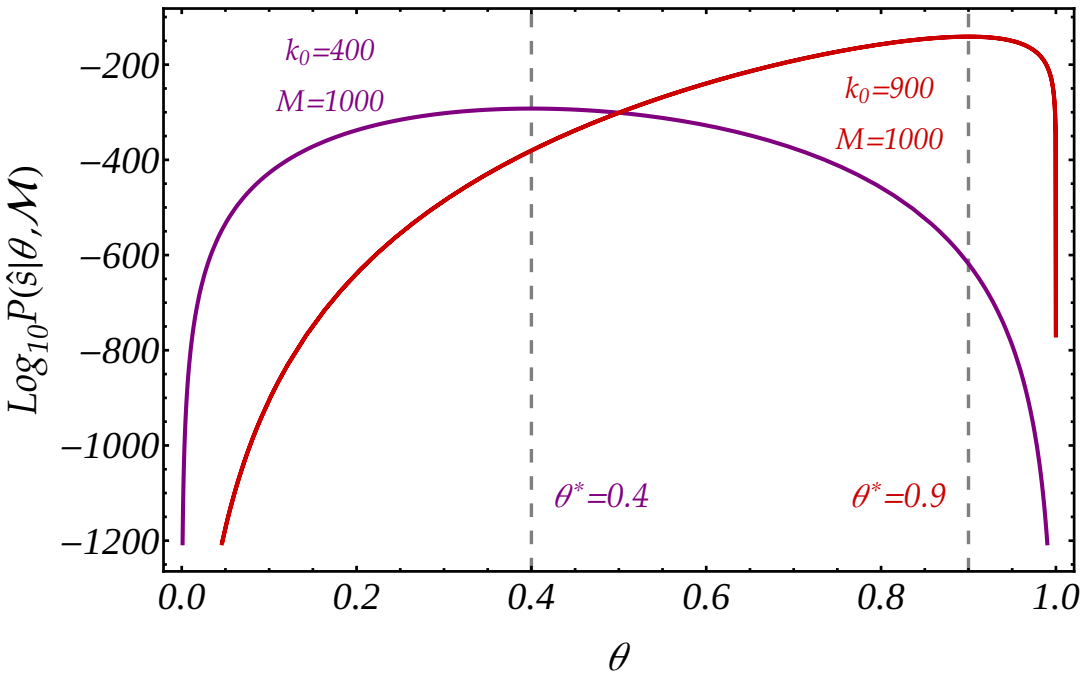

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Recipe for how to update our (un)certainty about a model – *hypothesis* – given some data.

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$$\{\mathcal{M}_\alpha\}_{\alpha=1}^N \xrightarrow[\hat{s}]{\text{Bayes}} \left\{P(\mathcal{M}_\alpha|\hat{s})\right\}_{\alpha=1}^N \Longrightarrow \mathcal{M}^* = \arg\max_{\mathcal{M}_\alpha} P(\mathcal{M}_\alpha|\hat{s})$$

Binary models: Likelihoods and inference

$\hat{s} = \texttt{0100110101...1110100101}$, $|\hat{s}| = M$ bits, with k_0 and k_1 the frequencies of '0's and '1's; $k_0 + k_1 = M$.

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$P(\theta|\mathcal{M}_0) = \delta(\theta - \frac{1}{2})$

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Binary models: phase diagrams

Posterior:

$$P(\mathcal{M}_\alpha|\hat{s}) = \frac{P(\hat{s}|\mathcal{M}_\alpha)P_0(\mathcal{M}_\alpha)}{P_0(\hat{s})}$$
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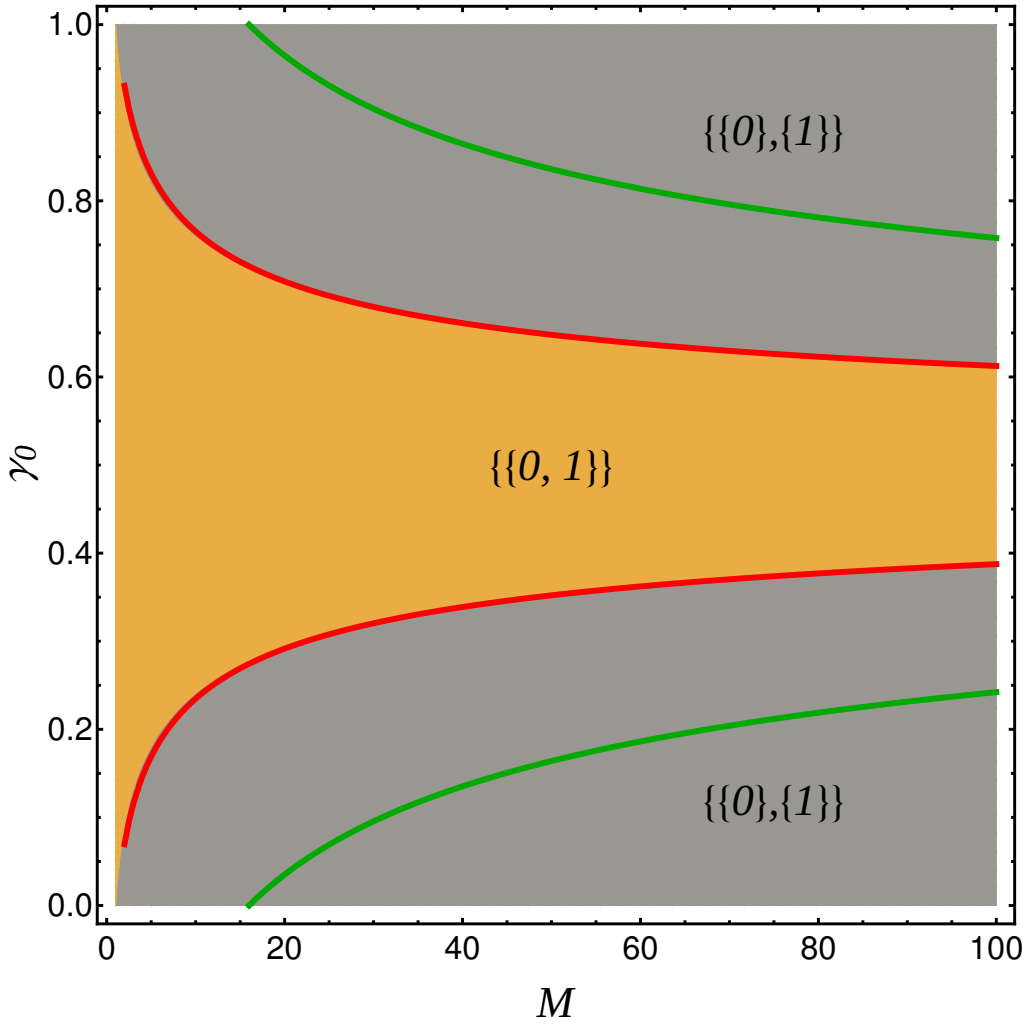
Region where $\frac{P(\mathcal{M}_0|\hat{s})}{P(\mathcal{M}_1|\hat{s})} > 1$

Region allowed by Borel Normality



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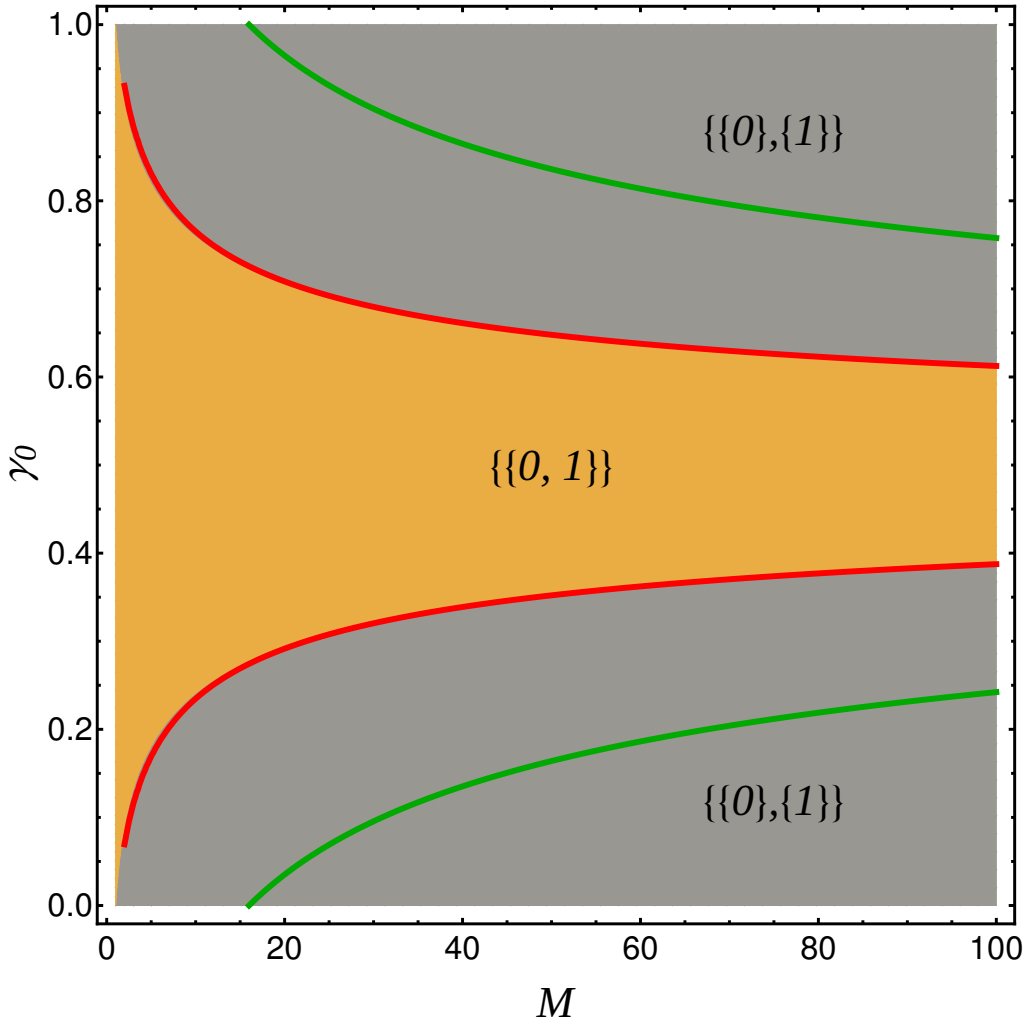

$$\Xi = \{0,1\} \longrightarrow 2 \text{ partitions} \Longleftrightarrow 2 \text{ models}$$

$\mathcal{M}_0 \longleftrightarrow \{\{0,1\}\}$ $\qquad$ $\mathcal{M}_1 \longleftrightarrow \{\{0\},\{1\}\}$

Partitions as a tool to identify regularities

What about the sequence $\hat{s}_{\text{reg}} = \texttt{0101010101...010101}$? ¿¿¿It's random???

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e.g., with $\beta = 2$ there are *six* partitions of $\Xi_2 = \{\texttt{0,1,2,3}\}$ in $K = 3$ subsets.

$$\alpha_3^{(1)} = \{\{\texttt{0}\}, \{\texttt{1}\}, \{\texttt{2,3}\}\}; \quad \alpha_3^{(2)} = \{\{\texttt{0}\}, \{\texttt{2}\}, \{\texttt{1,3}\}\}; \quad \alpha_3^{(3)} = \{\{\texttt{0}\}, \{\texttt{3}\}, \{\texttt{1,2}\}\}$$

$$\alpha_3^{(4)} = \{\{\texttt{1}\}, \{\texttt{2}\}, \{\texttt{0,3}\}\}; \quad \alpha_3^{(5)} = \{\{\texttt{1}\}, \{\texttt{3}\}, \{\texttt{0,2}\}\}; \quad \alpha_3^{(6)} = \{\{\texttt{2}\}, \{\texttt{3}\}, \{\texttt{0,1}\}\}$$

Partitions as a tool to identify regularities

What about the sequence $\hat{s}_{\text{reg}} = \texttt{0101010101...010101}$? ¿¿¿It's random???

$$\hat{s} = \underbrace{\texttt{110}}_{\beta}\texttt{100101011...10110} \quad \xrightarrow[\text{simultaneously}]{\text{“read” } \beta \text{ bits}} \quad \begin{cases} \beta = 1, & \hat{s} = \texttt{110100101011...010110} \\ \beta = 2, & \hat{s} = \texttt{310223...112} \\ \beta = 3 & \hat{s} = \texttt{6453...15} \end{cases}$$

$$\beta = 2 \implies \hat{s}_{\text{reg}} = \texttt{11111...111}.$$

One partition $\iff$ one model

Partitions represent different ways to assign *biases* to strings.

$$\mathcal{M}_{\alpha_K^{(l)}} : \quad p_j = \frac{\theta_r}{\left|\omega_K^{(l)}(r)\right|}; \quad \forall j \in \omega_K^{(l)}(r); \quad r = 1, \ldots, K\,.$$

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$$\alpha_3^{(3)} = \{\{\texttt{0}\}, \{\texttt{3}\}, \{\texttt{1,2}\}\} \quad \implies \quad \mathcal{M}_{\alpha_3^{(3)}} : \quad p_0 = \theta_0 \neq p_3 = \theta_1 \neq p_1 = p_2 = \frac{\theta_2}{2}\,;$$

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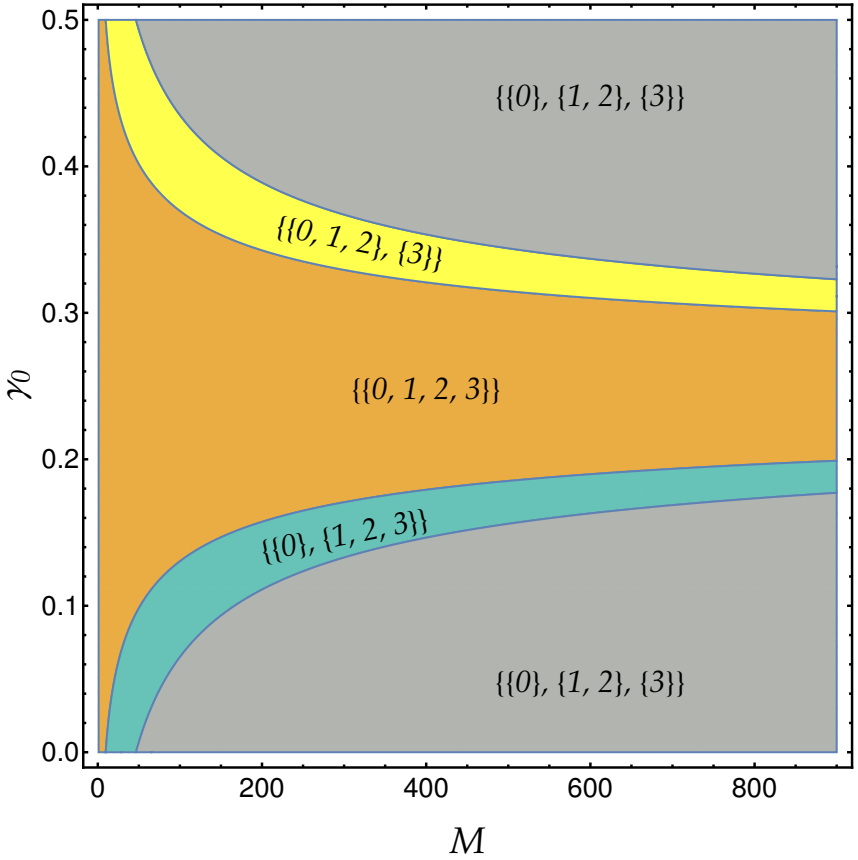
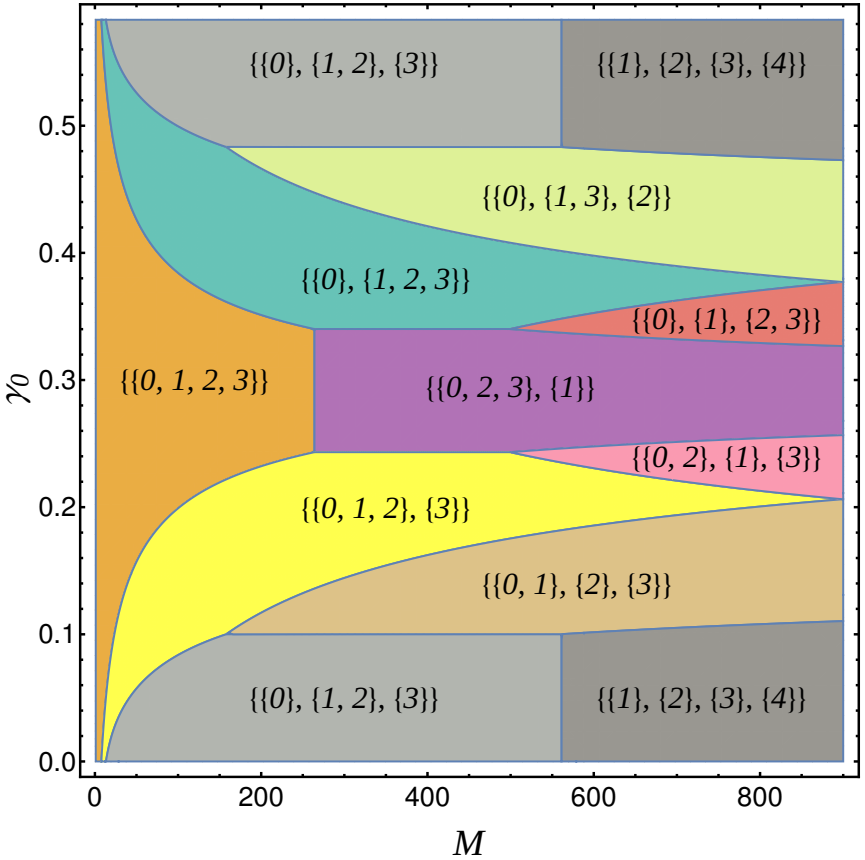
α_1 is **unique** $\implies \mathcal{M}_{\alpha_1} \equiv \mathcal{M}_{\text{sym}} : \; p_j = \frac{1}{2^\beta}$ is the **only** model describing a *maximally random* process .

Phase diagrams: $\beta = 2$

We can perform the model selection using only information about the frequencies $\{\gamma_i = \frac{\beta k_i}{M}\}_{i=0}^{2^\beta - 1}$ and the sequence length, M.



(a) $\gamma_1 = \gamma_2 = 1/4$.

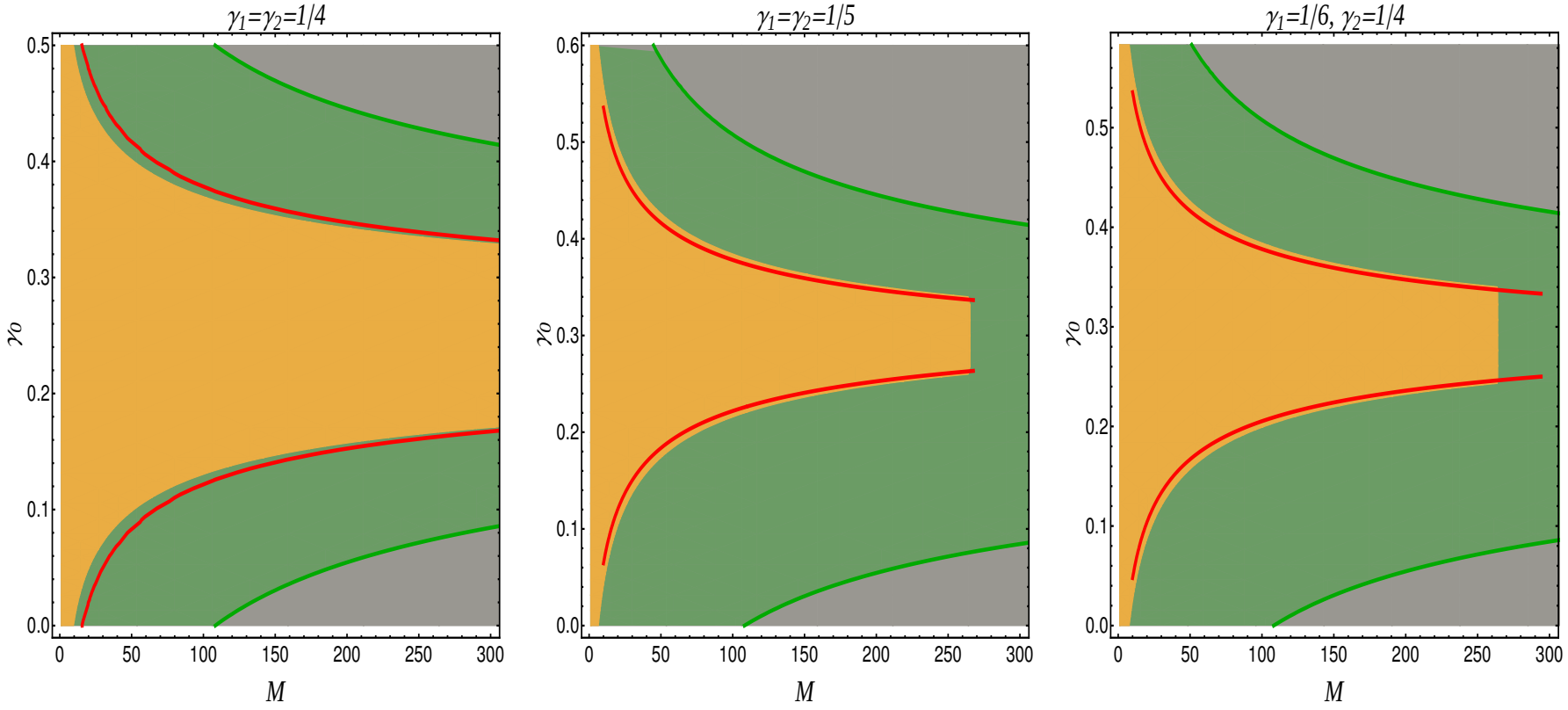


(b) $\gamma_1 = 1/6;\ \gamma_2 = 1/4$.

Comparing with Borel's Normality bounds

A sequence $\hat{s}$ is Borel-Normal if:

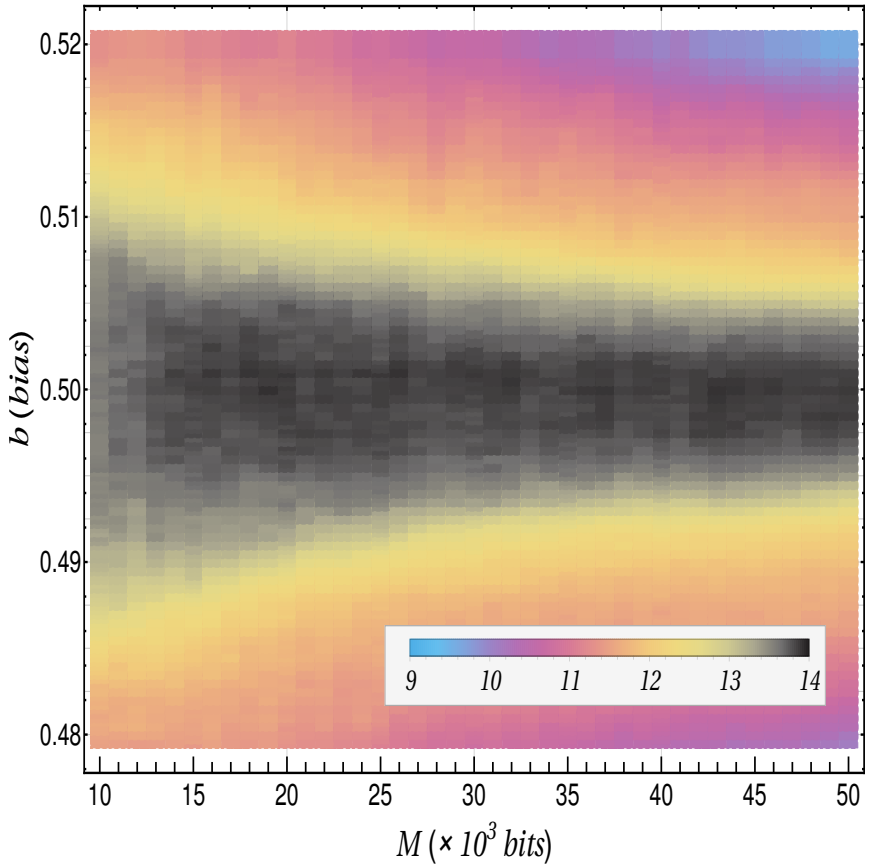
$$\left|\gamma_j^{(\beta)} - \frac{1}{2^\beta}\right| < \sqrt{\frac{\log_2 M}{M}}, \qquad \beta \leq \log_2 \log_2 M$$



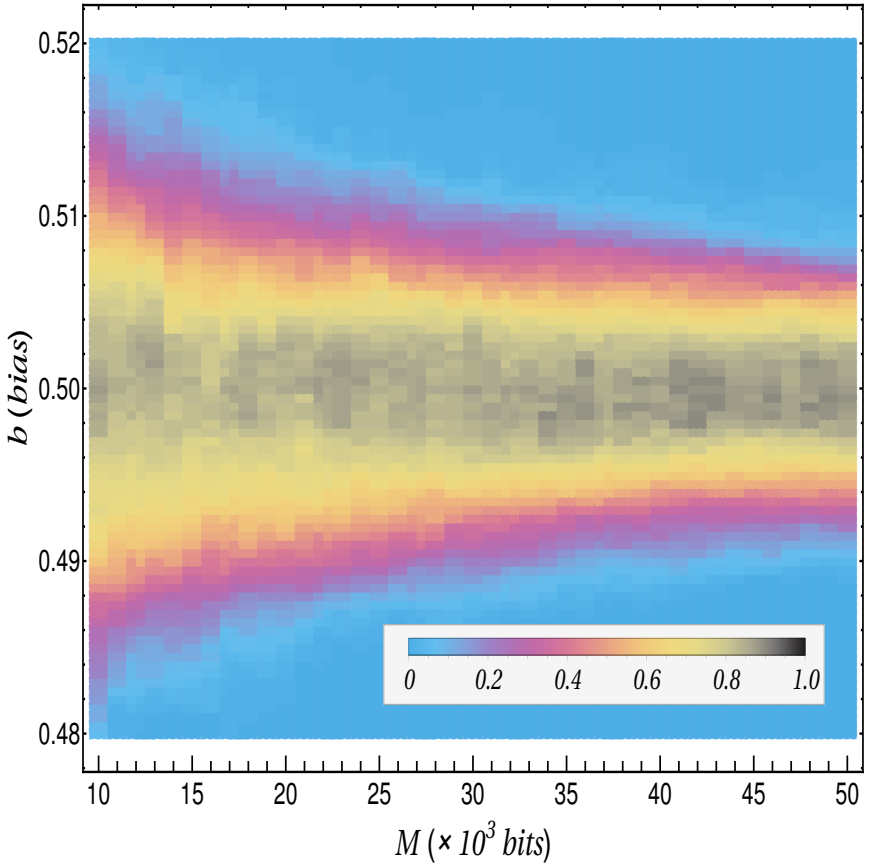
Region where $\mathcal{M}_{\text{sym}}$ is the likeliest , region allowed by Borel Normality test, and Approximated bounds obtained by an expansion of $\log \mathcal{M}_{\text{sym}}/\mathcal{M}_{\alpha_2}$.

Comparing with NIST tests

Used a RNG (Mathematica) to generate 100 bit sequences of different length, with $p_0 = b \in (0.48, 0.52)$.

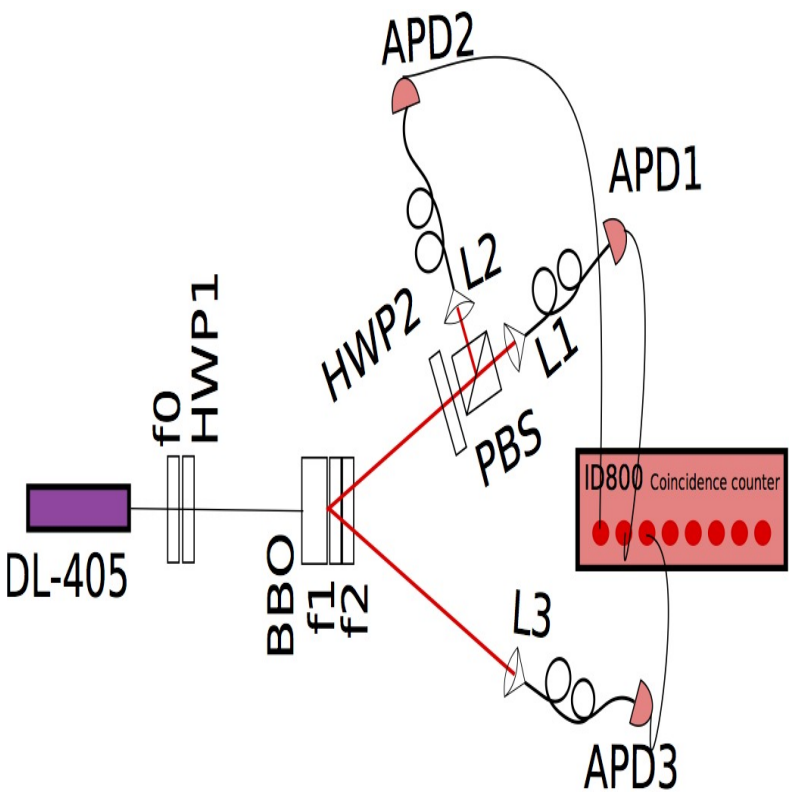


(a) Average of tests passed



(b) Fraction of times $\mathcal{M}_{\text{sym}}$ was selected using $\beta = \{1, 2, 3\}$.

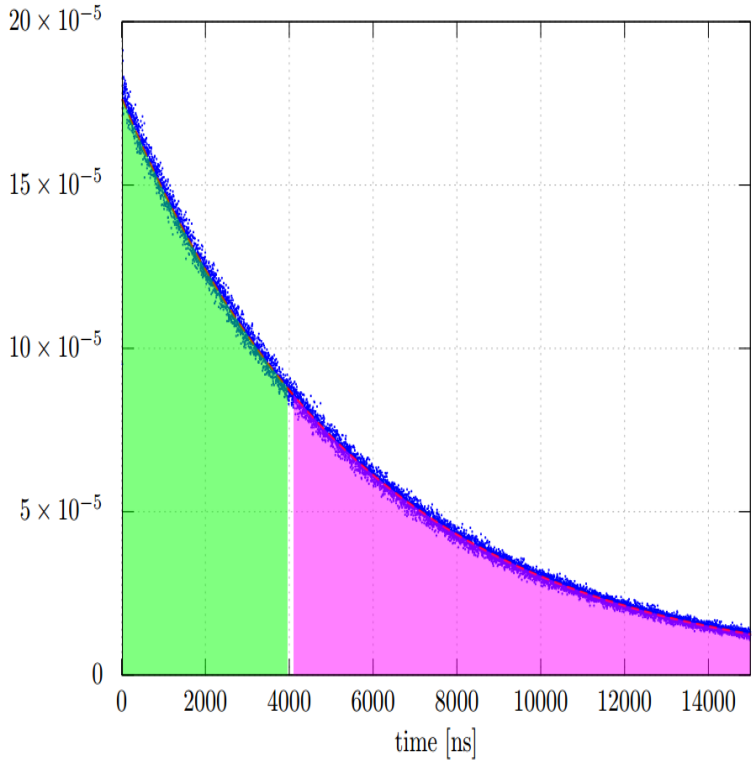
Analysing an experimentally generated sequence



Experimental setup to produce correlated photons.

A. Solis et al., "How random are random numbers generated using photons?", Physica Scripta **90**, 074034 (2015)

Analysing an experimentally generated sequence



Distribution of Δt between the simultaneous detections. It is used to generate strings of 0 and 1.

$$M = 4 \times 10^9 \text{ bits!}$$

$$\beta_{max} \lesssim 5$$

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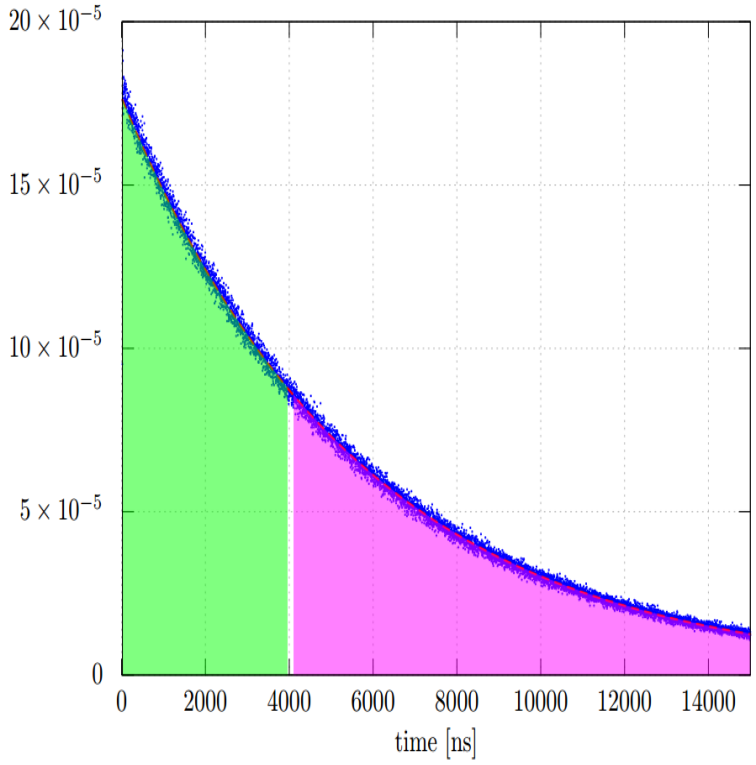

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β	$P(\mathcal{M}_{\text{sym}}\|\hat{s})$	$\log_{10} \text{BF}_{\text{sym},\alpha}$
1	0.999965	4.45
2	0.999562	≥ 3.72
3	0.968353	≥ 2.01
4	0.46718	≥ 3.46

For $\beta = 4$, only models associated to partitions into $K = 2$ subsets were considered.

$$\left\{ \begin{matrix} 2^4 \\ 2 \end{matrix} \right\} = 32,767 \quad \Longrightarrow \quad P(\mathcal{M}) = 3 \times 10^{-5} \, .$$

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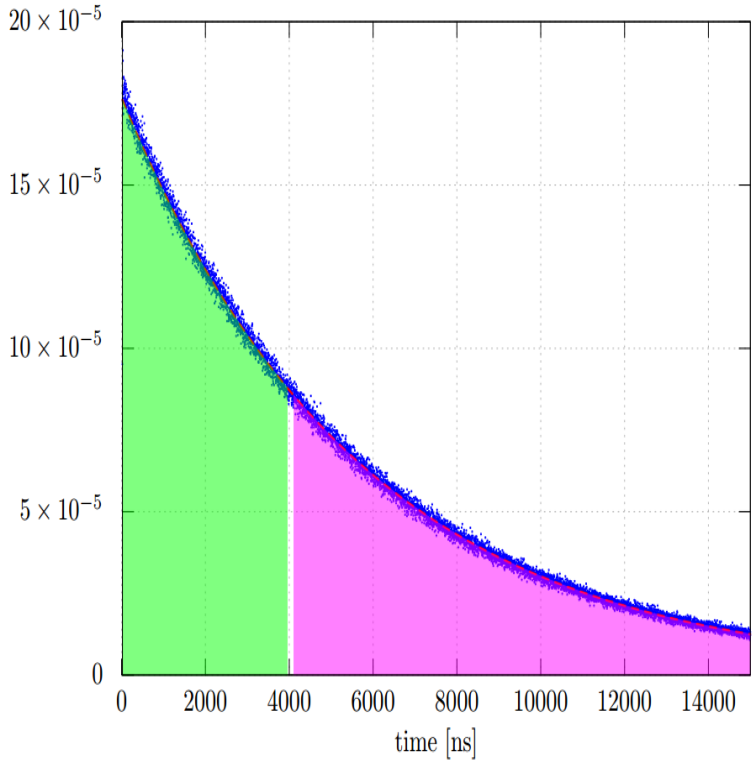

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The device functions as a random *source*.

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Improving randomness characterization through Bayesian model selection

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Thank you!