Chern-Simons and Heterotic Superpotentials

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Based on work with Lara Anderson, Andre Lukas and Juntao Wang

Chern-Simons terms in Heterotic Superpotentials.

- Heterotic compactifications have a Bianchi Identity:

$$dH = \alpha' \left(\mathrm{tr} R \wedge R - \mathrm{tr} F \wedge F \right)$$

- And a Gukov-Vafa-Witten superpotential:

$$W \supset \int_X H \wedge \Omega \supset \alpha' \int_X \left(\omega_3^{\mathrm{L}} - \omega_3^{\mathrm{YM}} \right) \wedge \Omega$$

- Here we have:

$$\omega_3^{\mathrm{YM}} = \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

and similarly for ω_3^{L} .

Apruzzi, Gautason, Parameswaran and Zagerman arXiv:1410.2603
Gukov, Kachru, Liu, McAllister hep-th/0310159

Definition from the mathematics literature:

Thomas, J. Diff. Geom., 53 (1999) 367-438

- Take two connections on the same bundle, A and A_0 and define $A - A_0 = a$.
- Then the Chern-Simons invariant is given as

$$CS(A)_{A_0} = \frac{1}{4\pi^2} \int_X \mathrm{tr} \left(\frac{1}{2} \overline{D}_{A_0} a \wedge a + \frac{1}{3} a \wedge a \wedge a \right) \wedge \Omega$$

- A useful property of this object is that it vanishes if A and A_0 are related by a holomorphic deformation.

But how is this the same as the objects we see in heterotic string theory? In particular, this is about a difference of two connections on the same bundle…

- Two E8 bundles are real isomorphic iff their second Chern characters match.

Witten Int. Jour. Mod. Phys. A Vol. 1 No. 1 (1986) 39-64

- By the Bianchi Identity, then, the gauge bundle and the tangent bundle of heterotic compactifications are isomorphic as real objects.
- The Chern-Simons terms in the heterotic superpotential then matches the mathematical definition of the invariant, if the Bianchi Identity is satisfied.

For which known heterotic standard models does this superpotential term obviously vanish?

Only the one which is a holomorphic deformation of the standard embedding:

Braun, Candelas, Davies and Donagi arXiv:1112.1097

An example of a real bundle isomorphism.

- Consider the following two bundles on $\mathbb{P}^1$

  $V = \mathcal{O}(-1) \oplus \mathcal{O}(1)$ and $\tilde{V} = \mathcal{O} \oplus \mathcal{O}$

  (using the standard notation where $c_1(\mathcal{O}(p)) = pJ_{\mathbb{P}^1}$)

- These are clearly different as holomorphic objects.
  - For example $\tilde{V}$ has nowhere vanishing holomorphic sections, whereas V does not.

- But in fact at the level of real structure, the bundles are the same.

How to see this explicitly?

- $\mathbb{P}^1$ has two patches in the canonical open cover. We will call the affine coordinates on these patches z and w.
- The transition functions for our two bundles are as follows.

$$T_{1,0} = \mathrm{diag}(z, 1/z) \qquad \tilde{T}_{1,0} = \mathrm{diag}(1,1)$$

- We can now define a bundle morphism

$$f : V \to \tilde{V}$$

by specifying its action on each patch of the base.

$$P_0 = \begin{pmatrix} 1 & \frac{\bar{z}}{1+|z|^2} \\ -z & \frac{1}{1+|z|^2} \end{pmatrix} \qquad P_1 = \begin{pmatrix} w & \frac{1}{1+|w|^2} \\ -1 & \frac{\bar{w}}{1+|w|^2} \end{pmatrix}$$

Lets check this works and look at its properties…

- First note that

$$P_1 \cdot T_{1,0} \cdot P_0^{-1} = \tilde{T}_{1,0}$$

so that this morphism does indeed map one bundle into the other.

- Properties of this morphism:
  - It is well defined and invertible on both patches.
  - It is not holomorphic in structure (it depends on z and $\overline{z}$ for example).
  - You can think of this as just a non-holomorphic change of fiber basis.

- Thus this is an explicit real bundle isomorphism between V and $\tilde{V}$.

Example for use in Heterotic:

- Work on the tetra-quadric Calabi-Yau threefold:

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \\ \mathbb{P}^1 & 2 \end{array} \right]$$

- With the bundle:

$$0 \to V \to \mathcal{O}(1,0,0,0)^{\oplus 2} \oplus \mathcal{O}(0,1,0,0)^{\oplus 2} \ldots \to \mathcal{O}(2,2,2,2) \to 0$$

(actually a holomorphic deformation of $TX \oplus \mathcal{O}^{\oplus 4}$)

- This bundle is real equivalent to:

$$0 \to \tilde{V} \to \mathcal{O}(0,0,1,-1) \oplus \mathcal{O}(2,0,0,0) \oplus \mathcal{O}(0,0,-1,1) \oplus \mathcal{O}(0,2,0,0) \ldots$$

$$\to \mathcal{O}(2,2,2,2) \to 0$$

- How to explicitly see this `equivalence':

$$0 \to V \to \mathcal{O}(1,0,0,0)^{\oplus 2} \oplus \mathcal{O}(0,1,0,0)^{\oplus 2} \ldots \to \mathcal{O}(2,2,2,2) \to 0$$

$$\Big\downarrow \text{ Holomorphic deformation}$$

$$\mathcal{O}(1,0,0,0)^{\oplus 2} \oplus \mathcal{O}(0,1,0,0)^{\oplus 2} \oplus U$$

$$\Big\downarrow \text{ Real bundle isomorphism}$$

$$\mathcal{O}(0,0,1,-1) \oplus \mathcal{O}(2,0,0,0) \oplus \mathcal{O}(0,0,-1,1) \oplus \mathcal{O}(0,2,0,0) \oplus U$$

$$\Big\downarrow \text{ Holomorphic deformation}$$

$$0 \to \tilde{V} \to \mathcal{O}(0,0,1,-1) \oplus \mathcal{O}(2,0,0,0) \oplus \mathcal{O}(0,0,-1,1) \oplus \mathcal{O}(0,2,0,0) \ldots \to \mathcal{O}(2,2,2,2) \to 0$$

In the above we have defined the following for convenience:

$$0 \to U \to \mathcal{O}(0,0,1,0)^{\oplus 2} \oplus \mathcal{O}(0,0,0,1)^{\oplus 2} \to \mathcal{O}(2,2,2,2) \to 0$$

- So this is a case where the bundles are suitable for use in a heterotic compactification, and where we explicitly know some relevant real bundle isomorphisms.

- Lets now compute the Chern-Simons invariants:
  - The bundle V is a holomorphic deformation of TX so its Chern-Simons invariant is zero.
  - What about the second, real equivalent bundle $\tilde{V}$? Well we know how it is related to V and Chern-Simons invariants have the property:

$$CS(\tilde{A})_{A_0} - CS(A)_{A_0} = CS(\tilde{A})_A$$

    so we can simply compare the connections on these two bundles and obtain the physical answer we want.

- Note that the holomorphic deformations would not affect Chern-Simons invariants – so we can just work with the simpler split objects.

Computing the Chern-Simons invariant of $\tilde{V}$

- Explicitly write down the real isomorphism.
- Write down an appropriate Chern connection on the sum of line bundles in the deformation of V.
- Map this connection across to $\tilde{V}$ using our explicit real bundle isomorphism.
- Write down an appropriate Chern connection on the sum of line bundles in the deformation of $\tilde{V}$.
- Plug these two into the formula for the Chern-Simons invariant and perform the integral (similar to period integrals) and…..

Get zero.

At this stage you ask why...

- Theorem (Thomas, 1999)

  Suppose that the Calabi-Yau 3-fold X is a smooth effective anticanonical divisor in a 4-fold Y defined by $s \in H^0(K_Y^{-1})$. *If* $E \to X$ *is a bundle that extends to a bundle* $\mathbb{E} \to Y$*, then for a* $\bar{\partial}$*-operator* A *on* E*, let* $\mathbb{A}$ *be any* $\bar{\partial}$*-operator on* $\mathbb{E}$ *extending* A*. We have, modulo periods,*

$$CS(A) = \frac{1}{4\pi^2} \int_Y \mathrm{tr} F_{\mathbb{A}}^{0,2} \wedge F_{\mathbb{A}}^{0,2} \wedge s^{-1} \ .$$

- For the case we just looked at, because of the favorable nature of the manifold, the relevant bundles extend holomorphically to the ambient space, as do the Chern connections we have used, and so we get zero.

This result leads to zeros more generally than you might think…

- Thomas’ theorem can be used in the case of complete intersections (for example by applying all but one defining relation and calling the result Y).

  Before quotienting and adding Wilson lines we believe that all know Line Bundle Standard Models will have zero Chern-Simons term.

- Thomas’ theorem can often be applied to quotients despite the fact that in such cases the ambient space is typically singular. For example this can be the case if the ambient space can be resolved without affecting the Calabi-Yau manifold.

- Note the power of this theorem: If the Calabi-Yau manifold admits any description where the conditions of the theorem hold and the connection can be extended holomorphically then the Chern-Simons term is zero.

  We still expect there to be cases where the Chern-Simons term does not vanish, which we should be able to compute with this formalism. Building examples of these is what we are working on now.

Conclusions

- We have looked at Chern-Simons terms in heterotic superpotentials.
- The physics and mathematics definitions agree, at least in simple cases.
- We have a method for computing the Chern-Simons terms in some interesting cases based upon real bundle isomorphisms.
- A general theorem of Thomas could prove to be quite powerful in the context of constructions of Calabi-Yau manifolds and bundles often used in physics.