Anomalies and dimensional reduction

Iñaki García Etxebarria

Durham University

Department of Mathematical Sciences

Based on work in progress with
P. Corvilain, M. Montero, D. Regalado.

Anomalies

The textbook view on anomalies is that anomalies arise whenever we have a symmetry of the classical Lagrangian that is not a symmetry of the full quantum theory.

The problem is particularly serious whenever we are talking about gauge transformations: if a gauge transformation is anomalous then the theory is inconsistent.

The canonical example is the theory of a Weyl fermion in four dimensions charged under a $U(1)$ gauge symmetry

$$\mathcal{L} = \frac{1}{2g} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \psi^\dagger (i\partial_\mu - A_\mu) \sigma^\mu \psi$$

which looks fine classically, but is inconsistent quantum-mechanically.

The uses of anomalies

Anomalies give **constraints on the spectrum**. For instance, if we were to find a "fourth-generation right-handed electron" ζ_R (generalising the e_R, μ_R and τ_R) we would immediately know that some other particles need to exist, since this particle is, by itself, anomalous.

The uses of anomalies

Anomalies give **constraints on the spectrum**. For instance, if we were to find a "fourth-generation right-handed electron" ζ_R (generalising the e_R, μ_R and τ_R) we would immediately know that some other particles need to exist, since this particle is, by itself, anomalous.

Anomalies also give **constraints on the dynamics**, via 't Hooft anomaly matching: if we believe that we have two dual descriptions of the same quantum theory, the symmetries, and their anomalies, should match.

The Dai-Freed approach to anomalies

Recent developments [Dai, Freed '94], [Witten '15] have shed new light on this old topic.

These recent developments are geared towards condensed matter, but we have recently argued in [I.G.-E., Montero '18] (see also [Hsieh '18], [Wang, Wen '18], [Wang, Wen, Witten '18]) that there are also interesting consequences for particle physics:

- In the (vanilla + ν_R) Standard Model there is a $\mathbb{Z}_4$ symmetry at high energies, acting as

$$Q(A_\mu) = 0 \quad ; \quad Q(\psi_R) = +1 \quad ; \quad Q(H) = +2 \,.$$

  For this symmetry to be gaugeable — or more precisely, for being able to twist the spacetime spin bundle by this $\mathbb{Z}_4$ — we need 16 fermions per generation. General lore is that it *should* be gaugeable.

- For baryon triality (a $\mathbb{Z}_3$ symmetry) to be gaugeable we need the number of generations to be a multiple of 3.

Anomalies and string universality (I)

Another application of our modern understanding of anomalies is to the question of *string universality* (See Wati's talk):

String Universality (Question)

Given a *supersymmetric* gravitational theory in d-dimensions, which is free of anomalies, is there always a string compactification that reproduces the massless spectrum of the theory at low energies?

Anomalies and string universality (I)

Another application of our modern understanding of anomalies is to the question of *string universality* **(See Wati's talk)**:

String Universality (Question)

Given a *supersymmetric* gravitational theory in d-dimensions, which is free of anomalies, is there always a string compactification that reproduces the massless spectrum of the theory at low energies?

Note that the question is slightly different to the usual swampland questions we have heard much about, but there might be a connection between both questions once we start thinking about anomalies in quantum gravitational theories.

Anomalies and string universality (II)

- In more than 11 dimensions every susy Lagrangian theory includes excitations of helicity larger than two.

Anomalies and string universality (II)

- In more than 11 dimensions every susy Lagrangian theory includes excitations of helicity larger than two.
- In 11d there is a single choice, 11d supergravity, which arises as a limit of string theory (strongly coupled IIA). So string theory is **universal** in 11d.

Anomalies and string universality (II)

- In more than 11 dimensions every susy Lagrangian theory includes excitations of helicity larger than two.
- In 11d there is a single choice, 11d supergravity, which arises as a limit of string theory (strongly coupled IIA). So string theory is **universal** in 11d.
- In 10d we have IIA and IIB sugra in the $\mathcal{N}=2$ sector. Pure $\mathcal{N}=1$ gauge theory is anomalous. If we couple it to supergravity we can cancel the anomalies using the Green-Schwarz mechanism [Green, Schwarz '84]. This is possible for the gauge algebras

$$\mathfrak{g} \in \{\mathfrak{e}_8 \oplus \mathfrak{e}_8, \mathfrak{so}(32), \mathfrak{e}_8 \oplus \mathfrak{u}(1)^{248}, \mathfrak{u}(1)^{496}\}\,.$$

  The first two possibilities are realized by string theory, while the second two cannot be made consistent supersymmetrically [Adams, DeWolfe, Taylor '10], [Kim, Shiu, Vafa '19]. So string theory is also universal in 10d.

Anomalies and string universality (III)

The situation in lower dimensions is murkier. For $d \in \{9, 8, 7\}$ all fermions are in the adjoint, so there are no local gauge anomalies. Any gauge group seems to be allowed!

Anomalies and string universality (III)

The situation in lower dimensions is murkier. For $d \in \{9, 8, 7\}$ all fermions are in the adjoint, so there are no local gauge anomalies. Any gauge group seems to be allowed!

Nevertheless the list of theories that we know how to construct is very limited in 9d and 8d!

For 9d [Aharony, Komargodski, Patir '07]:

- **Rank 2 (a):** M-theory on the Klein bottle.
- **Rank 2 (b):** IIA with $O8^+$ and $O8^-$.
- **Rank 10:**
  - M-theory on Möbius band.
  - CHL string. [Chaudhury, Hockney, Lykken '95]
- **Rank 18:**
  - M-theory on the cylinder.
  - Heterotic on S^1.
  - IIA with two $O8^-$ planes and 16 D8s.

Anomalies and string universality (III)

The situation in lower dimensions is murkier. For $d \in \{9, 8, 7\}$ all fermions are in the adjoint, so there are no local gauge anomalies. Any gauge group seems to be allowed!

Nevertheless the list of theories that we know how to construct is very limited in 9d and 8d!

For 8d we have

- **Rank 4:** IIB with two $\mathrm{O7}^-$ and two $\mathrm{O7}^+$.
- **Rank 12:** IIB with three $\mathrm{O7}^-$, one $\mathrm{O7}^+$ and 8 D7s.
- **Rank 20:** IIB with four $\mathrm{O7}^-$ and 16 D7s.

Various patterns emerge here (similar patterns in 9d):

- The ranks jump in units of 8.
- There is a maximal rank of 20. (Relation to [Kim, Shiu, Vafa '19], [Lee, Weigand '19]?)
- Additionally, no known compactifications have algebra $\mathfrak{f}_4$, $\mathfrak{g}_2$ or $\mathfrak{so}(2N+1)$.

Anomalies and string universality (IV)

We partially solved the last problem in [I.G.-E., Hayashi, Ohmori, Tachikawa, Yonekura '17]. We found that $\mathfrak{f}_4$ and $\mathfrak{so}(2N+1)$ gauge groups are **anomalous** in 8d $\mathcal{N}=1$ theories $\implies$ some of the missing gauge groups are simply inconsistent. (The algebra $\mathfrak{g}_2$ survived the analysis; the status of this case is still unknown.)

Somewhat surprisingly, we found a subtlety in $USp(2N)$ theories: if we compactify this theory on S^4, with the minimal instanton number, the resulting four dimensional theory is inconsistent!

Anomalies and string universality (IV)

We partially solved the last problem in [I.G.-E., Hayashi, Ohmori, Tachikawa, Yonekura '17]. We found that $\mathfrak{f}_4$ and $\mathfrak{so}(2N+1)$ gauge groups are **anomalous** in 8d $\mathcal{N}=1$ theories $\implies$ some of the missing gauge groups are simply inconsistent. (The algebra $\mathfrak{g}_2$ survived the analysis; the status of this case is still unknown.)

Somewhat surprisingly, we found a subtlety in $USp(2N)$ theories: if we compactify this theory on S^4, with the minimal instanton number, the resulting four dimensional theory is inconsistent!

What this talk is about

- When does the inconsistency of the compactified theory indicate an inconsistency of the parent theory?
- More generally, how do we relate anomalies in different dimensions?
- I will follow the modern approach to anomalies, where things are clearer. (Dimensionally reduce a TQFT in one dimension higher!)

Review of anomalies (I)

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A for G, and compute the path integral

$$Z(A) = \int [D\psi] e^{-S(A,\psi)}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

Review of anomalies (I)

Consider a (Lagrangian) theory $\mathcal{T}$ with some global symmetry G. We can introduce a background connection A for G, and compute the path integral

$$Z(A) = \int [D\psi] e^{-S(A,\psi)}$$

where ψ are some fundamental fields. (Only the fermionic fields, and the connection they couple to, matter for my discussion.)

A definion of anomalies

Denote by $\mathcal{M}$ the space of all A. We say a theory is non-anomalous iff $Z(A)$ (in particular its phase) is well defined as a function on $\mathcal{M}/G$.

Review of anomalies (II)

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

Review of anomalies (II)

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

Local anomalies are easy to describe: the object that encodes the curvature of $Z(A)$ on $\mathcal{M}/G$ is the "anomaly polynomial"

$$\mathcal{I}_{d+2} = \mathrm{ch}(F)\hat{A}(R)|_{d+2}$$

Review of anomalies (II)

- Non-invariance under small loops (curvature) in $\mathcal{M}/G$: *local anomaly*.
- Non-invariance under parallel transport for non-trivial loops in $\mathcal{M}/G$: *global anomalies*.

Local anomalies are easy to describe: the object that encodes the curvature of $Z(A)$ on $\mathcal{M}/G$ is the "anomaly polynomial"

$$\mathcal{I}_{d+2} = \mathrm{ch}(F)\hat{A}(R)|_{d+2}$$

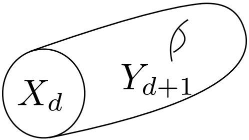
When there are no local anomalies

$$\mathcal{I}_{d+2} = 0$$

and the connection on $\mathcal{M}/G$ is flat. We can still have global anomalies, if any monodromies of the flat bundle are non-trivial.

The Dai-Freed viewpoint on anomalies

Consider the case that your space-time X_d is the boundary of some manifold Y_{d+1}, over which all the relevant structures on X_d extend.



We define the path integral of a fermion ψ on X_d as [Dai, Freed '04]

$$Z_\psi = |Z_\psi| e^{-2\pi i\, \eta(\mathcal{D}_{Y_{d+1}})}$$

with

$$\eta(\mathcal{D}_{Y_{d+1}}) = \frac{\dim\ker \mathcal{D}_{Y_{d+1}} + \sum_{\lambda\neq 0} \operatorname{sign}(\lambda)}{2}\,.$$

[*] For the experts, this is the same η that appears in the APS index theorem. More on this soon.

Why is this prescription useful

The η invariant is, in general, very difficult to compute. We only know expressions for it in a handful of examples.

Why is this prescription useful

The η invariant is, in general, very difficult to compute. We only know expressions for it in a handful of examples.

Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

$$e^{2\pi i\,\eta(\mathcal{D}_A)} = e^{-2\pi i\,\eta(\mathcal{D}_{\overline{A}})}$$

Why is this prescription useful

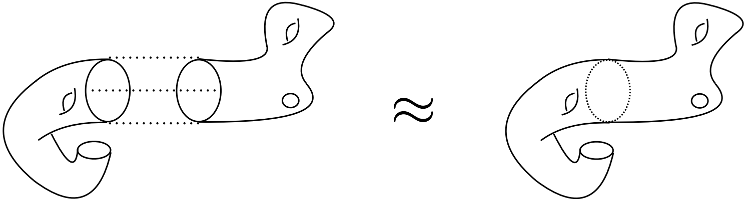
The η invariant is, in general, very difficult to compute. We only know expressions for it in a handful of examples.

Nevertheless, it has very nice properties: if we change the orientation of the manifold the phase of the partition function changes sign:

$$e^{2\pi i\,\eta(\mathcal{D}_A)} = e^{-2\pi i\,\eta(\mathcal{D}_{\overline{A}})}$$

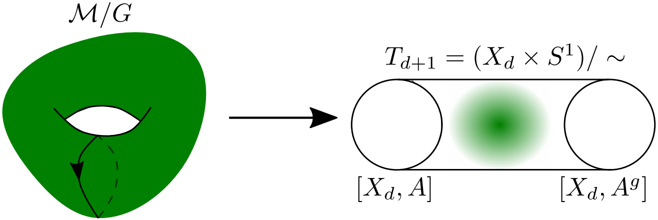
and it is "local", in the sense that η behaves nicely under gluing:

$$e^{2\pi i\eta(\mathcal{D}_A)}e^{2\pi i\,\eta(\mathcal{D}_B)} = e^{2\pi i\,\eta(\mathcal{D}_{A+B})}$$

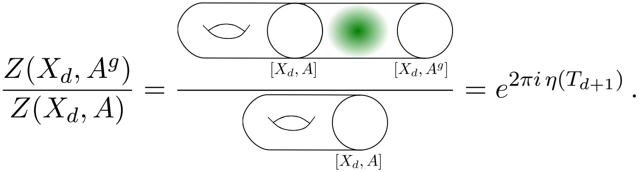


Computing anomalies

In order to compute the anomaly in this framework associated to a given path in $\mathcal{M}/G$, we construct a "mapping torus" that implements the transformation:



The partition function transforms as



The global anomaly for real fermions

Consider for example the case in which the fermions are *real*. This means that the mass coupling

$$m\psi\psi = 0$$

does not break G, but it identically vanishes.
We can add an extra copy of the fermions, and introduce a mass coupling

$$m\psi_1\psi_2 \neq 0$$

This implies that $Z(A)^2$ is well defined, so the anomaly is $\mathbb{Z}_2$-valued (i.e. $e^{i\mathcal{A}} = \pm 1$ at most). It is, in particular, invariant under smooth deformations of the configuration on W_{d+1}.

A 4d example: Witten's $SU(2)$ anomaly

An example of real fermions are 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes), since the fundamental of $SU(2)$ is pseudoreal, and the Weyl spinor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ is pseudoreal.

A 4d example: Witten's $SU(2)$ anomaly

An example of real fermions are 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes), since the fundamental of $SU(2)$ is pseudoreal, and the Weyl spinor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ is pseudoreal.

Famously [Witten '82], this system has a global anomaly:

$$Z(A) = -Z(A^g)$$

for $[g]$ the non-trivial generator of $\pi_4(SU(2)) = \mathbb{Z}_2$.

A 4d example: Witten's $SU(2)$ anomaly

An example of real fermions are 4d Weyl fermion ψ_1 in the fundamental of $SU(2)$. This is a real fermion (the mass term is allowed, but it identically vanishes), since the fundamental of $SU(2)$ is pseudoreal, and the Weyl spinor of $\mathrm{Spin}(4) = SU(2) \times SU(2)$ is pseudoreal.

Famously [Witten '82], this system has a global anomaly:

$$Z(A) = -Z(A^g)$$

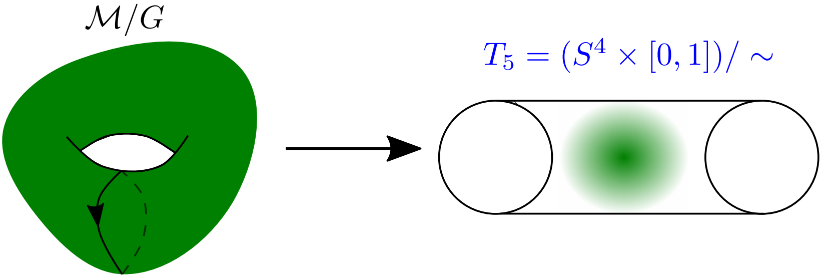
for $[g]$ the non-trivial generator of $\pi_4(SU(2)) = \mathbb{Z}_2$. This implies that the theory becomes ill-defined when we try to gauge the $SU(2)$ group:

$$Z = \int_{\mathcal{M}} [DA] Z(A) e^{-\mathrm{Tr}(F^2)} = \int_{\mathcal{M}/G} [DA] K(1 + (-1)) Z(A) e^{-\mathrm{Tr}(F^2)} = 0\,.$$

and similarly with insertions of gauge invariant operators, so

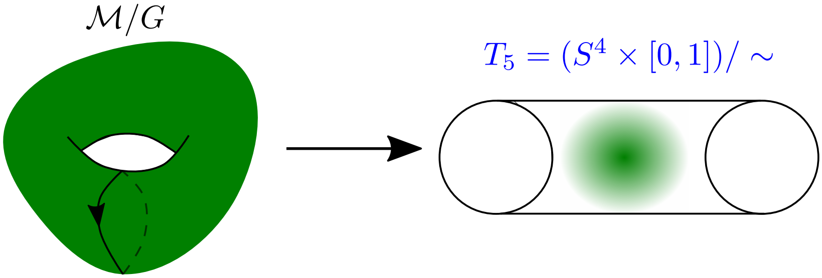
$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \frac{0}{0}\,.$$

Witten's $SU(2)$ anomaly (II)



$$\frac{Z(A^g)}{Z(A)} = e^{2\pi i\,\eta(T_5)} = -1$$

Witten's $SU(2)$ anomaly (II)



$$\frac{Z(A^g)}{Z(A)} = e^{2\pi i\,\eta(T_5)} = -1$$

Note

This anomaly is global on $\mathcal{M}/G$, but local in the target spacetime: we are working on a neighbourhood of a point, and we do not care about the topology of the ambient d-dimensional manifold.

K-theory tadpole cancellation

(See Blumenhagen's and Brinkmann's talks.)

I would now like to argue that if K-theory tadpoles do not cancel in type I then there is a global anomaly: the partition function, and any gauge-invariant correlators, vanish.

We are in $d = 10$, and our fermions are *real* (in the adjoint of $SO(32)$, and Majorana). So there is a possibility of a global $\mathbb{Z}_2$ anomaly, like in 4d.

In fact, both things can be related [Uranga '00]. I will try to refine this argument in a way that clears up some of the subtleties.

K-theory tadpole cancellation (II)

The probe argument

Assume that we have type I string theory on a T^2, and we introduce an $SO(32)$ bundle on the T^2 with non-trivial $\mathbb{Z}_2$ K-theory charge (a dissolved $\widehat{\mathrm{D}7}$).

K-theory tadpole cancellation (II)

The probe argument

Assume that we have type I string theory on a T^2, and we introduce an $SO(32)$ bundle on the T^2 with non-trivial $\mathbb{Z}_2$ K-theory charge (a dissolved $\widehat{\text{D7}}$).

[Uranga '00] showed that an inconsistency can be detected on a probe D5 wrapping the T^2. The worldvolume theory on the D5, in the limit of small T^2, is a $d = 4$ $\mathcal{N} = 2$ theory with gauge group $SU(2)$, and a fundamental Weyl fermion. (Coming from $\widehat{\text{D7}}$-D5 strings, had we not dissolved the $\widehat{\text{D7}}$.)

We just saw that this theory has a global anomaly!

K-theory tadpole cancellation (II)

The probe argument

Assume that we have type I string theory on a T^2, and we introduce an $SO(32)$ bundle on the T^2 with non-trivial $\mathbb{Z}_2$ K-theory charge (a dissolved $\widehat{\text{D}7}$).

[Uranga '00] showed that an inconsistency can be detected on a probe D5 wrapping the T^2. The worldvolume theory on the D5, in the limit of small T^2, is a $d = 4$ $\mathcal{N} = 2$ theory with gauge group $SU(2)$, and a fundamental Weyl fermion. (Coming from $\widehat{\text{D}7}$-D5 strings, had we not dissolved the $\widehat{\text{D}7}$.)

We just saw that this theory has a global anomaly!

One common objection to this argument: is it the theory or the probe that is sick? I will now argue that it is the theory that is sick, by constructing a explicit 11 dimensional manifold, connecting a configuration **without** probes to a gauge transform of itself, that flips the sign of the partition function.

K-theory tadpole cancellation (III)

Note

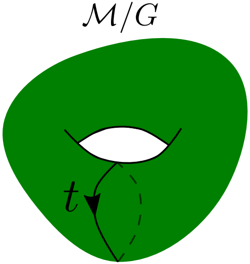
Before we start, recall that the final result is necessarily ± 1, so it is invariant under smooth deformations, and we only need to care about statements in topology.

K-theory tadpole cancellation (III)

Note

Before we start, recall that the final result is necessarily ± 1, so it is invariant under smooth deformations, and we only need to care about statements in topology.

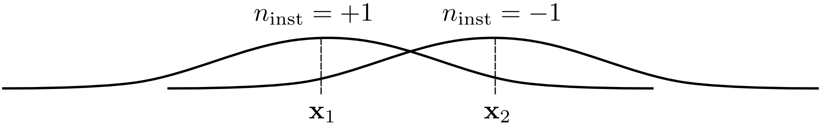
I will describe the 11d configuration as a time-dependent process, where time is the 11th dimension. We start with the configuration with the dissolved $\widehat{\mathrm{D7}}$ on $\mathbb{R}^4 \times S^4 \times T^2$.



K-theory tadpole cancellation (III)

Step 1

Start creating a widely separated instanton-anti-instanton pair on $\mathbb{R}^4$. No total charge, so we can do this smoothly.

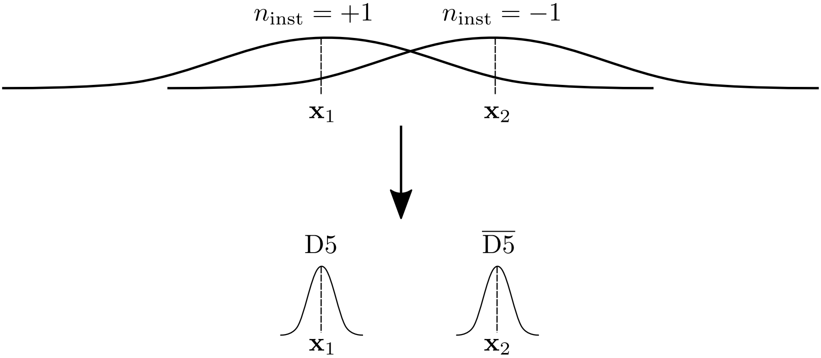


For example, start with very separated instanton-anti-instanton pair $|\mathbf{x}_1 - \mathbf{x}_2| = \ell \gg \Lambda^{-1}$, with Λ the cutoff for the 10d field theory description. Additionally, we take $\rho_1 \approx \rho_2 \gg \ell$ for the size of the instantons.

K-theory tadpole cancellation (III)

Step 2

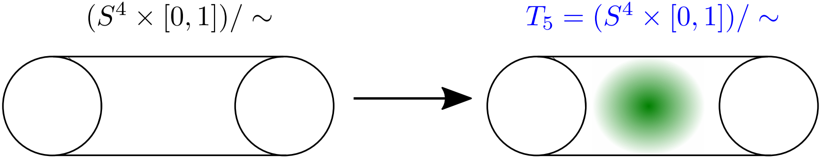
We shrink the instantons, so we have a D5 wrapping $\{\mathbf{x}_1\} \times S^4 \times T^2$ and an $\overline{\text{D5}}$ wrapping $\{\mathbf{x}_1\} \times S^4 \times T^2$. On each D5 we have $SU(2)$, with a fundamental.



K-theory tadpole cancellation (III)

Step 3

Along an interval $\mathcal{I}$ in the 11th dimension where nothing happens, the D5 wraps $\mathcal{I} \times T^2 \times S^4 \times \{\mathbf{x}_1\}$. If we cut a small ball around $\mathbf{x}_1$ in $\mathbb{R}_4$ and a point in S^4, the boundary is $\mathcal{I} \times T^2 \times S^3 \times S^4$. We glue in a manifold with $\mathcal{I} \times S^4$ replaced with T_5 (Witten's mapping torus for the $SU(2)$ anomaly).



Crucially, we can do this around $\mathbf{x}_1$ without affecting the $\overline{\text{D5}}$ at $\mathbf{x}_2$. The η invariant behaves nicely under gluing, to this cutting-and-gluing leads to $\eta \to \eta + \frac{1}{2}$ (so an extra sign in Z).

K-theory tapdole cancellation (III)

Step 4

Finally, we recombine the instanton-anti-instanton pairs. We are at a gauge transform of the original configuration, but we picked up a minus sign along the way, when we glued in T_5.

K-theory tapdole cancellation (III)

Step 4

Finally, we recombine the instanton-anti-instanton pairs. We are at a gauge transform of the original configuration, but we picked up a minus sign along the way, when we glued in T_5.

All of this together implies that the IIB partition function vanishes if we do not cancel K-theory charges, at least in this example. More generally, the same result follows if

- The charge can be detected by a probe.
- One can create probe-antiprobe pairs from the vacuum smoothly.
- The probes can be dissolved into the background.

The anomaly for the dissolved D5

Note that, except in step 2 (the instanton $\to$ D5 contraction) we never left the regime of effective field theory. It would be more satisfactory if we could do everything for smooth bundles, so every intermediate configuration is manifestly smooth.

The anomaly for the dissolved D5

Note that, except in step 2 (the instanton $\to$ D5 contraction) we never left the regime of effective field theory. It would be more satisfactory if we could do everything for smooth bundles, so every intermediate configuration is manifestly smooth.

We can do this using a result in [Atiyah, Patodi, Singer '76]. Given an odd dimensional manifold X and an even-dimensional manifold Y then

$$\eta(X \times Y) = \eta(X)\,\mathrm{ind}(Y)\,.$$

In our discussion above $X = T_5$ and $Y = (\mathbb{R}^4)_1 \times T^2$ with a point-like instanton on $(\mathbb{R}^4)_1$ and the non-trivial K-theory bundle on T^2. But clearly the result still holds if we dissolve the instanton on Y, since the index does not change under smooth deformations.

Anomalies and dimensional reduction

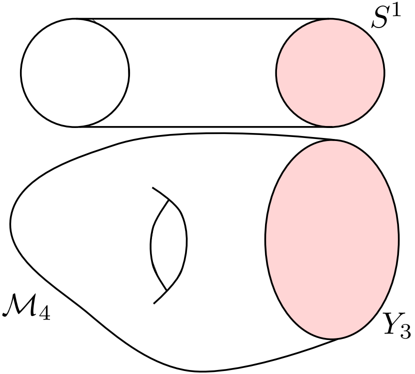
We just saw that the formula

$$\eta(X \times Y) = \eta(X)\,\mathrm{ind}(Y)$$

allowed us to "uplift" four dimensional anomalies to 10 dimensions. I will now discuss some further applications of this formula in different settings.

Anomalies and circle reduction

As studied in [Grimm, Kapfer '15], [Grimm, Kapfer, Klevers '15], [Corvilain, Grimm, Regalado '17] (see also [Jensen, Loganayagam, Yarom '13], [di Pietro, Komargodski '14]), it is quite interesting in the context of F/M-theory to understand what happens as we compactify a 4d F-theory model on a circle (as we can match this to Chern-Simons terms one obtains from M-theory).



Anomalies and circle reduction

For a compactification on S^1, we can choose the 5d anomaly manifold as $\mathcal{M}_4 \times S^1$, with boundary $Y_3 \times S^1$ (we have $Y_3 = \partial\mathcal{M}_4$). From the APS formula one finds

$$\eta(\mathcal{M}_4 \times S^1) = \eta(S^1) \int_{\mathcal{M}_4} \hat{A}(R)\,\mathrm{ch}(F) = \eta(S^1) \int_{Y_3} \Omega\,.$$

with $d\Omega = \hat{A}(R)\,\mathrm{ch}(F)$ and

$$\eta(S^1) = \lim_{s\to 0} \sum_{\lambda} \lambda^{-s}\,\mathrm{sign}(\lambda)$$

which reproduces the result of previous analyses. (A similar application: anomalies of class-$\mathcal{S}$ theories [Alday, Benini, Tachikawa '09], [Bah, Beem, Bobev, Wecht '12], [Bah, Nardoni '18].)

When probes don't kill theories

We now come back to our original example: an N D7 branes on top of an O7$^+$, compactified on S^4, in the presence of the minimal instanton:

$$\eta(S^4 \times X_5) = \eta(X_5)\,\mathrm{ind}(S^4) = \eta(X_5)$$

By choosing $X_5 = T_5$, we can uplift Witten's anomaly to the 8d configuration.

When probes don't kill theories

We now come back to our original example: an N D7 branes on top of an O7$^+$, compactified on S^4, in the presence of the minimal instanton:

$$\eta(S^4 \times X_5) = \eta(X_5)\,\mathrm{ind}(S^4) = \eta(X_5)$$

By choosing $X_5 = T_5$, we can uplift Witten's anomaly to the 8d configuration.

But in this case we *cannot* conclude that the 8d theory is inconsistent! The instanton is a "half-instanton": it is stuck to the 7-brane stack. We cannot nucleate a pair away from the stack and then bring one for probing the anomaly.

When probes don't kill theories

We now come back to our original example: an N D7 branes on top of an O7$^+$, compactified on S^4, in the presence of the minimal instanton:

$$\eta(S^4 \times X_5) = \eta(X_5)\,\mathrm{ind}(S^4) = \eta(X_5)$$

By choosing $X_5 = T_5$, we can uplift Witten's anomaly to the 8d configuration.

But in this case we *cannot* conclude that the 8d theory is inconsistent! The instanton is a "half-instanton": it is stuck to the 7-brane stack. We cannot nucleate a pair away from the stack and then bring one for probing the anomaly.

And a single instanton on the S^4 could be forbidden by coupling to a TQFT with Lagrangian (on the S^4):

$$\lambda w_4(E)\,.$$

Conclusions

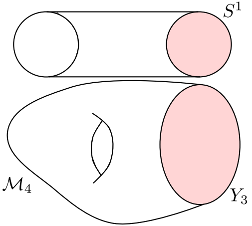
There is a systematic and simple way of dimensionally reducing anomalies, and in fact we can also sometimes "uplift" anomalies from lower dimensions to higher dimensions.

- We can understand why this is possible most naturally in terms of the Dai-Freed viewpoint, with our world a boundary, and a topological theory on the bulk: dimensional "reduction" does not do much in a topological theory.
- Useful for clarifying probe arguments.
- We could efficiently compute the anomaly structure after compactification.

Further directions

Many interesting questions when we take gravity into account:

How much information about anomalies do we lose when compactifying a gravitational theory, where the full dimensional theory is typically accessible somewhere in moduli space?



Relatedly, in string theory many different compactifications are connected by dualities and geometric transitions. What do we learn if we impose that the partition function is well-defined as we take monodromies in this more general space?