# M5 branes and Theta Functions

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M5 branes and Theta Functions, Haghighat, RS, 18'
D-type fiber-base duality, Haghighat, Kim, Yan, Yau, 18'
ADE String Chains and Mirror Symmetry, Haghighat, Yan, Yau, 17'
Categorical Mirror Symmetry: The Elliptic Curve, Polishchuk, Zaslow, 98'

# Outline

# Introduction

- M5 branes have been studied from various aspects and led to surprising insights into string theory and SCFT in various dimensions. However, an action for M5 theory is not clearly derived yet.
- In order to circumvent the fact that there is no Lagrangian description available, Witten proposed to view partition functions of M5 branes as vectors in quantum Hilbert space.
- This Hilbert space arises by realizing the M5 brane world-volume as a boundary of a 7d theory whose path integral with a suitable boundary condition gives the corresponding state, and M5 brane states as sections of a certain line bundle. [Witten' 96]

# Review of Witten's construction

- Witten showed that partition functions of M5 branes on a six-manifold $W$ can be understood as wave-functions which depend on the value of a certain background gauge three-form $C$.
- The fact that the two-form on the M5-brane worldvolume is chiral implies that these wave-functions are holomorphic. This holomorphy plus gauge-invariance under complexified gauge transformations then together imply that $C$ defines a point in $H^3(W, \mathbb{R})$.
- Dividing further by gauge transformations we then see that this space descends to $J_W = H^3(W, \mathbb{R})/H^3(W, \mathbb{Z})$, which is known as the *intermediate Jacobian* of $W$.
- Thus partition functions or states of the M5 brane are theta functions as sections of certain line bundle $L$ over the over the intermediate Jacobian $J_W$ of the world-volume manifold $W$.

# Quantum States of LSTs

- Whether Witten's approach can be generalized to the case of 6d $(1,0)$ theories where there is equally no action principle available?
- Such theories have recently enjoyed a resurgence due to the discovery of a vast geometric classification through F-theory. [Heckman, Mirrison, Vafa, 13']

  First steps towards this direction were undertaken by constructing the defect groups for various 6d SCFTs. [DelZotto, Heckman, Park, Rudelius, 15']
- We further generalize this to the case of 6d $(1,0)$ Little String Theories, by looking at 6d theories arising from M5 branes probing ADE singularities. The symmetry enhancement leads to surprising structures involving Riemann theta functions on complex 4-tori.
- The backgrounds we look at are provided by M5 branes on a certain limit of the Omega background [Nekrasov, Witten, 10'] probing $S^1 \times \mathbb{C}^2/\Gamma_{A,D}$ where the singularity is either of A-type or of D-type.

# ADE Strings

## Calabi-Yau threefolds admitting elliptic fibrations

- Calabi-Yau threefolds with elliptic fibrations are consists of non-compact threefolds which are elliptic fibrations over a non-compact base $B$. $B$ is a complex two-dimensional space which is obtained by blowing up an ADE singularity.
- It has 2-cycles $C^i$ which are $\mathbb{P}^1$'s with negative intersection matrix $\eta^{ij} = -C^i \cdot C^j$ being equal to the Cartan matrix of a simply laced gauge group of ADE type.
- The structure of the elliptic fiber is such that above each $C^i$ we let it degenerate according to an $I_{N_i}$ Kodaira singularity. To maintain the Calabi-Yau condition the $N_i$ have to be proportional to the Dynkin label of the corresponding node in the ADE Dynkin diagram.

- Topological string partition functions for such geometries were computed:

$$Z^{top}(t_{b,1},\ldots,t_{b,r},\tau,\vec{t}_{f,i},\lambda)=\sum_{\vec{n}} e^{2\pi i\vec{n}\cdot\vec{t}_b} Z^{\vec{n}}(\tau,\vec{t}_{f,i},\lambda),$$

with $\tau$ as the modulus of the elliptic fiber, $t_{b,i}$, $i=1,\ldots,r$ ($r$: rank of the ADE Lie algebra) are the moduli of the $C^i$ curves in the base, and $\vec{t}_{f,i}$ are moduli of the degenerate elliptic fiber above node $i$.
[Haghighat, Iqbal, Kozaz, Lockhart, Vafa,13'; Gadde, Haghighat, Kim, Kim,15']

- The mirror Calabi-Yau manifold of $X_{N,\widehat{\mathfrak{g}}}$ is given by equation

$$uv=F^{\mathrm{open}}(z_1,z_2),$$

where $F^{\mathrm{open}}$ is the open Gromov-Witten potential. And the equation

$$F^{\mathrm{open}}=0$$

is known as the so called *mirror curve*. [Haghighat, Yan, Yau, 17']

# M5 branes probing A-type singularities

In this case we have open Gromov-Witten potential (taking $\mathfrak{g}$ to be $A_{N-1}$) [Kanazawa, Lau, 16'; Haghighat, Yan, Yau, 17']

$$F^{\text{open}} = \sum_{i=0}^{r}\sum_{l=0}^{N-1} K_{i,l}\Delta_{i,l}\Theta_2\left[\begin{array}{c}(\frac{i}{r+1},\frac{l}{N})\\(\frac{-(r+1)\rho}{2},-\frac{N\tau}{2})\end{array}\right]\left((r+1)z_1, Nz_2;\left[\begin{array}{cc}(r+1)\rho & \sigma\\ \sigma & N\tau\end{array}\right]\right)$$

The theta function here is the genus 2 theta function. $\Delta_{i,l}$ are open Gromov-Witten generating functions and $K_{i,l}$ are given by

$$K_{i,l} = q_\rho^{\frac{i}{2}-\frac{i^2}{2(r+1)}}\, q_\tau^{\frac{l}{2}-\frac{l^2}{2N}}\,.$$

This defines a conic fibration over the abelian surface $\mathbb{C}^2/(\mathbb{Z}^2\oplus\Omega\mathbb{Z}^2)$ with discriminant being the genus $(r+1)N+1$ curve $F^{\text{open}}=0$, and $u$ and $v$ are sections of suitable line bundles over the abelian surface.

# Unimodular Transformation

Under shifts $\vec{z} \mapsto \vec{z} + \frac{1}{p}\Omega\vec{\mu} + \vec{\lambda}$, the theta function transforms as

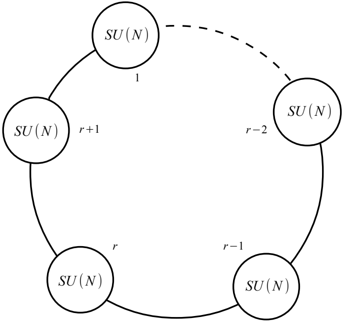
$$\theta\left[\begin{array}{c} (\frac{i}{r+1}, \frac{l}{N}) \\ (0,0) \end{array}\right]\left(\Omega, \begin{pmatrix} r+1 & 0 \\ 0 & N \end{pmatrix} \cdot \left(\vec{z} + \frac{1}{p}\Omega\vec{\mu} + \vec{\lambda}\right)\right)$$

$$= \sum_{\vec{n}\in\mathbb{Z}^2} \exp\left[\frac{1}{2}\vec{n'}^t\Omega\vec{n'} + \vec{n'} \cdot \left(\begin{pmatrix} r+1 & 0 \\ 0 & N \end{pmatrix}(\vec{z} + \frac{1}{p}\Omega\vec{\mu} + \vec{\lambda})\right)\right], \text{ where } \vec{n'} = \vec{n} + \begin{pmatrix} \frac{i}{r+1} \\ \frac{l}{N} \end{pmatrix}$$

$$= \sum_{\vec{n}\in\mathbb{Z}^2} \exp\left[\frac{1}{2}\left(\vec{n'} + \begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix}\right)^t \Omega \left(\vec{n'} + \begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix}\right) - \frac{1}{2}\begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix}^t \Omega \begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix}\right.$$

$$\left. + \left(\vec{n'} + \begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix}\right) \cdot \begin{pmatrix} (r+1)(z_1+\lambda_1) \\ N(z_2+\lambda_2) \end{pmatrix} - \begin{pmatrix} \frac{r+1}{p}\mu_1 \\ \frac{N}{p}\mu_2 \end{pmatrix} \cdot \begin{pmatrix} (r+1)(z_1+\lambda_1) \\ N(z_2+\lambda_2) \end{pmatrix}\right]$$

Studying theta function mathematically via abelian varieties, and perform a unimodular transformation, we mathematically prove the duality observed in "Dual Little Strings from F-Theory and Flop Transitions" is a unimodular transformation. [Hohenegger, Iqbal, Rey, 16'] [Birkenhake, Lange]




$$SU(N) \to SU\big((r+1)N/\gcd(r+1, N)\big)$$

# M5 branes probing D-type singularities

- Here our discussion will be enriched by a fundamental new symmetry, the $\mathbb{Z}_2$ reflection symmetry of the root lattice of $D_n$, which acts on our wave functions and under which they are invariant. [Haghighat, Kim, Yan, Yau, 18']
- Consider in the following the LST arising from a single M5 brane probing a singularity of type $D_4$, i.e. the perpendicular space to the M5 brane is $S^1 \times \mathbb{C}^2/\Gamma_{D_4}$. We denote this theory by $\mathcal{T}_{1,D_4}$. It was observed the M5 brane splits into two fractional $\frac{1}{2}\mathrm{M5}$ branes. [Del Zotto, Heckmann, Tomasiello, Vafa, 14']
- Then our $\mathbb{Z}_2$ symmetry will act now on these fractional branes and their wavefunctions will be either odd or even under this action. The reason is that only their square which was our original M5 brane has to be invariant.

Let us denote the $\frac{1}{2}\mathrm{M5}$ wavefunctions by $|\psi_{\pm}^{\frac{1}{2}}\rangle$ to keep track of the $\mathbb{Z}_2$ Eigenvalues. This can be schematically depicted in Figure 1.

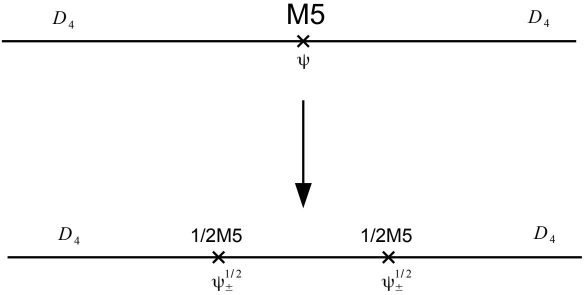



Figure: M5 branes probing D-type singularities.

# Construction of the Hilbert space

- Let us now apply the above results to our picture of a single M5 brane probing a $D_4$ singularity. In the following we shall construct the space of ground states of the M5 brane.
- We have two $\frac{1}{2}\mathrm{M5}$ branes and their corresponding operators/states are given by the following two theta functions

$$\begin{aligned}\theta_1(\tau, z_1) &= \theta\left[\begin{array}{c}1/2\\1/2\end{array}\right](\tau, z_1)\\ \theta_4(\tau, z_1) &= \theta\left[\begin{array}{c}0\\1/2\end{array}\right](\tau, z_1)\end{aligned} \tag{3.1}$$

- The first state is odd under the $\mathbb{Z}_2$ reflection while the second one is even. They thus correspond to our wavefunctions $\psi_{\pm}^{\frac{1}{2}}$. $\theta_1$ and $\theta_4$ are sections of *different* line bundles over $\mathbb{T}_\tau$.

$\mathbb{Z}_2$-invariant states appear only at the level $\left(\psi_{\pm}^{\frac{1}{2}}\right)^2$ and indeed the squares of $\theta_1$ and $\theta_4$ belong to the same line bundle

$$\begin{aligned} &\theta_1(\tau, z_1)^2 \\ = \ & \theta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](2\tau, 0)\theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](2\tau, 2z_1) - \theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](2\tau, 0)\theta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](2\tau, 2z_1) \\ &\theta_4(\tau, z_1)^2 \\ = \ & \theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](2\tau, 0)\theta\left[\begin{array}{c} 0 \\ 0 \end{array}\right](2\tau, 2z_1) - \theta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](2\tau, 0)\theta\left[\begin{array}{c} 1/2 \\ 0 \end{array}\right](2\tau, 2z_1) \end{aligned}$$

For T-dual picture of our LST, one can first compactify on one of the internal directions of our M5 brane and then perform T-duality along the perpendicular $S^1$ anoalogous to the already discussed $A$-type case.

We prove that the Seiberg-Witten curve written via theta functions matches exactly the result obtained in [Haghighat, Kim, Yan, Yau, 18']

$$0 = \frac{\theta_3(\tau,0)^2\theta_2(\tau,0)^2}{4\eta(\tau)^6} \cdot \sum_{n=0}^{4} a_n X(\rho,z_2)^n - \left(\frac{\theta_3(\tau,0)^4+\theta_2(\tau,0)^4}{12} + \frac{X(\tau,z_1)}{4}\right) \cdot \frac{c_0}{64} Y(\rho,z_2)^2$$

where $X$ and $Y$ are the Weierstrass functions

$$\begin{aligned} X(\rho,z) &= \theta_3(\rho,0)^2\theta_2(\rho,0)^2\frac{\theta_4(\rho,z)^2}{\theta_1(\rho,z)^2} - \frac{\theta_3(\rho,0)^4+\theta_2(\rho,0)^4}{3} \\ Y(\rho,z)^2 &= 4X(\rho,z)^3 - \frac{4}{3}E_4(\rho)X(\rho,z) - \frac{8}{27}E_6(\rho) \end{aligned}$$

with $E_4$, $E_6$ and $\eta$ the Eisenstein series of index 4 and 6 respectively and the Dedekind eta function.

# Conclusion & Outlook

- Constructing quantum states of M5 branes probing A-type and D-type singularities in terms of theta functions, we analyzed different duality frames of M5 branes probing A-type singularities. [Hohenegger, Iqbal, Rey, 16']
- In D-type, utilizing theta functions properties, we derived Seiberg-Witten curves for our LSTs which match with previously obtained results. [Haghighat, Kim, Yan, Yau, 18']
- We observe that D-type quantum states have a similar form as A-type states for the massless limit in a particular region in moduli space, it would be very interesting to explore this further. [Bastian, Hohenegger, 18']
- Indirect derivations and attempts for constructing M5 brane theory action [Cordova, Jafferis, 13, 16'] [Braden, Seamann, et al.]

*Thank you very much!*