Topological Formulae for Line Bundle Cohomology

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1808.09992, 1906.08769, 1906.08730 [hep-th]

1906.08363 [math-AG]

Three ways of computing cohomology

- view X as a topological space: cohomology can be computed by combinatorial data: it is determined by the combinatorics of a good covering by open sets
- view X as a differentiable manifold: use differential forms to compute cohomology
- view X as an algebraic variety: compute cohomology in terms of a complex of modules

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Less impressive: except for very simple cases, none of the above is practicable.

In an ideal world...

on a smooth complex manifold X, the cohomology groups of $V \to X$ would be computed from a topological quantity, similar to the Euler characteristic

$$\sum_{i=0}^{3}(-1)^i h^i(X,V) = \int_X \mathrm{ch}(V)\,\mathrm{Td}(X)$$

To what extent is it actually possible to have

$$h^i(X,V) = \int_X \mathrm{ch}(\tilde{V})\,\mathrm{Td}(X) \ ?$$

Important special case: line bundles.

Line bundles

Line bundles are classified by their first Chern class

$$c_1(L) = k_1 J_1 + k_2 J_2 + \ldots + k_{\mathrm{Pic}(X)} J_{\mathrm{Pic}(X)} \ ,$$

where k_i are integers and $\{J_i\}$ is a basis of $H^2(X, \mathbb{Z})$.

Notation: $L = \mathcal{O}_X(k_1, \ldots, k_{\mathrm{Pic}(X)})$. Line bundles form an abelian group under tensor product, the trivial bundle being denoted by $\mathcal{O}_X$.

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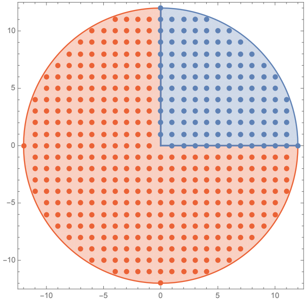
Zero locus of a global section: divisor (linear combination of irreducible subvarieties of codimension 1 with integral coefficients).

The ample cone



- cohomological characterisation: $h^0(L) = \mathrm{ind}(L)$, $h^{i>0}(L) = 0$
- algebraic characterisation: the linear system of divisors associated with L has a ‘small’ base locus (very ample: vanishing base locus)

The base locus of a linear system of divisors is the set of points shared by all divisors in the linear system (all divisors pass through the base locus; different name: fixed part).

The idea

If L is

- non-effective, then $h^0(X, L) = 0$
- effective and $L \otimes K_X^*$ is ample, $h^0(L) = \mathrm{ind}(L)$
- effective and $L \otimes K_X^*$ is not-ample, then alter L without changing the dimension of the zeroth cohomology until you bring it into the ample cone

$$L \to \tilde{L}$$

Main message: this can be achieved, at least in some cases.

Idea: reduce the base locus of $|D|$. This doesn't change the count of global sections, but changes the divisor class (the line bundle). If the base locus is made small enough, the cohomology of the 'reduced' line bundle is given by the Euler characteristic.

Difficulty: detect the components of the base locus.

Surfaces

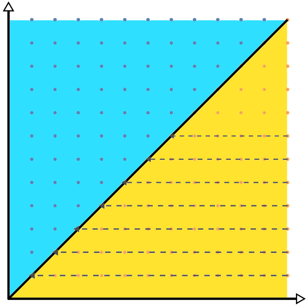
Surfaces are special in several ways:

- knowledge of $h^0(X, L)$ throughout the entire Picard lattice determines all higher cohomologies:

$$\begin{aligned} h^2(X, L) &= h^0(X, K_X \otimes L^*) \\ \mathrm{ind}(L) &= h^0(X, L) - h^1(X, L) + h^2(X, L) \end{aligned}$$

- divisors are curves
- if the base locus contains rigid curves, these can be detected by intersection theory

Surface example: dP$_1$: blow-up of $\mathbb{P}^2$ in a point



$$h^0(\mathcal{O}(D)) = \operatorname{ind}(D - \# D_{-1}),$$

$$\# = \theta(-D \cdot D_{-1})(D \cdot D_{-1})$$

Three-fold examples

Example 1: blow-up of $\mathbb{P}^3$ in a point: similar stratification of the Picard group; there is a rigid divisor which plays a similar role as for dP_1.

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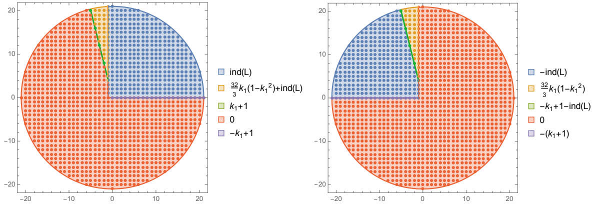
Example 2: a generic hypersurface of type $(2,4)$ in $\mathbb{P}^2 \times \mathbb{P}^4$



- there are no rigid divisors in this case, so the stratification observed must be due to something else

Experimental results

- CICY three-folds [Buchbinder, Constantin, Lukas 2013; Constantin, Lukas 2018; Larfors, Schneider 2019]
- (hypersurfaces in) toric varieties [Klaewer, Schlechter 2018]
- del Pezzo surfaces, Hirzebruch surfaces, toric surfaces [Brodie, Constantin, Deen, Lukas 2019]

Outlook and questions

- analytic formulae for line bundle cohomology facilitate a bottom-up approach to model building
- what is the mathematics behind the stratification observed for CICY three-folds; higher dimensional manifolds?
- the formulae depend on topology, but also on the complex-structure of X, via the map $L \to \tilde{L}$: study jumping loci for cohomology (relevant e.g. for moduli stabilisation)

Thank You!