Index Formulae for Line Bundle Cohomology on Complex Surfaces

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Based on 1906.08363, 1906.08730, and 1906.08769

with Andrei Constantin, Rehan Deen, and Andre Lukas

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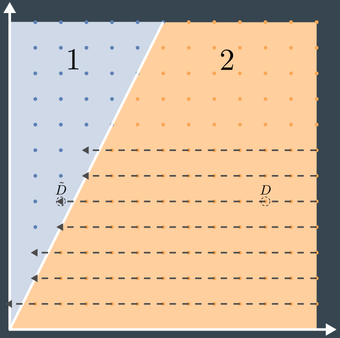
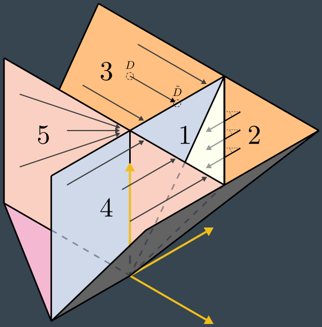
Understand cohomology structure illustrated in these pictures:





Motivation

Andrei's preceding talk motivated understanding formulae for line bundle cohomology, but . . .

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Why (complex) surfaces?

- Surfaces are building blocks for CYs (toric, del Pezzos, Hirzebruchs . . .)

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- Simpler arena to understand cohomology

  $\Rightarrow$ learn about CY_3 cohomology structure too?

Line bundles and divisors

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Key idea for us: some divisor parts can be rigid - always in complete linear system, don't contribute to deformations.

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Dropping rigid pieces doesn't affect deformations, so zeroth cohomology of associated bundle is unaffected,

$$h^0(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(D - D_{\text{rigid}}))\,,$$

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Why is this useful? If we throw away enough rigid pieces, often resulting bundle has simpler cohomology, specifically

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But this is only useful in practice if we can detect rigid pieces. Happily, rigid pieces can be detected by intersections,

$$D \cdot D_{\text{rigid}} < 0 \quad \Rightarrow \quad D_{\text{rigid}} \subset |D|_{\text{def}}\,.$$

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If a divisor D contains D^- as a part, then there are negative contributions to intersection $(D^-) \cdot D$.

But intersection theory is defined up to equivalence: so doesn't care about individual deformations.

So negative intersection $(D^-) \cdot D < 0$ means every deformation contains the piece D^-.

Question:

How many rigid pieces can be detected with intersections?

General theorems

Theorem

Let D be an effective divisor on a smooth compact complex projective surface S, with associated line bundle $\mathcal{O}_S(D)$. Let $\mathcal{I}$ be the set of irreducible negative self-intersection divisors. Then the following map,

$$D \to \tilde{D} = D - \sum_{C \in \mathcal{I}} \theta(-D \cdot C)\, \mathrm{ceil}\left(\frac{D \cdot C}{C^2}\right) C\,,$$

where θ is the step function, preserves the zeroth cohomology,

$$h^0\big(S, \mathcal{O}_S(\tilde{D})\big) = h^0\big(S, \mathcal{O}_S(D)\big)\,.$$

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Corollary

Write $\underline{\tilde{D}}$ for the divisor that is the result of iterating the map $D \to \tilde{D}$, until stabilisation after a finite number of steps. Then $\underline{\tilde{D}}$ is a nef divisor such that $h^0\big(S, \mathcal{O}(D)\big) = h^0\big(S, \mathcal{O}(\underline{\tilde{D}})\big)$.

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Divisor D mapped to new divisor $\underline{\tilde{D}}$: $D \to \tilde{D} \to \ldots \to \underline{\tilde{D}}$.

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If higher cohomologies vanish for $\underline{\tilde{D}}$, then h^0 reduces to index computation (simpler, topological),

$$\text{if } h^1(S, \mathcal{O}_S(\underline{\tilde{D}})) = h^2(S, \mathcal{O}_S(\underline{\tilde{D}})) = 0\,,$$

$$\text{then } h^0(S, \mathcal{O}_S(D)) = h^0(S, \mathcal{O}_S(\underline{\tilde{D}})) = \chi(S, \mathcal{O}_S(\underline{\tilde{D}}))\,.$$

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Can we make general statements on vanishing for $\underline{\tilde{D}}$?
Yes when there are vanishing theorems.

Corollary

If a vanishing theorem ensures that $h^q(S, \mathcal{L}) = 0$ for $q > 0$ for all nef bundles $\mathcal{L}$, then all zeroth cohomology is given by an index,

$$h^0(S, \mathcal{O}_S(D)) = \chi(S, \mathcal{O}_S(\underline{\tilde{D}}))\,.$$

Example: Compact toric surfaces

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For the important class of compact toric surfaces there is a powerful vanishing theorem (Demazure) for nef bundles.

Corollary

Let S be a compact toric surface, and D an effective divisor. Then

$$h^0\big(S, \mathcal{O}(D)\big) = \chi(\underline{\tilde{D}})\,,$$

where the divisor $\underline{\tilde{D}}$ is obtained from D by iterating the shifts $D \to \tilde{D}$.

Single-shift cases

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If there is a vanishing theorem for the nef cone,
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Corollary

If $\tilde{D}$ is always nef and $h^q(S_S, \mathcal{O}_S(\underline{\tilde{D}})) = 0$ for $q > 0$ for all $\tilde{D}$, then for effective D we have the closed-form expression,

$$h^0(S, \mathcal{O}_S(D)) = \chi\left(D - \sum_{C \in \mathcal{I}} \theta(-D \cdot C)\,\mathrm{ceil}\left(\frac{D \cdot C}{C^2}\right) C\right).$$

Surprisingly, there are many interesting examples of such surfaces, including Hirzebruch and del Pezzo surfaces.

Example: Hirzebruch surfaces $\mathbb{F}_n$

Vanishing theorem for nef bundles ✓ (Demazure)
Shift $D \to \tilde{D}$ terminates in one step ✓

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where C is the unique negative self-intersection curve.

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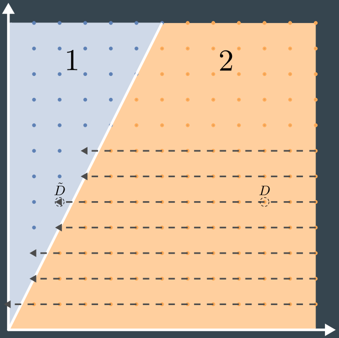

Figure shows situation for $\mathbb{F}_2$ (2d Picard lattice)

Region 1 is the nef cone

Example: del Pezzo surfaces dP_n

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where C_i are the exceptional curves.

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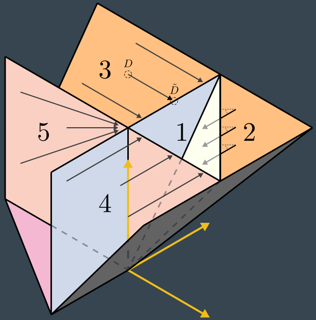

Figure shows situation for dP_2 (3d Picard lattice)

Region 1 is the nef cone

Results

- Understanding of structure for h^0 for surfaces.
- For compact toric surfaces: algorithm to get h^0 as index.
- For dP_n and $\mathbb{F}_n$: closed-form index formulae for h^0.

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- Use surfaces as building blocks for CY_3 and lift h^0 to CY_3.
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Extensions

- Extend to other surfaces? K3, general type, ...
- Extend proofs to higher dimensions? 3-folds, 4-folds, ...
- Extend results to higher cohomologies? h^1, h^2, ...

Thank you for your attention