Slides

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(Birational) Geometry and String Pheno

CERN, 2019

Example: $X$ Kummer

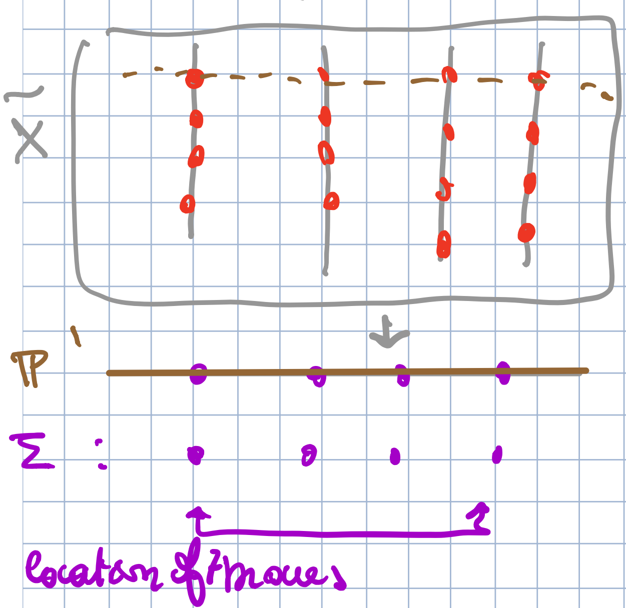
- resolution $X$ :

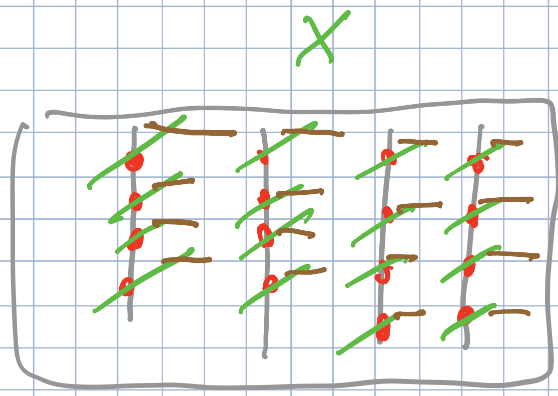
- $(T^2 \times T^2)/\mathbb{Z}/2\mathbb{Z} = \bar{X}$

$$\begin{array}{c} (T^2 \times T^2)/\mathbb{Z}/2\mathbb{Z} \\ \downarrow \\ \mathbb{P}^1 = T^2/\mathbb{Z}/2\mathbb{Z} \end{array}$$

Action:

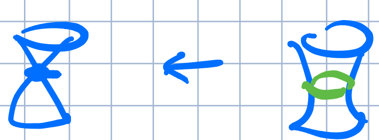
$(x, y) \mapsto (-x, -y)$




sections.

(F-theory)



R. Miranda - elliptic surfaces



## I.4: The List of possible singular fibers. Kodaira

Before beginning to address the issues mentioned above, I would like to give a series of examples to illustrate some of the features of the theory to the reader. It will be useful for the purposes of illustration and communication for the reader to know the possible singular fibers which can occur, and Kodaira's names for them. This I present below, without proof, simply so that I can speak of them intelligently in the examples to follow.

(I.4.1)Table of possible singular fibers of a smooth minimal elliptic surface. The names are those used by Kodaira.

| Name | Fiber |
|---|---|
| $I_0$ | smooth elliptic curve |
| $I_1$ | nodal rational curve |
| $I_2$ | two smooth rational curves meeting transversally at two points |
| $I_3$ | three smooth rational curves meeting in a cycle; a triangle |
| $I_N, N\geq 3$ | N smooth rational curves meeting in a cycle, i.e., meeting with dual graph $\tilde{A}_N$ |
| $I_N^*, N\geq 0$ | N+5 smooth rational curves meeting with dual graph $\tilde{D}_{N+4}$ |
| II | a cuspidal rational curve |
| III | two smooth rational curves meeting at one point to order 2 |
| IV | three smooth rational curves all meeting at one point |
| $IV^*$ | 7 smooth rational curves meeting with dual graph $\tilde{E}_6$ |
| $III^*$ | 8 smooth rational curves meeting with dual graph $\tilde{E}_7$ |
| $II^*$ | 9 smooth rational curves meeting with dual graph $\tilde{E}_8$ |
| ${}_M I_N, N\geq 0$ | topologically an $I_N$, but each curve has multiplicity N |

All components of reducible fibers have self-intersection -2; the irreducible fibers have self-intersection 0, of course.

The dual graphs referred to above are those of the ~~extended~~ Dynkin diagrams. For ease of reference I'll give below tables of the Dynkin diagrams and the extended Dynkin diagrams.

# 2. Algebras, representations, geometry

| Singular fiber, up to birational equivalence | $I_1$ | $II$ | $I_n$ | $I_2$ | $III$ | $IV$ | $I_n^*$ | $IV^*$ | $III^*$ | $II^*$ | ${}_mI_b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| algebra: A-D-E | $\{e\}$ | $\{e\}$ | $\mathrm{su}(n)$ | $\mathrm{su}(2)$ | $\mathrm{su}(2)$ | $\mathrm{su}(3)$ | $\mathrm{so}(2n{+}8)$ | $e_6$ | $e_7$ | $e_8$ | **?** |

If there is a section: Kodaira's $\overset{\text{almost}}{\leftrightarrow}$ algebras

Exceptions: $I_1$ and $II$; $I_2$ and $III$, ${}_m I_b$

Distinguish these?

Generic, "Movable" in Kodaira's canonical divisor formula, !! – $\begin{cases} \text{NOT in} \\ \text{NHC} \end{cases}$

Analysis to distinguish them:
(Morsification: G.-Halverson-Shaneson; G.-Halverson-Long-Shaneson-Tian; Grassi-Weigand)

# 2. Algebras, representations, geometry

For $\dim X = 2$, Kodaira's classification of fibers on $X$

| Singular fiber, up to birational equivalence | $I_1$ | $II$ | $I_n$ | $I_2$ | $III$ | $IV$ | $I_n^*$ | $IV^*$ | $III^*$ | $II^*$ | ${}_mI_b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| algebra: A-D-E | $\{e\}$ | $\{e\}$ | $\mathrm{su}(n)$ | $\mathrm{su}(2)$ | $\mathrm{su}(2)$ | $\mathrm{su}(3)$ | $\mathrm{so}(2n+8)$ | $e_6$ | $e_7$ | $e_8$ | **?** |

Singularities of $W$, Weierstrass model: Klein, Dynkin, Coxeter...

Question posed by Arnold, 1976:
*to find a common origin of all the A-D-E classification theorems, and to substitute a a priori proofs to a posteriori verifications of a parallelism of the classifications*

Conjectures-Theorems (slice): *Grothendieck-Brieskorn* 1970; Slodowy

(From Algebra to Geometry)

Our results.

algebras & their reps and singularities

**The formula $\mathcal{R}$: local to global (Grassi-Morrison; Grassi-Weigand.)**
**Application: Global to local, Birational Kodaira classification of "singular fibers".**

| Number | Type | $\mathfrak{g}$ | $\rho_0$ | $\rho_{Q_1^\ell}$ | $\rho_{Q_2^\ell}$ | $(\dim \mathrm{adj})_{ch}$ | $(\dim \rho_0)_{ch}$ | $\dim(\rho_{Q_1^\ell})_{ch}$ | $\dim(\rho_{Q_2^\ell})_{ch}$ | $\tau(P_1)$ | $\tau(P_2)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $I_1$ | $\{e\}$ | | – | – | 0 | 0 | 0 | 0 | 0 | 1 |
| 7 | $II$ | $\{e\}$ | | – | | 0 | 0 | 0 | | 2 | |
| 2 | $I_2$ | su(2) | | – | fund | 2 | 0 | 0 | 2 | | |
| 8 | $III$ | su(2) | | $2 \times$ fund | | 2 | 0 | 4 | | | |
| 3 | $I_3$ | su(3) | | – | fund | 6 | 0 | 0 | 3 | | |
| 4 | $I_{2k},\ k \geqslant 2$ | sp($k$) | $\Lambda_0^2$ | – | fund | $2k^2$ | $2k^2-2k$ | 0 | $2k$ | | |
| 5 | $I_{2k+1},\ k \geqslant 1$ | sp($k$) | $\Lambda^2 + 2 \times$ fund | $\frac{1}{2}$ fund | fund | $2k^2$ | $2k^2+2k$ | $k$ | $2k$ | 1 | 0 |
| 6 | $I_n,\ n \geqslant 4$ | su($n$) | | $\Lambda^2$ | fund | $n^2-n$ | 0 | $\frac{1}{2}(n^2-n)$ | $n$ | | |
| 9 | $IV$ | sp(1) | $\Lambda^2 + 2 \times$ fund | $\frac{1}{2}$ fund | | 2 | 4 | 1 | | | |
| 10 | $IV$ | su(3) | | $3 \times$ fund | | 6 | 0 | 9 | | | |
| 11 | $I_0^*$ | $\mathfrak{g}_2$ | **7** | – | | 12 | 6 | 0 | | | |
| 12 | $I_0^*$ | spin(7) | vect | – | spin | 18 | 6 | 0 | 8 | | |
| 13 | $I_0^*$ | spin(8) | | vect | $\mathrm{spin}_\pm$ | 24 | 0 | 8 | 8 | | |
| 14 | $I_1^*$ | spin(9) | vect | – | spin | 32 | 8 | 0 | 16 | | |
| 15 | $I_1^*$ | spin(10) | | vect | $\mathrm{spin}_\pm$ | 40 | 0 | 10 | 16 | | |
| 16 | $I_2^*$ | spin(11) | vect | – | $\frac{1}{2}$ spin | 50 | 10 | 0 | 16 | | |
| 17 | $I_2^*$ | spin(12) | | vect | $\frac{1}{2}\,\mathrm{spin}_\pm$ | 60 | 0 | 12 | 16 | | |
| 18 | $I_n^*,\ n \geqslant 3$ | so($2n+7$) | vect | – | NM | $2(n+3)^2$ | $2n+6$ | 0 | NM | | |
| 19 | $I_n^*,\ n \geqslant 3$ | so($2n+8$) | | vect | NM | $2(n+3)(n+4)$ | 0 | $2n+8$ | NM | | |
| 20 | $IV^*$ | $\mathfrak{f}_4$ | **26** | – | | 48 | 24 | 0 | | | |
| 21 | $IV^*$ | $\mathfrak{e}_6$ | | **27** | | 72 | 0 | 27 | | | |
| 22 | $III^*$ | $\mathfrak{e}_7$ | | $\frac{1}{2}$**56** | | 126 | 0 | 28 | | | |
| 23 | $II^*$ | $\mathfrak{e}_8$ | | NM | | 240 | 0 | NM | | | |

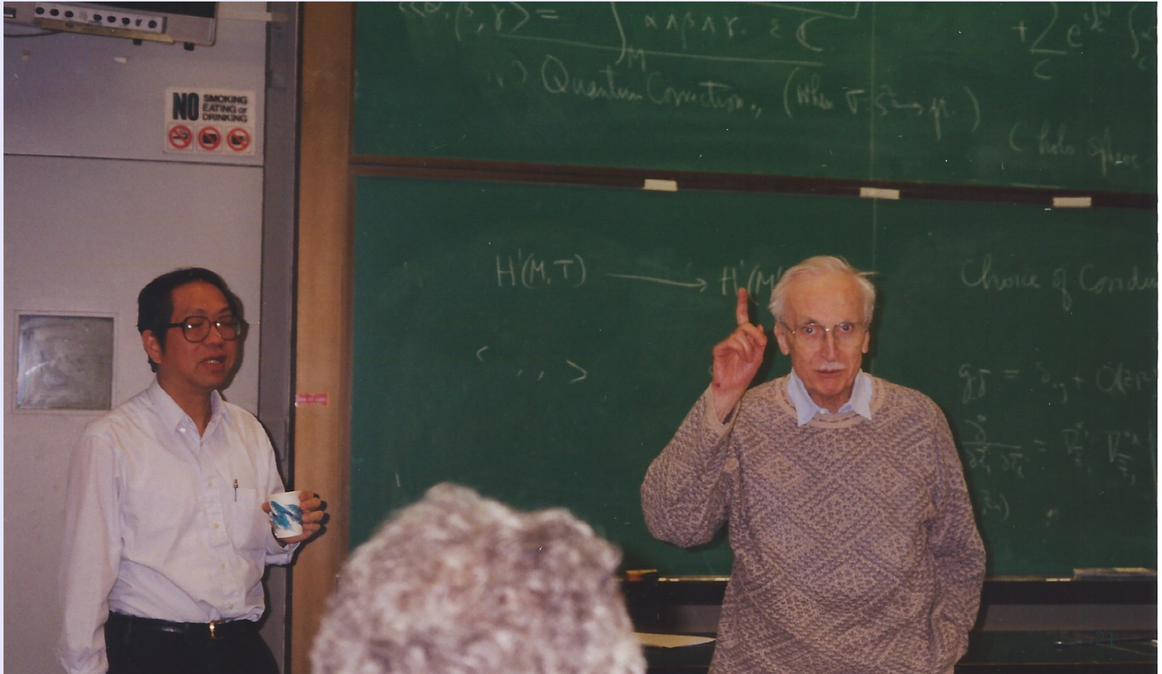


birationally Distinguish $(I_1, II)$ $(III, I_2)$ $(\hat{IV}, I_3)$
su(2) su(3)

**And** more general codim 2 singularities
in the discriminant

(bi-reps etc... )

Also: multiple fibers too $\leftrightarrow$
discrete symmetries and !! codim 1.


NO SMOKING EATING or DRINKING