Stochastic inflation beyond slow roll

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Overview

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2. Stochastic Approach
3. What can we use stochastic inflation for?
4. Beyond slow roll
5. Summary

Inflation and Slow-roll

Slow-roll inflation

Classical equation of motion for inflaton is

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0\,.$$

Assuming field is potential dominated (plus de Sitter), so

$$\dot{\phi}^2 \ll V(\phi)\,,$$

is **slow roll**.

Then eom simplifies to

$$\dot{\phi}_{\rm SR} \simeq -\frac{V'(\phi)}{3H}\,.$$

Stochastic Approach

Stochastic Formalism

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Split the inflaton field into two parts,

$$\phi = \bar{\phi} + \phi_{\rm s} \, .$$

$\bar{\phi}$ is the coarse-grained field (locally FLRW), and $\phi_{\rm s}$ contains all short-wavelength modes.

Plug split field into eom, and to linear order in the short-wavelength parts of the fields, we have the Langevin equation

$$\frac{\mathrm{d}\bar{\phi}}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi\left(N\right) \,.$$

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ξ is a source function related to the short-wavelength parts of the fields, where $\langle\xi\left(N\right)\rangle = 0$, $\langle\xi\left(N\right)\xi\left(N'\right)\rangle = \delta\left(N - N'\right)$, $k < aH$, and $H = \frac{8\pi G}{3}V(\bar{\phi})$

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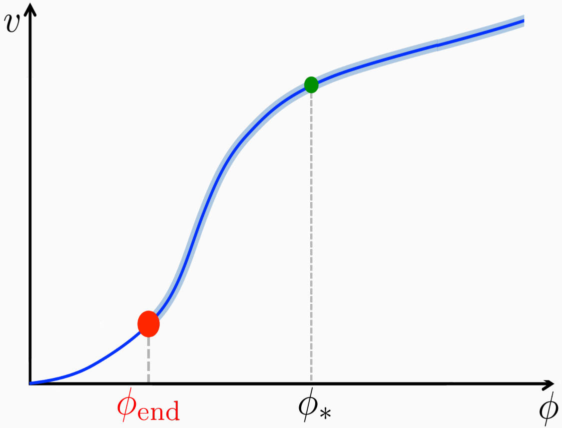
We interpret these equations as **classical and stochastic**, rather than quantum, including viewing ξ as a noise term.

Inflaton evolves under Langevin equation until ϕ reaches ϕ_{end} where inflation ends.



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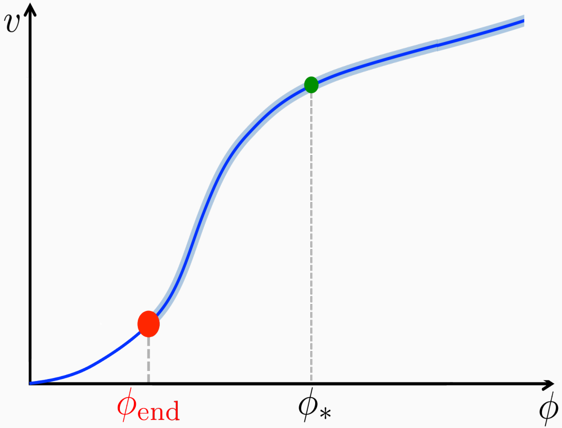

Important note for later: The amplitude of the noise is determined by $\langle \delta\phi^2 \rangle$, which is usually calculated in the **spatially-flat gauge**.

What can we use stochastic inflation for?

Example: Primordial black holes (PBHs)

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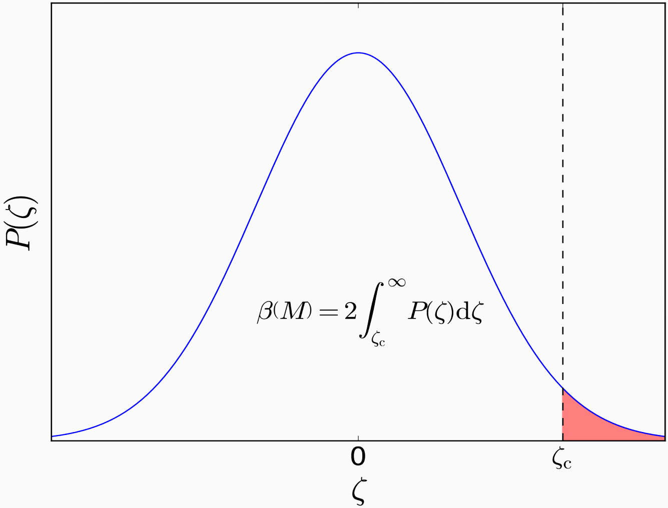
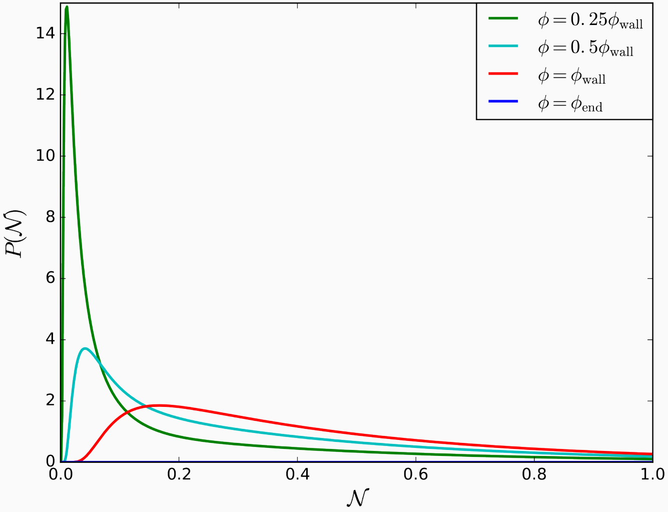
- Large density fluctuations during inflation can collapse to form PBHs.
- Such large fluctuations need a non-perturbative approach - the δN formalism.
- We use stochastic-δN to study how likely PBHs are to form (Pattison et al, 2017).

Gaussian Example

Typically assumed curvature perturbations have Gaussian distribution.



Including large quantum effects...



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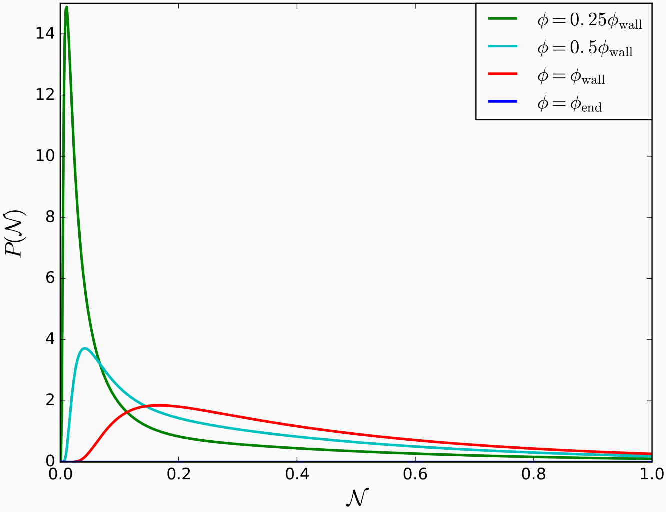

Quantum effects = produce more PBHs. Use this to constrain inflaton potential. However...

Beyond slow roll

Slow-roll violation

- Many models that produce PBHs also violate slow-roll!
- This means stochastic formalism needs to be extended to include these situations.
- Need to check that stochastic is a valid formalism beyond slow roll (Pattison et al, 2019).

Requirements of stochastic inflation

There are three criterion that we need fulfil for stochastic inflation to be a valid approach.

- **Quantum-to-classical transition:** In order to replace quantum quantities with stochastic ones, we need to assume all operators act classically.

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- In slow roll, these gauges are the same so it is no problem!
- In other cases such as ultra slow-roll - which can be stable and long-lived (Pattison et al, 2018) - this gauge correction could be large
- We calculate the gauge correction explicitly in common examples and show it is always small

Summary

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- Stochastic inflation models quantum fluctuations during inflation, using classical equations
- It is sensitive to large-scale quantum kicks, coming from new modes exiting the horizon
- The quantum effects can be important for astrophysical objects such as PBHs
- Although slow-roll is often useful, formalism needs to be extended beyond this, for example in models that produce PBHs

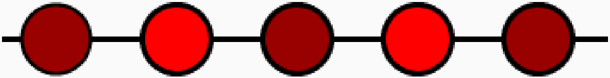
Questions?

Uniform expansion?

Separate universe if safe, but we used spatially flat gauge to show this.

Stochastic inflation implicitly uses the uniform expansion gauge in it's Langevin equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}N} = -\frac{V'}{3H^2} + \frac{H}{2\pi}\xi\left(N\right) \ ,$$



Amplitude of the noise is $\propto \left\langle \delta\phi^2 \right\rangle$, so if there is a gauge transformation from SF to UE, then this needs to be corrected.

The perturbation $\delta\phi$ transforms as

$$\delta\phi \to \underbrace{\tilde{\delta\phi}}_{\text{UE}} = \underbrace{\delta\phi}_{\text{SF}} + \frac{\partial\phi}{\partial\eta} \underbrace{\alpha}_{\text{transf.}},$$

If we solve for general α, we find

$$\alpha = \frac{1}{3\mathcal{H}} \int_{0^-}^{\eta} S(\eta') \exp\left[\frac{k^2}{3} \int_{\eta'}^{\eta} \frac{\mathrm{d}\eta''}{\mathcal{H}(\eta'')}\right] \mathrm{d}\eta'$$

Consistency check: slow roll

It is well known that in slow roll the spatially flat and uniform expansion gauges coincide.

Let us check we reproduce this result!

We find $S \propto P_{\mathrm{nad}}$, so S should vanish as slow roll is an attractor.

Solving for α, we see

$$\alpha_{\mathrm{SR}} \sim (-k\eta)^3 \,,$$

or the corrected field fluctuation is

$$\delta\phi_{\mathrm{UE}} = \delta\phi_{\mathrm{SF}} \left(1 - \frac{\epsilon_1}{6}(-k\eta)^2\right)$$

so in the large scale limit $\alpha \to 0$.

Ultra-slow roll

Take the case of $V' = 0$ in

$$\ddot{\phi} + 3H\dot{\phi} = 0\,.$$

This is "ultra-slow-roll" inflation.

USR isn't necessarily an attractor (but can be, Pattison et al, 2018), so there is no reason, *a priori*, that S or α should be small.

Turning the handle, we find

$$\alpha_{\mathrm{USR}} \sim (-k\eta)^4 \to 0\,,$$
$$\delta\phi_{\mathrm{UE}} = \delta\phi_{\mathrm{SF}}\left(1 + \frac{\epsilon_1}{3}(-k\eta)^6\right)$$

In fact, $\alpha \to 0$ even faster in USR than SR.

Starobinsky model (Starobinsky, 1992)

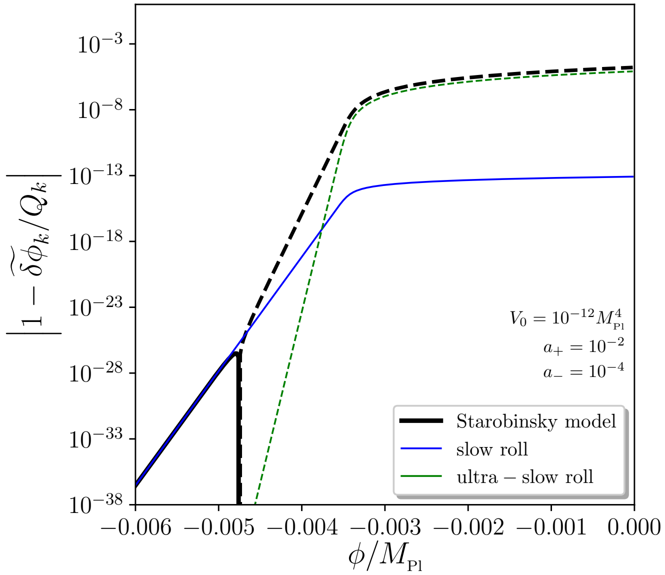
This is a model made of two linear parts

$$V(\phi) = \begin{cases} V_0 \left(1 + a_+ \frac{\phi}{M_{\rm Pl}}\right) & \text{for } \phi > 0 \\ V_0 \left(1 + a_- \frac{\phi}{M_{\rm Pl}}\right) & \text{for } \phi < 0\,. \end{cases}$$

and we assume $a_-/a_+ \ll 1$.

Then this model interpolates between slow roll and USR, with a transition phase in between.

We can again calculate the gauge corrections here, and it is always small!



$\left|1-\widetilde{\delta\phi_k}/Q_k\right|$
10^{-3}
10^{-8}
10^{-13}
10^{-18}
10^{-23}
10^{-28}
10^{-33}
10^{-38}
-0.006 -0.005 -0.004 -0.003 -0.002 -0.001 0.000
$\phi/M_{\rm Pl}$
$V_0 = 10^{-12} M_{\rm Pl}^4$
$a_+ = 10^{-2}$
$a_- = 10^{-4}$
Starobinsky model
slow roll
ultra − slow roll

Field: $\phi = \bar{\phi} + \phi_{\mathrm{s}}$.

Coarse-grained field $\bar{\phi}$ contains all wavelengths that are much larger than the Hubble radius ($k \ll \sigma a H$, where $\sigma \ll 1$).

Acts as a shifting background, continually absorbing new long-wavelength modes as they exit the horizon.

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Short-wavelength parts of the fields, ϕ_{s} are

$$\phi_{\mathrm{s}} = \int_{\mathbb{R}^3} \frac{\mathrm{d}\mathbf{k}}{(2k)^{\frac{3}{2}}} W\left(\frac{k}{\sigma a H}\right) \left[e^{-i\mathbf{k}\cdot\mathbf{x}} \delta\phi_{\mathbf{k}}(N) a_{\mathbf{k}} + e^{i\mathbf{k}\cdot\mathbf{x}} \delta\phi^*_{\mathbf{k}}(N) a^\dagger_{\mathbf{k}}\right]$$

W is a window function with $W \simeq 0$ for $k/(\sigma a H) \ll 1$ and $W \simeq 1$ for $k/(\sigma a H) \gg 1$.

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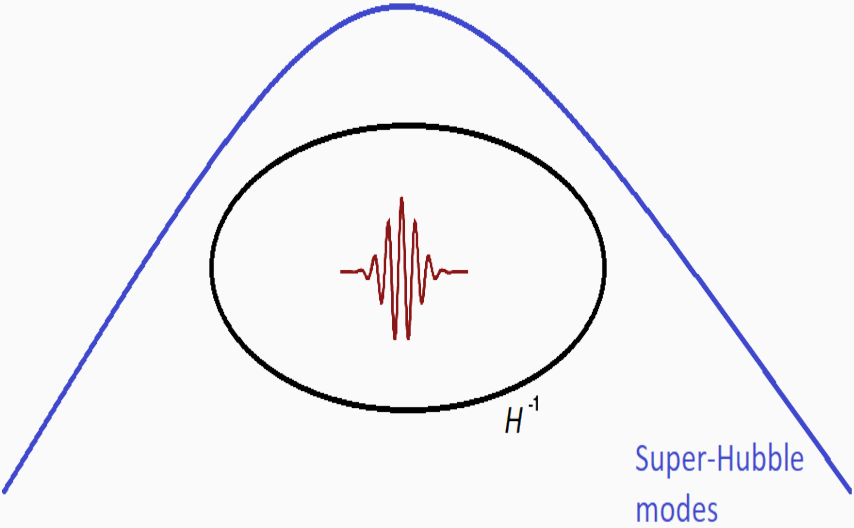

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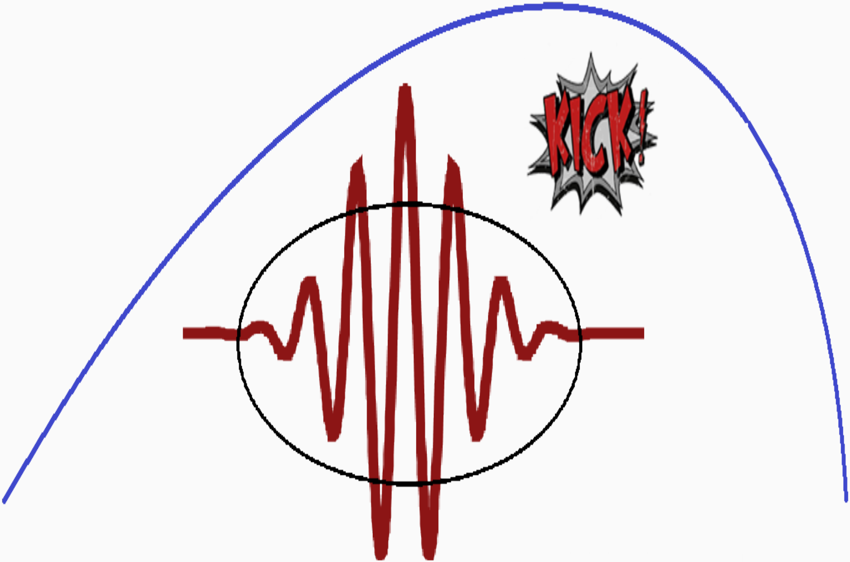
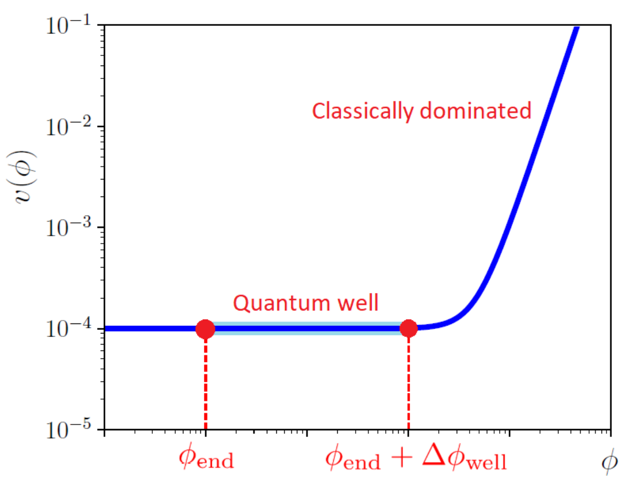


Stochastic Limit

Inflationary models that can produce $\zeta > \zeta_c$ are well approximated by a flat potential at the end of inflation, so $v \simeq v_0$.



Separate Universe (Wands et al, 2000) and δN

The primordial curvature perturbation ζ is

$$\zeta(t, \mathrm{x}) = N(t, \mathrm{x}) - N_0(t) \equiv \delta N \,,$$

where N is the local number of e-folds of inflation, and N_0 is the amount of expansion in an unperturbed universe.

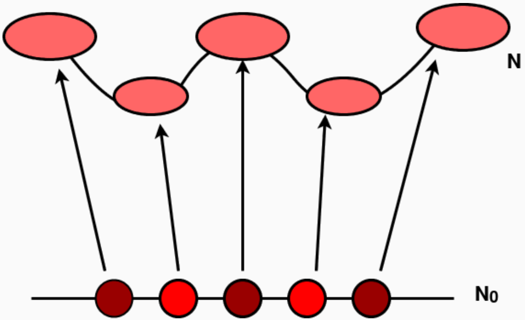


Figure 1: N_0 is at a zero curvature surface, final slice is constant density.