Baryon Physics and Tight Coupling Approximation in Boltzmann Code

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arXiv:1906.06831
4th Sept, 2019

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Introduction

- Modern cosmology heavily relies on linear perturbations, which are solved by Boltzmann codes.
- There are several examples CLASS, CAMB, CMBEASY, CMBFAST, etc.
- There are several future experiments to understand the dark sector of our universe like EUCLID, DESI and LSST.
- All these experiments aim for higher precision.
- The Boltzmann codes need to be precise enough and any inconsistency in the code should be fixed to take full advantage of these experiments.

https://class-code.net/
https://camb.info/

Covariance Problem in Current Boltzmann Codes

Gauge Incompatibility
The equations of motion for baryons in Newtonian and Synchronous gauges are not related. This breaks general covariance. The two equations of motion are intrinsically different.

Breaking Bianchi Identity
This gauge incompatibility breaks the Bianchi identity, i.e. solutions from the Einstein equations would be in general not consistent with the conservation law.

No Known Covariant Action for fluid with $c_s^2 \neq 0$
Introducing back the gauge-compatibility, there should be counter terms at the order of $c_s^2 k^2$. We need to introduce an action/eom beyond dust approximation.

Covariance Problem in Current Boltzmann Codes

$$\dot{\delta}_b = -\theta_b - 3\dot{\zeta},$$
$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + k^2\alpha + c_s^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T \left(\theta_\gamma - \theta_b\right).$$

[CP. Ma, E. Bertschinger, Astrophys.J. 455 (1995) 7-25]

Baryon equation in Newtonian gauge
$\delta_b \equiv (\rho_b - \bar{\rho}_b)/\bar{\rho}_b$, θ_γ & θ_b are the scalar-part of photon and baryon velocity perturbation respectively.

$$\delta_b^{\rm NG} = \delta_b + \frac{\dot{\bar{\rho}}_b}{a\bar{\rho}_b}\chi - \frac{\dot{\bar{\rho}}_b}{\bar{\rho}_b}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right),$$
$$\theta_b^{\rm NG} = \theta_b + \frac{k^2}{a}\chi - k^2\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right),$$
$$\theta_\gamma^{\rm NG} = \theta_\gamma + \frac{k^2}{a}\chi - k^2\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right),$$
$$\zeta^{\rm NG} = \zeta + \frac{\dot{a}}{a^2}\chi - \frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right),$$
$$\alpha^{\rm NG} = \alpha + \frac{1}{a}\dot{\chi} - \frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - \frac{\partial^2}{\partial\tau^2}\left(\frac{E}{a^2}\right).$$

Gauge invariant combination. Choosing $\chi = E = 0$ reduces to Newtonian gauge fields.

Gauge Invariant combination

$$\dot{\delta}_b = -\theta_b + k^2\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right) - 3\dot{\zeta},$$
$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2\left[\delta_b + 3\frac{\dot{a}}{a}\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)\right] + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a n_e \sigma_T \left(\theta_\gamma - \theta_b\right).$$

Resulting baryon equation of motion in Synchronous gauge. Missing in current Boltzmann codes.

Where we define $ds^2 = -(1+2\alpha)\,a^2\,d\tau^2 + 2a\partial_i\chi\,d\tau\,dx^i + a^2\left[(1+2\zeta)\,\delta_{ij} + \frac{2\partial_i\partial_j E}{a^2}\right]dx^i dx^j$.

Ideal Gas

Here we consider a model of perfect fluid for baryon with $c_s^2 \neq 0$.

Equation of state

$p = n\,T\,,$

$\rho = \mu_g\,n + \frac{3}{2}\,n\,T\,.$

p, n, ρ and T are the pressure, number density, energy density and temperature of the fluid respectively. μ_g is the mass of the fluid particle.

The equation of state $p = \frac{\rho T}{\mu_g + 3T/2}$, i.e. $p = p\,(\rho, T)$, shows the fluid is non-barotropic.

The speed of propagation at I$^{\rm st}$ order in $\frac{T}{\mu_g}$,
$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{T}{\mu_g}\left(1 - \frac{1}{3}\,\frac{d\ln T}{d\ln a}\right).$

When the baryons interact with photons there is an exchange of entropy with the photon fluid. Using first law of thermodynamics we get

$\dot{T} = -2\,\frac{\dot{a}}{a}\,T + \frac{8}{3}\,\frac{\rho_\gamma}{\rho}\,\frac{\mu_g}{m_e}\,a n_e \sigma_T (T_\gamma - T)$

First law for general perfect fluid $d\rho = \mu\,dn + n\,T\,ds$ where μ is enthalpy per particle $\mu = \frac{\rho + p}{n}$.

To understand the dynamics of baryons, we switch off the interaction, i.e. $\sigma_T = 0$.

Baryon Equation of Motion

Perfect Fluid Action

$$S_m = -\int d^4x\,\sqrt{-g}\,[\rho(n,s) + J^\mu\,(\partial_\mu\phi + \vartheta\,\partial_\mu s)]$$

[BF. Schutz R. Sorkin, 1977]

J^μ, $g_{\mu\nu}$, ϕ, ϑ, s are fundamental variables, and $n = \sqrt{-J^\mu J^\nu g_{\mu\nu}}$.

Background eom

$\phi \Rightarrow \nabla_\mu J^\mu = 0\,,$

$\vartheta \Rightarrow u^\mu \partial_\mu s = 0\,,$

$s \Rightarrow u^\mu \partial_\mu \vartheta = T\,.$

Eom for J^μ will give the relation between the fluid 4-velocity and the fields ϕ, ϑ, s. On defining $J^\mu \equiv n u^\mu$ we get the constraint $u^\mu u_\mu = -1$. u^μ represents 4-velocity of the fluid.

Linear Perturbation

$$s = s_0 + \delta s Y\,,$$

$$\phi = -\int^\tau d\eta\, a(\eta)\,\overline{\rho}_{,n} + \delta\phi\, Y\,,$$

$$\vartheta = \int^\tau d\eta\, a^4\,\overline{\rho}_{,s}/N_0 + \delta\vartheta\, Y\,,$$

$$J^0 = \frac{N_0}{a^4}(1 + W_0\, Y)\,,$$

$$J^i = \frac{W}{a^2}\,\gamma^{ij}\, Y_{|j}\,,$$

$$ds^2 = -(1+2\alpha Y)\,a^2\, d\tau^2 + 2a\chi\, Y_{|i}\, d\tau\, dx^i + [a^2(1+2\zeta Y)\gamma_{ij} + 2E\, Y_{|ij}]\, dx^i dx^j\,.$$

N_0 is the total number of fluid particles. Y is determined by the property $\gamma^{ij} Y_{|ij} = -k^2 Y$.

[H. Kodama, M. Sasaki, 1992]

For FLRW background $\frac{\partial \delta s}{\partial \tau} = 0$, i.e. δs is gauge invariant. Assuming an adiabatic fluid, we choose the initial condition $\delta s(\vec{x}) = 0$.

Baryon Equation of Motion

We have the equations of motion

$$\dot{\theta}=-\frac{\dot{a}}{a}\,\theta+k^2\alpha+\frac{\rho}{\rho+p}\,c_s^2k^2\left(\delta+3\,\frac{\dot{a}}{a}\,\frac{\rho+p}{\rho}\,\frac{\theta}{k^2}\right),$$
$$\frac{\partial}{\partial\tau}\left(\frac{\rho}{\rho+p}\,\delta\right)=-\theta-3\dot{\zeta}+k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)-\frac{k^2}{a}\,\chi\,.$$

Where $c_s^2=(\partial p/\partial\rho)_s=\dot{p}/\dot{\rho}=n\rho_{,nn}/\rho_{,n}$.

We redefine the field variables.

$$\delta\phi=\rho_{,n}\,v-\vartheta(\tau)\,\delta s\,,$$
$$\delta\vartheta=\delta\vartheta_v-\frac{\rho_{,s}}{n}\,v\,,$$
$$W_0=\frac{\rho}{n\rho_{,n}}\,\delta-\alpha-\frac{\rho_{,s}}{n\rho_{,n}}\,\delta s\,,$$
$$v=-\frac{a}{k^2}\,\theta\,.$$

Non - Relativistic Limit

We assume $\frac{\rho_\gamma}{\rho}\,\frac{n_e\sigma_T}{H}\,\frac{T_\gamma-T}{m_e}\ll 1$.

For the first order we have $c_s^2\simeq T/\mu_g$.

eom for baryons.

$$\dot{\delta}_b=-\frac{6}{5}\,c_s^2\,\frac{\dot{a}}{a}\,\delta_b-\left(1+\frac{3}{5}\,c_s^2\right)\left[\theta_b+3\dot{\zeta}-k^2\,\frac{\partial}{\partial\tau}\left(\frac{E}{a^2}\right)+\frac{k^2}{a}\,\chi\right],$$
$$\dot{\theta}_b=-\frac{\dot{a}}{a}\,\theta_b+k^2\alpha+c_s^2\,k^2\left(\delta_b+3\,\frac{\dot{a}}{a}\,\frac{\theta_b}{k^2}\right)+R\,an_e\sigma_T(\theta_\gamma-\theta_b)\,.$$

Here we add the standard collision term with photons $\sigma_T\neq 0$.

Where $R\equiv\frac{4}{3}\,\rho_\gamma/\rho_b$.

Tight Coupling Approximation

- In the regime before the recombination the baryons and photons were coupled to form a stiff photon-baryon fluid. This system becomes numerically unstable.
- The interaction time scale of the photons and baryons is $\tau_c \equiv (a n_e \sigma_T)^{-1}$ where σ_T is the Thomson scattering amplitude.
- This time scale is much shorter than sub-Horizon and super-Horizon scales.
- The idea is to solve the system perturbatively in τ_c for the terms which are considerably small in the limit $\tau_c \to 0$.
- Since we have new covariant baryon equations of motion, we need to recalculate the Tight Coupling Approximation.

[D. Blas, J. Lesgourgues, T. Tram, 2011]

Tight Coupling Approximation

From the Photon Boltzmann hierarchy equation and baryon equation without choosing any gauge, we get

$$\tau_c\left[\dot{\Theta}_{\gamma b}-\mathcal{H}\,\theta_b+c_s^2\,k^2\left(\delta_b+3\,\mathcal{H}\,\frac{\theta_b}{k^2}\right)-\frac{k^2}{4}\,\delta_\gamma+k^2\sigma_\gamma\right]+(1+R)\,\Theta_{\gamma b}=0\,,$$

$$\Theta_{\gamma b}=\theta_\gamma-\theta_b\,,$$

$$\sigma_\gamma=\frac{\tau_c}{9}\left\{\frac{8}{3}\left[\theta_\gamma+\frac{k^2}{a}\,\chi-k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right]-3\,k\,F_{\gamma 3}-10\dot{\sigma}_\gamma\right\}+\frac{1}{18}\left(G_{\gamma 0}+G_{\gamma 2}\right),$$

$$\dot{\theta}_b=-\frac{1}{1+R}\left\{\mathcal{H}\,(1-3c_s^2)\,\theta_b-(1+R)\,k^2\alpha-Rk^2\left(\frac{1}{4}\,\delta_\gamma-\sigma_\gamma\right)-c_s^2\,k^2\delta_b+R\dot{\Theta}_{\gamma b}\right\}\,.$$

Where $\mathcal{H}\equiv\frac{\dot{a}}{a}$.

Our method for finding a perturbative solution is by assuming a solution order by order up to second order.

$$G_{\gamma 1}=G_{\gamma 1}^{(0)}+\tau_c G_{\gamma 1}^{(1)}+\tau_c^2 G_{\gamma 1}^{(2)}+\dots\,,$$
$$F_{\gamma 1}=F_{\gamma 1}^{(0)}+\tau_c F_{\gamma 1}^{(1)}+\tau_c^2 F_{\gamma 1}^{(2)}+\dots\,,$$
$$\sigma_\gamma=\tau_c\sigma_\gamma^{(1)}+\tau_c^2\sigma_\gamma^{(2)}+\dots\,,$$
$$\dot{\Theta}_{\gamma b}=\dot{\Theta}_{\gamma b}^{(1)}+\dot{\Theta}_{\gamma b}^{(2)}+\dots\,.$$

[D. Blas, J. Lesgourgues, T. Tram, 2011]

Tight Coupling Approximation

Slip equation

$$\dot{\Theta}_{\gamma b} = \left(\frac{\dot{\tau}_c}{\tau_c} - \frac{2\mathcal{H}}{R+1}\right)\Theta_{\gamma b} + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{O}(\tau_c^3)$$

$$\mathcal{T}_1 \equiv \frac{\tau_c}{R+1}\left[\left(\mathcal{H}^2 + \dot{\mathcal{H}}\right)\theta_b + \left(\frac{\dot{\delta}_\gamma}{4} - c_s^2\dot{\delta}_b + \mathcal{H}\alpha + \frac{\mathcal{H}\delta_\gamma}{2} - \bar{c}_s^2\delta_b\right)k^2\right]$$
$$- \frac{3\tau_c}{R+1}\left[\frac{\left\{3\mathcal{H}^2 c_s^4 + [\dot{\mathcal{H}}(R+1) - \mathcal{H}^2]c_s^2 + \mathcal{H}\bar{c}_s^2(R+1)\right\}\theta_b}{R+1} + \mathcal{H}c_s^2 k^2\left(\alpha + \frac{1}{4}\frac{R\delta_\gamma}{R+1} + \frac{c_s^2\delta_b}{R+1}\right)\right]$$

We also have the assumption $\mathcal{H}\tau_c \ll 1$ and $\Theta_{\gamma b} \ll 1$.

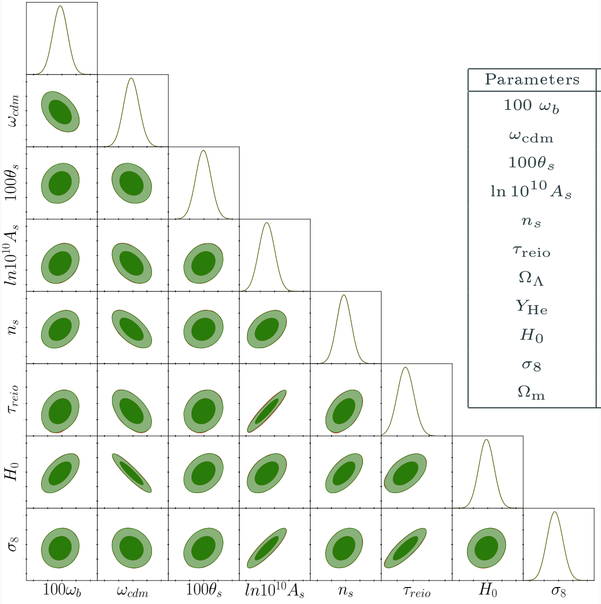
Like this we find up to second order.

Code Implementation and Results

- We choose CLASS Boltzmann solver to implement the correct baryon equations of motion and tight coupling approximation.
- We implement and perform 4 approximation schemes in the CLASS code, namely `first_order_MB` $\tau_c \propto a^2$, $c_s^2 \propto a^{-1}$, `first_order_CAMB` $c_s^2 \propto a^{-1}$, `first_order_CLASS`, `second_order_CLASS`.
- 3072 chains, each chain using 4 cores in parallel. For each chain we have set 13000 steps.
- We found no numerical inconsistency and the code does not become slower or stiffer.
- The acceptance rate for MCMC analysis is increased by about 0.95%.
- The likelihood of the covariantly corrected code get slightly improved to $-\ln \mathcal{L}_{\min} = 5984.11$, the old covariance-breaking code has $-\ln \mathcal{L}_{\min} = 5984.45$.

[D. Blas, J. Lesgourgues, T. Tram, 2011] [T. Brinckmann, J. Lesgourgues, 2018]

Code Implementation and Results



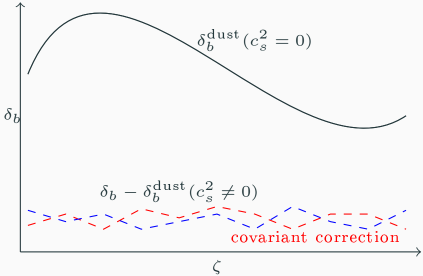
Parameters	Best fit: new (old)
$100\ \omega_b$	$2.236^{+0.041}_{-0.038}\left(2.238^{+0.04}_{-0.04}\right)$
$\omega_{\rm cdm}$	$0.1176^{+0.0021}_{-0.0025}\left(0.117^{+0.0027}_{-0.0019}\right)$
$100\theta_s$	$1.042^{+0.001}_{-0.001}\left(1.042^{+0.001}_{-0.001}\right)$
$\ln 10^{10} A_s$	$3.086^{+0.046}_{-0.051}\left(3.085^{+0.047}_{-0.05}\right)$
n_s	$0.969^{+0.0102}_{-0.0074}\left(0.9726^{+0.0067}_{-0.0111}\right)$
$\tau_{\rm reio}$	$0.07879^{+0.02461}_{-0.02672}\left(0.08009^{+0.02301}_{-0.02841}\right)$
Ω_Λ	$0.6989^{+0.0144}_{-0.0124}\left(0.7022^{+0.0111}_{-0.0157}\right)$
$Y_{\rm He}$	$0.2478^{+0.0002}_{-0.0001}\left(0.2478^{+0.0002}_{-0.0001}\right)$
H_0	$68.35^{+1.13}_{-0.98}\left(68.58^{+0.92}_{-1.19}\right)$
σ_8	$0.8209^{+0.0174}_{-0.0198}\left(0.8194^{+0.0188}_{-0.0184}\right)$
$\Omega_{\rm m}$	$0.301^{+0.0124}_{-0.0144}\left(0.2977^{+0.0158}_{-0.0111}\right)$

Conclusion

- The covariant action introduced fixes all the three issues. There are additional terms in the equations of motion at the order of c_s^2 which makes the baryon equations of motion gauge compatible and obeying general covariance.

- In the previous code there where approximations
$k \gg aH,\ k^2\delta_b \gg aH\theta_b,\ \delta_b \gg \eta(\text{any scale})$, during raidation domination $(z \geq 10^6)$
$k \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)h\text{Mpc}^{-1},\ k^2\delta_b \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)h\text{Mpc}^{-1}\theta_b$.
This approximation generally fails for large redshifts considering $10^{-2}\text{Mpc}^{-1} \leq k \leq 10\text{Mpc}^{-1}$.

- This covariantly corrected implementation does not make huge difference in the parameters but will be important in the future.



BACKUP - 01: Photon Boltzmann Hierarchy

Photon Boltzmann hierarchy equations

$$
\begin{aligned}
\dot{\delta}_\gamma &= -\frac{4}{3}\,\theta_\gamma - \frac{4}{3}\,\frac{k^2}{a}\,\chi + \frac{4}{3}\,\partial_\tau\left(\frac{k^2 E}{a^2}\right) - 4\dot{\zeta} \\
\dot{\theta}_\gamma &= \frac{k^2}{4}\,\delta_\gamma - k^2\sigma_\gamma + k^2\alpha - \frac{1}{\tau_c}\,(\theta_\gamma - \theta_b) \\
2\dot{\sigma}_\gamma &= \frac{8}{15}\left[\theta_\gamma + \frac{k^2}{a}\,\chi - k^2\,\partial_\tau\left(\frac{E}{a^2}\right)\right] - \frac{3}{5}\,k\,F_{\gamma 3} \\
&\quad -\frac{9}{5\tau_c}\,\sigma_\gamma + \frac{1}{10\tau_c}\,(G_{\gamma 0} + G_{\gamma 2}) \\
\dot{F}_{\gamma l} &= \frac{k}{2l+1}\left[lF_{\gamma(l-1)} - (l+1)\,F_{\gamma(l+1)}\right] - \frac{1}{\tau_c}F_{\gamma l}\,, \qquad l \geq 3 \\
\dot{G}_{\gamma l} &= \frac{k}{2l+1}\left[lG_{\gamma(l-1)} - (l+1)\,G_{\gamma(l+1)}\right] \\
&\quad + \frac{1}{\tau_c}\left[-G_{\gamma l} + \frac{1}{2}\,(F_{\gamma 2} + G_{\gamma 0} + G_{\gamma 2})\left(\delta_{l0} + \frac{\delta_{l2}}{5}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\tau_c\left[\dot{\theta}_\gamma - \tfrac{k^2}{4}\,\delta_\gamma + k^2\sigma_\gamma - k^2\alpha\right] + \Theta_{\gamma b} = 0 \\
&\tau_c\left[-\dot{\theta}_b - \tfrac{\dot{a}}{a}\,\theta_b + k^2\alpha + c_s^2\,k^2\left(\delta_b + 3\,\tfrac{\dot{a}}{a}\,\tfrac{\theta_b}{k^2}\right)\right] + R\,\Theta_{\gamma b} = 0
\end{aligned}
$$

BACKUP - 02: Equations in Synchronous Gauge

Synchronous gauge field redefinition

$$\zeta = -\eta$$
$$k^2\left(\frac{E}{a^2}\right) = -\frac{1}{2}(h+6\eta)$$

Synchronous gauge Einstein Equations

$$k^2\eta - \frac{1}{2}\frac{\dot{a}}{a}\dot{h} = -4\pi G a^2 \delta\rho$$
$$k^2\dot{\eta} = 4\pi G a^2(\bar{\rho}+\bar{P})\theta$$
$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - 2k^2\eta = -8\pi G a^2 \delta T^i_i$$
$$\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h}+6\dot{\eta}) - 2k^2\eta = -24 G a^2(\bar{\rho}+\bar{P})\sigma$$

Baryon equation in Synchronous gauge

$$\dot{\delta} = -\theta - \frac{1}{2}\dot{h} - \frac{6}{5}c_s^2\frac{\dot{a}}{a}\delta_b - \frac{3}{5}c_s^2\left(\theta_b + \frac{1}{2}\dot{h}\right)$$
$$\dot{\theta}_b = -\frac{\dot{a}}{a}\theta_b + c_s^2 k^2\delta_b + 3c_s^2\frac{\dot{a}}{a}\theta_b + R a n_e \sigma_T(\theta_\gamma - \theta_b)$$

BACKUP - 03: Baryon equation missing terms

We can write Baryon equations as
$\ddot{\delta}_b + \mathcal{H}\dot{\delta}_b + c_s^2 k^2 \delta_b + \ldots = 4\pi G a^2(\delta\rho + 3\delta P)$
dots includes terms of the order of $c_s^2 k^2 \eta$, $aHc_s^2\theta_b$, $a^2H^2c_s^2\delta_b$

Missing terms correspond the approximations
$k \gg aH$, $k^2\delta_b \gg aH\theta_b$, $\delta_b \gg \eta$
During the Radiation domination $H = H_0\sqrt{\Omega_{r0}}(1+z)^2$, where $H_0 = h\text{Mpc}^{-1}/(2997.9)$.
i.e. $k \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)h\text{Mpc}^{-1}$, and
$k^2\delta_b \gg (1+z)\sqrt{\Omega_{r0}}/(3\times 10^3)h\text{Mpc}^{-1}\theta_b$.

BACKUP - 04: Baryon equation

Temperature evolution can be written as

$$\frac{1}{aH}\frac{d}{d\tau}\left(\frac{T}{\mu_g}\right) = -2\,\frac{T}{\mu_g} + \frac{8}{3}\,\frac{\rho_\gamma}{\rho}\,\frac{n_e\sigma_T}{H}\,\frac{T_\gamma - T}{m_e}\,,$$

from equation of state we have

$$p = \frac{\rho}{\mu_g + 3T/2}$$

$$c_s^2 = \frac{\dot{p}}{\dot{\rho}} = \frac{n\rho_{,nn}}{\rho_{,n}} = \frac{10\,T}{3\,(5T + 2\mu_g)} \approx \frac{5}{3}\,\frac{T}{\mu_g}\,.$$

$$\delta p = \left(\frac{\partial p}{\partial \rho}\right)_s \delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \delta s,$$

which can be rewritten as $\delta p = c_s^2\,\delta\rho + \frac{4\mu_g p}{(6\mu_g + 15T)}\,\delta s$