Angular stability of a static and spherically symmetric solution in the Horndeski theory

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To distinguish GR from other theories

- Theoretical understanding of fundamental properties of modified gravity
  - Propagation of gravitational waves
  - Cosmological solution
  - Vicinity of BH as a strong field

- Perturbative behavior in general model
  - Static and spherically symmetric spacetime
  - Modified gravity: Horndeski theory (4 functional DOF)

Stability condition of perts.

- Two fundamental condition
  1. No ghost
     - Ghost: Negative kinetic energy
  2. Real propagation speed
     - To avoid gradient instability

- Perturbations in Horndeski
  - Odd-parity sector: T. Kobayashi, H. Motohashi, T. Suyama (2012)
    - Radial and angular propagation speed
  - Even-parity sector: T. Kobayashi, H. Motohashi, T. Suyama (2014)
    - Radial propagation speed

Horndeski theory

Horndeski 1974, Deffayet+ 2011, Kobayashi+ 2011

- The most general single scalar field theory with gravity having second-order EoMs.

→ We have $\phi, \dot{\phi}, \ddot{\phi}, g_{\mu\nu}, \dot{g}_{\mu\nu}, \ddot{g}_{\mu\nu}$
but don't have $\dddot{\phi}, \dddot{g}_{\mu\nu}$.

$\dddot{\phi}, \dddot{g}_{\mu\nu}$ always induce ghost.
[Ostrogradsky theorem]

Action

$$S = \int \mathrm{d}^4x\sqrt{-g}\sum_{i=2}^{5}\tilde{\mathcal{L}}_i,$$

$$\tilde{\mathcal{L}}_2 = G_2(\phi, X),$$
$$\tilde{\mathcal{L}}_3 = -G_3(\phi, X)\Box\phi,$$
$$\tilde{\mathcal{L}}_4 = G_4(\phi, X)\mathcal{R} + G_{4X}[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2],$$
$$\tilde{\mathcal{L}}_5 = G_5(\phi, X)\mathcal{G}_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5X}[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3]$$

$$\boxed{G_{5X} \equiv \frac{\partial G_5}{\partial X}}$$

$X \equiv -(\partial\phi)^2/2$: a canonical kinetic term

$G_2(\phi, X), G_3(\phi, X), G_4(\phi, X), G_5(\phi, X)$: arbitrary functions

It includes Brans-Dicke theory, f(R) theory, etc… as a specific model.

Background

- Static and spherically symmetric BG.

Unperturbed metric : $\bar{g}_{\mu\nu}$ BG scalar field: $\phi = \phi(r)$

$$\boxed{\bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -A(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{B(r)} + C(r)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)}$$

C(r) is redundant but helps to get the angular component of gravitational field eq.

- BG field equations ← Variation w.r.t. A, B, C and ϕ.

$$\mathcal{E}_A = 0, \quad \mathcal{E}_B = 0, \quad \mathcal{E}_C = 0, \quad \mathcal{E}_\phi = 0,$$

- Now we set $C(r) = 1$ without loss of generality.

Odd-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2012)

- Odd-type metric perturbations

Metric perturbations: $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$

$$h_{tt} = 0, \quad h_{tr} = 0, \quad h_{rr} = 0,$$

$$h_{ta} = \sum_{\ell,m} h_{0,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta,\varphi),$$

$$h_{ra} = \sum_{\ell,m} h_{1,\ell m}(t,r) E_{ab} \partial^b Y_{\ell m}(\theta,\varphi),$$

$$h_{ab} = \frac{1}{2} \sum_{\ell,m} h_{2,\ell m}(t,r) \left[E_a{}^c \nabla_c \nabla_b Y_{\ell m}(\theta,\varphi) + E_b{}^c \nabla_c \nabla_a Y_{\ell m}(\theta,\varphi) \right],$$

Antisymmetric tensor: $E_{ab} := \sqrt{\det \gamma}\, \epsilon_{ab}$ 2-dimensional metric: γ_{ab}

Antisymmetric symbol: $\epsilon_{\theta\varphi} = 1$

Odd-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2012)

- Second-order action

$$S^{(2)} = \int \mathrm{d}t\, \mathrm{d}r\, \mathcal{L}^{(2)}$$

← We omitted ℓ and m.

$$\frac{2\ell+1}{2\pi}\mathcal{L}^{(2)} = a_1 h_0^2 + a_2 h_1^2 + a_3\left(\dot{h}_1^2 - 2\dot{h}_1 h_0' + h_0'^2 + \frac{4}{r}\dot{h}_1 h_0\right)$$

$$\begin{aligned}
a_1 &= \frac{\ell(\ell+1)}{r^2}\left[\frac{d}{dr}\left(r\sqrt{\frac{B}{A}}\mathcal{H}\right) + \frac{\ell^2+\ell-2}{2\sqrt{AB}}\mathcal{F} + \frac{r^2}{\sqrt{AB}}\mathcal{E}_A\right], \\
a_2 &= -\ell(\ell+1)\sqrt{AB}\left[\frac{(\ell-1)(\ell+2)}{2r^2}\mathcal{G} + \mathcal{E}_B\right], \\
a_3 &= \frac{\ell(\ell+1)}{2}\sqrt{\frac{B}{A}}\mathcal{H},
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathcal{F} &:= 2\left(G_4 + \frac{1}{2}B\phi' X' G_{5X} - X G_{5\phi}\right), \\
\mathcal{G} &:= 2\left[G_4 - 2XG_{4X} + X\left(\frac{A'}{2A}B\phi' G_{5X} + G_{5\phi}\right)\right], \\
\mathcal{H} &:= 2\left[G_4 - 2XG_{4X} + X\left(\frac{B\phi'}{r}G_{5X} + G_{5\phi}\right)\right],
\end{aligned}}$$

Odd-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2012)

- After redefinition of variable, we get EOM for Q(t,r)

$$\frac{\mathcal{F}}{AB\mathcal{G}}\ddot{Q} - Q'' + \frac{\ell(\ell+1)\mathcal{F}}{r^2 B\mathcal{H}}Q + V(r)Q = 0$$

- Propagation speeds $c_r^2 = \frac{\mathcal{G}}{\mathcal{F}}, \quad c_\theta^2 = \frac{\mathcal{G}}{\mathcal{H}}$

Squared propagation speeds must be positive: $c_r^2 > 0$, $c_\theta^2 > 0$.

- No-ghost condition $\mathcal{G} > 0$

$$\therefore \quad \mathcal{F} > 0,\ \mathcal{G} > 0,\ \text{and}\ \mathcal{H} > 0$$

Even-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2014)

$$
\begin{aligned}
h_{tt} &= A(r)\sum_{\ell,m} H_{0,\ell m}(t,r)Y_{\ell m}(\theta,\varphi),\\
h_{tr} &= \sum_{\ell,m} H_{1,\ell m}(t,r)Y_{\ell m}(\theta,\varphi),\\
h_{rr} &= \frac{1}{B(r)}\sum_{\ell,m} H_{2,\ell m}(t,r)Y_{\ell m}(\theta,\varphi),\\
h_{ta} &= \sum_{\ell,m} \beta_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi),\\
h_{ra} &= \sum_{\ell,m} \alpha_{\ell m}(t,r)\partial_a Y_{\ell m}(\theta,\varphi),\\
h_{ab} &= \sum_{\ell,m} K_{\ell m}(t,r)g_{ab}Y_{\ell m}(\theta,\varphi) + \sum_{\ell,m} G_{\ell m}(t,r)\nabla_a\nabla_b Y_{\ell m}(\theta,\varphi)\,.
\end{aligned}
$$

$$
\phi(t,r,\theta,\varphi) = \phi(r) + \sum_{\ell,m}\delta\phi_{\ell m}(t,r)Y_{\ell m}(\theta,\varphi)
$$

Complete gauge fixing

$$
\boxed{G_{\ell m} = K_{\ell m} = \beta_{\ell m} = 0}
$$

8 – 3 = 5 vars.

Even-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2014)

Second-order action:

$$
\begin{aligned}
\frac{2\ell+1}{2\pi}\mathcal{L} \;=\;& H_0\left[a_1\delta\phi'' + a_2\delta\phi' + a_3 H_2' + j^2 a_4\alpha' + \left(a_5 + j^2 a_6\right)\delta\phi + \left(a_7 + j^2 a_8\right)H_2 + j^2 a_9\alpha\right] \\
& + j^2 b_1 H_1^2 + H_1\left(b_2\dot{\delta\phi}' + b_3\dot{\delta\phi} + b_4\dot{H}_2 + j^2 b_5\dot{\alpha}\right) \\
& + c_1\dot{H}_2\dot{\delta\phi} + H_2\left[c_2\delta\phi' + \left(c_3 + j^2 c_4\right)\delta\phi + j^2 c_5\alpha\right] + c_6 H_2^2 + j^2 d_1\dot{\alpha}^2 + j^2\alpha\left(d_2\delta\phi' + d_3\delta\phi\right) + j^2 d_4\alpha^2 \\
& + e_1\dot{\delta\phi}^2 + e_2\delta\phi'^2 + \left(e_3 + j^2 e_4\right)\delta\phi^2,
\end{aligned}
$$

28 Coefficients:

$$
\boxed{\begin{array}{c} a_1 \sim a_9 \\ b_1 \sim b_5 \\ c_1 \sim c_6 \\ d_1 \sim d_4 \\ e_1 \sim e_4 \end{array}}
$$

$$
\begin{aligned}
a_1 &= \sqrt{AB}\,\Xi, \\
a_2 &= \frac{\sqrt{AB}}{2\phi'}\left[2\phi'\Xi' - \left(2\phi'' - \frac{A'}{A}\phi'\right)\Xi + 2r\left(\frac{A'}{A} - \frac{B'}{B}\right)\mathcal{H} + \frac{2r^2}{B}\left(\mathcal{E}_B - \mathcal{E}_A\right)\right], \\
a_3 &= -\frac{\sqrt{AB}}{2}\left(\phi'\Xi + 2r\mathcal{H}\right), \\
a_4 &= \sqrt{AB}\,\mathcal{H}, \\
a_5 &= -\sqrt{\frac{A}{B}}r^2\frac{\partial\mathcal{E}_A}{\partial\phi} = a_2' - a_1'', \\
a_6 &= -\sqrt{\frac{A}{B}}\frac{1}{r\phi'}\left(r\mathcal{H}' + \mathcal{H} - \mathcal{F}\right), \\
a_7 &= a_3' + \frac{r^2}{2}\sqrt{\frac{A}{B}}\mathcal{E}_B, \\
a_8 &= -\frac{a_4}{2B}, \\
a_9 &= \frac{\sqrt{A}}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\sqrt{B}\mathcal{H}\right) = a_4' + \left(\frac{1}{r} - \frac{A'}{2A}\right)a_4, \\
b_1 &= \frac{1}{2}\sqrt{\frac{B}{A}}\mathcal{H}, \\
b_2 &= -2\sqrt{\frac{B}{A}}\Xi, \\
b_3 &= \sqrt{\frac{B}{A}}\frac{1}{\phi'}\left[\left(2\phi'' + \frac{B'}{B}\phi'\right)\Xi - 2r\left(\frac{A'}{A} - \frac{B'}{B}\right)\mathcal{H} + \frac{2r^2}{B}\mathcal{E}_A\right] = \frac{2}{A}\left(a_1' - a_2\right) + \frac{2r^2}{\sqrt{AB}\phi'}\mathcal{E}_B, \\
b_4 &= \sqrt{\frac{B}{A}}\left(\phi'\Xi + 2r\mathcal{H}\right), \\
b_5 &= -2b_1, \\
c_1 &= -\frac{1}{\sqrt{AB}}\Xi, \\
c_2 &= -\sqrt{AB}\left(\frac{A'}{2A}\Xi + r\Gamma - \frac{r^2\phi'}{X}\Sigma\right), \\
c_3 &= r^2\sqrt{\frac{A}{B}}\frac{\partial\mathcal{E}_B}{\partial\phi}, \\
c_4 &= \frac{1}{2}\sqrt{\frac{A}{B}}\Gamma, \\
c_5 &= -\frac{1}{2}\sqrt{AB}\left(\phi'\Gamma + \frac{A'}{A}\mathcal{H} + \frac{2}{r}\mathcal{G}\right), \\
c_6 &= \frac{r^2}{2}\sqrt{\frac{A}{B}}\left(\Sigma + \frac{A'B\phi'}{2r^2A}\Xi + \frac{B\phi'}{r}\Gamma - \frac{1}{2}\mathcal{E}_B + \frac{B}{r^2}\mathcal{G} + \frac{A'B}{rA}\mathcal{H}\right),
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= b_1, \\
d_2 &= \sqrt{AB}\,\Gamma, \\
d_3 &= \frac{\sqrt{AB}}{r^2}\left[\frac{2r}{\phi'}\left(\frac{A'}{A} - \frac{B'}{B}\right)\mathcal{H} - r^2\left(\frac{2}{r} - \frac{A'}{A}\right)\frac{\partial\mathcal{H}}{\partial\phi} + \frac{2}{B\phi'}\left(\mathcal{F} - \mathcal{G}\right)\right. \\
&\qquad \left. - \frac{r^2}{2\phi'}\left(2\phi'' + \frac{B'}{B}\phi'\right)\left(\Gamma_1 + \frac{2}{r}\Gamma_2\right) - \frac{2r^2}{B\phi'}\left(\mathcal{E}_A - \mathcal{E}_B\right)\right], \\
d_4 &= \frac{\sqrt{AB}}{r^2}\left(\mathcal{G} - r^2\mathcal{E}_B\right), \\
e_1 &= \frac{1}{2\sqrt{AB}}\left[\frac{r^2}{X}\left(\mathcal{E}_A - \mathcal{E}_B\right) - \frac{2}{\phi'}\Xi' + \left(\frac{A'}{A} - \frac{X'}{X}\right)\frac{\Xi}{\phi'} + \frac{2B}{X}\mathcal{F} - \frac{2rB}{X}\mathcal{H}' - \mathcal{H}\frac{B^2}{rXA}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2A}{B}\right)\right], \\
&= \frac{1}{AB\phi'}\left[\left(\frac{A'}{A} + \frac{B'}{2B}\right)a_1 + a_2 - 2a_1' - 2rBa_6\right], \\
e_2 &= -\sqrt{AB}\frac{r^2}{X}\Sigma, \\
e_3 &= r^2\sqrt{\frac{A}{B}}\frac{\partial\mathcal{E}_\phi}{\partial\phi}, \\
e_4 &= -\frac{1}{4}\sqrt{\frac{A}{B}}\tilde{e}_4, \\
\tilde{e}_4 &= \frac{2}{X}\left(\mathcal{E}_A - \mathcal{E}_B\right) - \frac{2}{\phi'}\Gamma' - \frac{2}{r^2X}\left(1 - rB\frac{A'}{A}\right)\mathcal{F} + \frac{2B}{r^2X}\mathcal{G} + \frac{2}{r^2X}\left(-2rB\frac{A'}{A} + 1 - B + rB'\right)\mathcal{H} \\
&\quad - \frac{2B}{rX}\mathcal{G}' - \frac{B}{X}\frac{A'}{A}\mathcal{H}' - \frac{2}{r\phi'}\left(2 - r\frac{A'}{A}\right)\frac{\partial\mathcal{H}}{\partial\phi} + \frac{1}{r^3\phi'}\left(2 - r\frac{A'}{A}\right)\left[-2(1-B) + rB\frac{A'}{A}\right]\Xi \\
&\quad + \left[-2\frac{A'}{A}\phi' + \frac{rB}{2}\left(\frac{A'}{A}\right)^2\phi' - \frac{B'}{B}\phi' + \frac{2}{r}(1-B)\phi' - 2\phi''\right]\frac{\Gamma_1}{\phi'^2} \\
&\quad + \left[-2\frac{A'}{A}\phi' - r(1-B)\left(\frac{A'}{A}\right)^2\phi' - 2\frac{B'}{B}\phi' + \frac{4}{r}(1-B)\phi' - 4\phi''\right]\frac{\Gamma_2}{r\phi'^2}.
\end{aligned}
$$

Even-parity perturbations

T. Kobayashi, H. Motohashi, T. Suyama (2014)

a new variable ψ: $H_2 = \frac{1}{a_3}\left(\psi - a_1\delta\phi' - j^2 a_4\alpha\right)$

Substituting EOMs for H_0 and H_1, we get

$$\frac{2\ell+1}{2\pi}\mathcal{L} = \frac{1}{2}\mathcal{K}_{ij}\dot{v}^i\dot{v}^j - \frac{1}{2}\mathcal{G}_{ij}v^{i\prime}v^{j\prime} - Q_{ij}v^i v^{j\prime} - \frac{1}{2}\mathcal{M}_{ij}v^i v^j$$

where i and j run from 1 to 2, and $v^1 := \psi,\ v^2 := \delta\phi.$

Propagation speeds along the *radial* direction are derived from the eigenvalues of the matrix $(\mathrm{AB})^{-1}\mathcal{K}^{-1}\mathcal{G}$

$$c_{r1}^2 = \frac{\mathcal{G}}{\mathcal{F}}\ ,\quad c_{r2}^2 = \frac{\mathcal{G}_s}{\mathcal{F}_s}$$

No-ghost condition: $2\mathcal{P}_1 - \mathcal{F} > 0$

Go further…

	Radial propagation	Axial propagation	No-ghost condition
Odd-parity	Checked	Checked	Checked
Even-parity	Checked	Not yet	Checked

To get the angular condition, we need take into account Q_{ij} and M_{ij}

$$\frac{2\ell+1}{2\pi}\mathcal{L}=\frac{1}{2}\mathcal{K}_{ij}\dot{v}^i\dot{v}^j-\frac{1}{2}\mathcal{G}_{ij}v^{i\prime}v^{j\prime}-\underline{Q_{ij}}v^iv^{j\prime}-\frac{1}{2}\underline{\mathcal{M}_{ij}}v^iv^j$$

To get speed of free propagation, we take high ℓ limit -> justifies ignoring Q_{ij}.

$$\frac{2\ell+1}{2\pi}\mathcal{L}=\frac{1}{2}\mathcal{K}_{ij}\dot{v}^i\dot{v}^j-\frac{1}{2}\mathcal{G}_{ij}v^{i\prime}v^{j\prime}-\frac{1}{2}\mathcal{M}_{ij}v^iv^j$$

Normal mode of coupled oscillator

$$\frac{2\ell+1}{2\pi}\mathcal{L} = \frac{1}{2}\mathcal{K}_{ij}\dot{v}^i\dot{v}^j - \frac{1}{2}\mathcal{G}_{ij}v^{i\prime}v^{j\prime} - \frac{1}{2}\mathcal{M}_{ij}v^i v^j$$

$$\rightarrow \left(-\mathcal{K}_{ij}\partial_t^2 + \mathcal{G}_{ij}\partial_r^2 - \mathcal{M}_{ij}\right)v^j = 0$$

EOM in matrix representation
We ignore r dependence of matrix

To get nontrivial solution $v^j \neq 0$, the matrix must not be invertible.

$$\rightarrow \mathrm{Det}\left[-\mathcal{K}_{ij}\partial_t^2 + \mathcal{G}_{ij}\partial_r^2 - \mathcal{M}_{ij}\right] = 0$$

or

$$\mathrm{Det}\left[\mathcal{K}_{ij}\omega^2 - \mathcal{G}_{ij}k_r^2 - \mathcal{M}_{ij}\right] = 0$$

To get simple representation

$$\mathrm{Det}\left[\mathcal{K}_{ij}\omega^2 - \mathcal{G}_{ij}k_r^2 - \mathcal{M}_{ij}\right] = 0$$

- To simplify the eqn, we look for simpler expression for the coefficients d_3 and $\tilde{e}_4$.

$$\begin{aligned} d_3 = \frac{\sqrt{AB}}{r^2}\Bigg[& \frac{2r}{\phi'}\left(\frac{A'}{A} - \frac{B'}{B}\right)\mathcal{H} - r^2\left(\frac{2}{r} - \frac{A'}{A}\right)\frac{\partial\mathcal{H}}{\partial\phi} + \frac{2}{B\phi'}(\mathcal{F} - \mathcal{G}) \\ & -\frac{r^2}{2\phi'}\left(2\phi'' + \frac{B'}{B}\phi'\right)\left(\Gamma_1 + \frac{2}{r}\Gamma_2\right) - \frac{2r^2}{B\phi'}(\mathcal{E}_A - \mathcal{E}_B)\Bigg], \end{aligned}$$

$$\begin{aligned} \tilde{e}_4 = & \frac{2}{X}(\mathcal{E}_A - \mathcal{E}_B) - \frac{2}{\phi'}\Gamma' - \frac{2}{r^2 X}\left(1 - rB\frac{A'}{A}\right)\mathcal{F} + \frac{2B}{r^2X}\mathcal{G} + \frac{2}{r^2X}\left(-2rB\frac{A'}{A} + 1 - B + rB'\right)\mathcal{H} \\ & -\frac{2B}{rX}\mathcal{G}' - \frac{B}{X}\frac{A'}{A}\mathcal{H}' - \frac{2}{r\phi'}\left(2 - r\frac{A'}{A}\right)\frac{\partial\mathcal{H}}{\partial\phi} + \frac{1}{r^3\phi'}\left(2 - r\frac{A'}{A}\right)\left[-2(1-B) + rB\frac{A'}{A}\right]\Xi \\ & + \left[-2\frac{A'}{A}\phi' + \frac{rB}{2}\left(\frac{A'}{A}\right)^2\phi' - \frac{B'}{B}\phi' + \frac{2}{r}(1-B)\phi' - 2\phi''\right]\frac{\Gamma_1}{\phi'^2} \\ & + \left[-2\frac{A'}{A}\phi' - r(1-B)\left(\frac{A'}{A}\right)^2\phi' - 2\frac{B'}{B}\phi' + \frac{4}{r}(1-B)\phi' - 4\phi''\right]\frac{\Gamma_2}{r\phi'^2}. \end{aligned}$$

$\downarrow$ Eliminate $\partial\mathcal{H}/\partial\phi$ $\downarrow$

$$\begin{aligned} d_3 = \frac{\sqrt{AB}}{r^2}\Bigg[& \frac{2}{B\phi'}(\mathcal{F} - \mathcal{G}) + \frac{2\mathcal{G}}{\phi'} - \frac{2\mathcal{H}}{\phi'}(1 - \frac{A'}{A}) - \frac{rB'}{B\phi'}(\mathcal{G} + \mathcal{H}) \\ & -\frac{r\mathcal{H}'}{\phi'}(2 - \frac{rA'}{A}) - \frac{r^2\Gamma}{2}(\frac{B'}{2B} + \frac{\phi''}{\phi'})\Bigg] \end{aligned}$$

$$\begin{aligned} \tilde{e}_4 = & \frac{2}{X}(\mathcal{E}_A - \mathcal{E}_B) + \frac{2}{r^2X}(\mathcal{G} - \mathcal{F}) + \frac{2B}{r^2X}(\mathcal{H} - \mathcal{G}) + \frac{BA'}{rXA}(2\mathcal{F} - \mathcal{G} - 3\mathcal{H}) \\ & + \frac{B'}{rX}(\mathcal{G} + \mathcal{H}) + \frac{2B}{rX}(\mathcal{H}' - \mathcal{G}') - \frac{2BA'\mathcal{H}'}{XA} - 2\frac{(\sqrt{AB}\Gamma\phi')'}{\sqrt{AB}\phi'^2} \end{aligned}$$

Equation to solve

$$\mathrm{Det}\left[\mathcal{K}_{ij}\omega^2 - \mathcal{G}_{ij}k_r^2 - \mathcal{M}_{ij}\right] = 0$$

$$\boxed{\begin{array}{c}\Delta Y_{\ell m} = -\ell(\ell+1)Y_{\ell m} \\ j^2 \equiv \ell(\ell+1)\end{array}}$$

$$\downarrow$$

$$\mathrm{Det}\left[\begin{pmatrix} 2(r\mathcal{H}+\Xi\phi') & 2r\mathcal{H}+\Xi\phi' \\ 2r\mathcal{H}+\Xi\phi' & 0 \end{pmatrix}\frac{\mathcal{F}}{r\mathcal{H}^2}\frac{\omega^2}{A} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{\mathcal{P}_1(2r\mathcal{H}+\Xi\phi')^2}{r^2\mathcal{H}^3}\frac{\omega^2}{A} - \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_2 \\ \mathcal{R}_2 & \mathcal{R}_3 \end{pmatrix}\frac{j^2}{r^2} - \begin{pmatrix} \mathcal{C}_1 & \mathcal{C}_2 \\ \mathcal{C}_2 & \mathcal{C}_3 \end{pmatrix}Bk_r^2\right] = 0$$

$$\mathcal{R}_1 \equiv \frac{r^4\mathcal{H}}{2\sqrt{AB}}\frac{d}{dr}\left[\frac{\sqrt{B}(A'\mathcal{H}+A\Gamma\phi')}{r^2\sqrt{A}\mathcal{H}^2}\right] + \frac{r\mathcal{F}(A'\mathcal{H}+A\Gamma\phi')}{A\mathcal{H}^2} + \left(\frac{1}{B} - 1 - \frac{rA'}{2A} + \frac{rB'}{2B} - \frac{r\mathcal{G}'}{\mathcal{G}}\right)\frac{\mathcal{G}}{\mathcal{H}}$$

$$\mathcal{R}_2 \equiv \frac{r^4\mathcal{H}}{2\sqrt{AB}}\frac{d}{dr}\left[\frac{\sqrt{B}(A'\mathcal{H}+A\Gamma\phi')}{r^2\sqrt{A}\mathcal{H}^2}\right] + \frac{r\mathcal{F}(A'\mathcal{H}+A\Gamma\phi')}{2A\mathcal{H}^2} + \left(\frac{1}{B} - 2 + \frac{\mathcal{F}}{\mathcal{H}} + \frac{rB'}{2B} - \frac{r\mathcal{H}'}{\mathcal{H}}\right)\frac{\mathcal{G}}{\mathcal{H}}$$

$$\mathcal{R}_3 \equiv \frac{r^4\mathcal{H}}{2\sqrt{AB}}\frac{d}{dr}\left[\frac{\sqrt{B}(A'\mathcal{H}+A\Gamma\phi')}{r^2\sqrt{A}\mathcal{H}^2}\right] + \left(\frac{1}{B} - 3 + \frac{rA'}{2A} + \frac{rB'}{2B} + \frac{r\mathcal{G}'}{\mathcal{G}} - 2\frac{r\mathcal{H}'}{\mathcal{H}}\right)\frac{\mathcal{G}}{\mathcal{H}}$$

$$\mathcal{C}_1 \equiv -\frac{2r^2\Sigma}{B\mathcal{H}} + \frac{\Gamma\Xi\phi'^2}{\mathcal{H}^2}, \quad \mathcal{C}_2 \equiv \mathcal{C}_1 + \frac{\mathcal{G}\Xi\phi'}{r\mathcal{H}^2}, \quad \mathcal{C}_3 \equiv \mathcal{C}_1 + 2\frac{\mathcal{G}\Xi\phi'}{r\mathcal{H}^2} + 2\frac{\mathcal{G}}{\mathcal{H}}$$

Dispersion relation

Final result

$$\left(\frac{\omega^2}{A} - \frac{\mathcal{G}}{\mathcal{F}}Bk_r^2 - \frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2}\right)\left(\frac{\omega^2}{A} - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}}Bk_r^2 - \frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}}\frac{j^2}{r^2}\right) = M\frac{j^4}{r^4}$$

Angular speed (scalar)

$$\frac{\mathcal{G}_\mathrm{S}}{\mathcal{H}_\mathrm{S}} = \delta - \frac{2r^2\mathcal{H}^3\mathcal{R}_3}{(2\mathcal{P}_1 - \mathcal{F})(2r\mathcal{H} + \Xi\phi')^2} - \frac{\mathcal{F}\mathcal{G}}{\mathcal{H}(2\mathcal{P}_1 - \mathcal{F})}$$

Radial speed (scalar)

$$\frac{\mathcal{G}_\mathrm{S}}{\mathcal{F}_\mathrm{S}} = \frac{\mathcal{V}}{(2\mathcal{P}_1 - \mathcal{F})(2r\mathcal{H} + \Xi\phi')^2}$$

Mixing term

$$M = \frac{\delta^2(2\mathcal{P}_1 - \mathcal{F})}{4\mathcal{F}}$$

Defined functions

$$\delta = \frac{2rA(\mathcal{F} - \mathcal{H})\mathcal{G}\mathcal{H} + r^2(\mathcal{G} - \mathcal{F})\mathcal{H}^2A' + 2r^2A\mathcal{G}\mathcal{H}(\mathcal{G}' - \mathcal{H}') - r^2A\mathcal{F}\mathcal{H}\Gamma\phi' + 2A\mathcal{F}\mathcal{G}\Xi\phi'}{A\mathcal{H}(2\mathcal{P}_1 - \mathcal{F})(2r\mathcal{H} + \Xi\phi')}$$

$$\mathcal{V} \equiv 2r^2\mathcal{H}\Gamma\Xi\phi'^2 - \mathcal{G}\Xi^2\phi'^2 - 4r^4\mathcal{H}^2\Sigma/B$$

Dispersion relation

Final result

$$\left(\frac{\omega^2}{A} - \frac{\mathcal{G}}{\mathcal{F}}Bk_r^2 - \frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2}\right)\left(\frac{\omega^2}{A} - \frac{\mathcal{G}_{\rm S}}{\mathcal{F}_{\rm S}}Bk_r^2 - \frac{\mathcal{G}_{\rm S}}{\mathcal{H}_{\rm S}}\frac{j^2}{r^2}\right) = M\frac{j^4}{r^4}$$

Mixing term

$$M = \frac{\delta^2(2\mathcal{P}_1 - \mathcal{F})}{4\mathcal{F}}$$

$$\delta = \frac{2rA(\mathcal{F}-\mathcal{H})\mathcal{G}\mathcal{H} + r^2(\mathcal{G}-\mathcal{F})\mathcal{H}^2A' + 2r^2A\mathcal{G}\mathcal{H}(\mathcal{G}'-\mathcal{H}') - r^2A\mathcal{F}\mathcal{H}\Gamma\phi' + 2A\mathcal{F}\mathcal{G}\Xi\phi'}{A\mathcal{H}(2\mathcal{P}_1 - \mathcal{F})(2r\mathcal{H} + \Xi\phi')}$$

$M = 0$ and $c_{GW} = \frac{\mathcal{G}}{\mathcal{F}} = \frac{\mathcal{G}}{\mathcal{H}} = 1$ if $G_{4X} = G_{5\phi} = G_{5X} = 0$.

If M=0, we get decoupled dispersions.

$$\frac{\omega^2}{A} - \frac{\mathcal{G}}{\mathcal{F}}Bk_r^2 - \frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2} = 0 \qquad \text{or} \qquad \frac{\omega^2}{A} - \frac{\mathcal{G}_{\rm S}}{\mathcal{F}_{\rm S}}Bk_r^2 - \frac{\mathcal{G}_{\rm S}}{\mathcal{H}_{\rm S}}\frac{j^2}{r^2} = 0$$

Stability conditions

Dispersion relation:

$$\left(\frac{\omega^2}{A}-\frac{\mathcal{G}}{\mathcal{F}}Bk_r^2-\frac{\mathcal{G}}{\mathcal{H}}\frac{j^2}{r^2}\right)\left(\frac{\omega^2}{A}-\frac{\mathcal{G}_{\mathrm{S}}}{\mathcal{F}_{\mathrm{S}}}Bk_r^2-\frac{\mathcal{G}_{\mathrm{S}}}{\mathcal{H}_{\mathrm{S}}}\frac{j^2}{r^2}\right)=M\frac{j^4}{r^4}$$

How can we read conditions to avoid instability?

$$\rightarrow \quad (\Omega-ax-by)(\Omega-cx-dy)=my^2$$

$$f(\Omega)=(\Omega-ax-by)(\Omega-cx-dy)-my^2$$

Phase speeds are always real

⇅

Roots of $f(\Omega)=0$ are always positive for any $x>0, y>0$

$$f(0)=(ax+by)(cx+dy)-my^2>0$$

$$D=(ax+by-cx-dy)^2+4my^2\geq 0 \quad \rightarrow \quad a>0,\ b>0,\ c>0,\ d>0,\ bd-m>0$$

$$ax+by+cx+dy>0$$

$$\frac{\mathcal{G}}{\mathcal{F}}>0,\ \frac{\mathcal{G}}{\mathcal{H}}>0,\ \frac{\mathcal{G}_s}{\mathcal{F}_s}>0,\ \underline{\frac{\mathcal{G}_s}{\mathcal{H}_s}>0}\ \text{and}\ \underline{\frac{\mathcal{G}}{\mathcal{H}}\cdot\frac{\mathcal{G}_s}{\mathcal{H}_s}>M}$$

Conclusion

- Theoretical understanding of fundamental properties of modified gravity is crucial to distinguish GR from other theories of gravity.
- We calculated the dispersion relations of perturbations within general model in the Horndeski theory.
- We derived two other conditions. It is easy to apply it to a specific model.