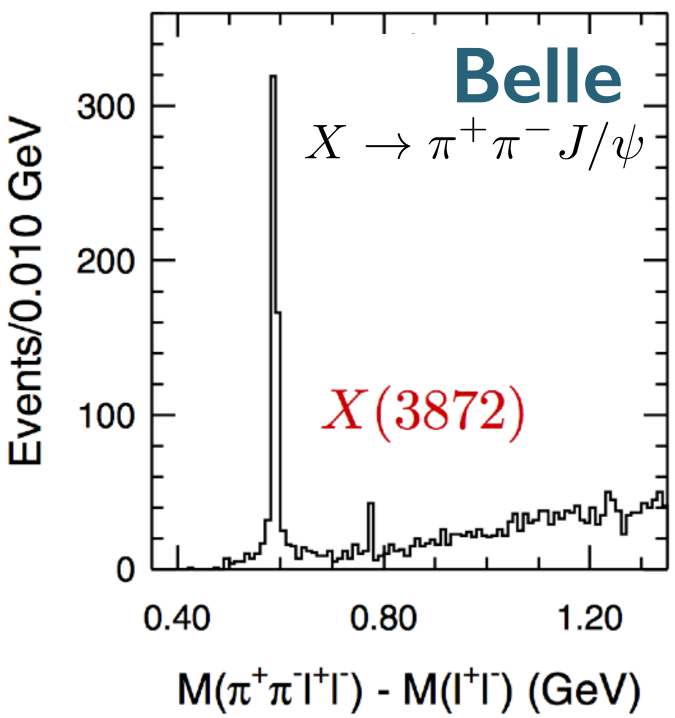
Real-time quantities in a finite volume

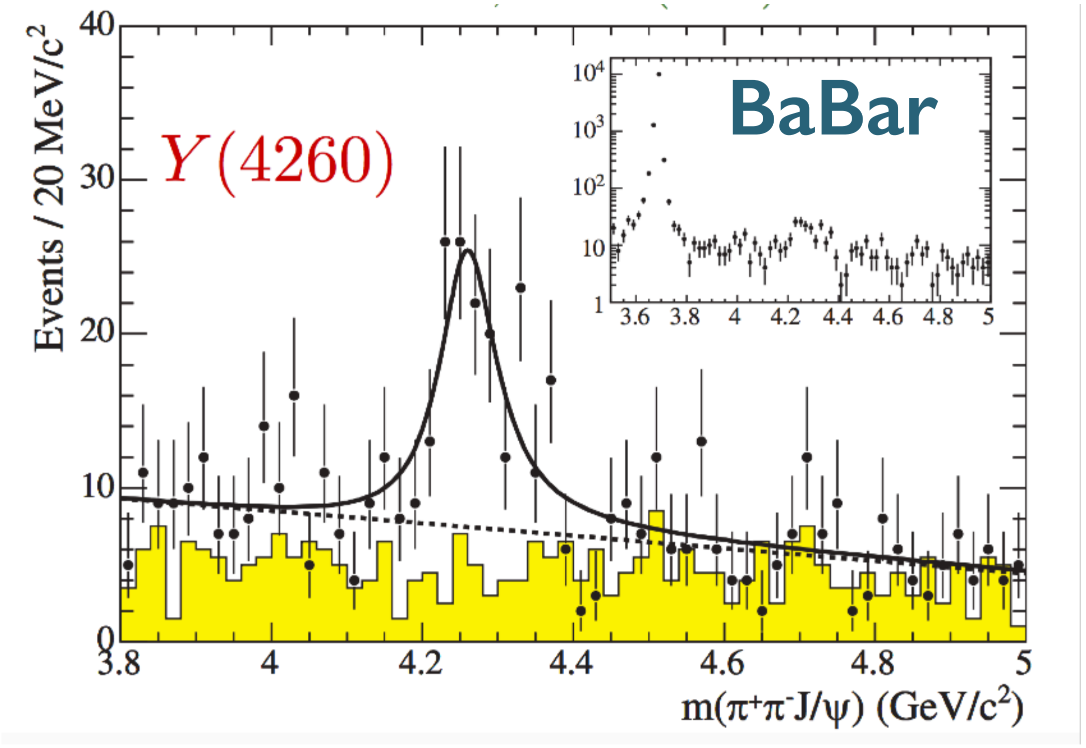
Maxwell T. Hansen

March 14th, 2019

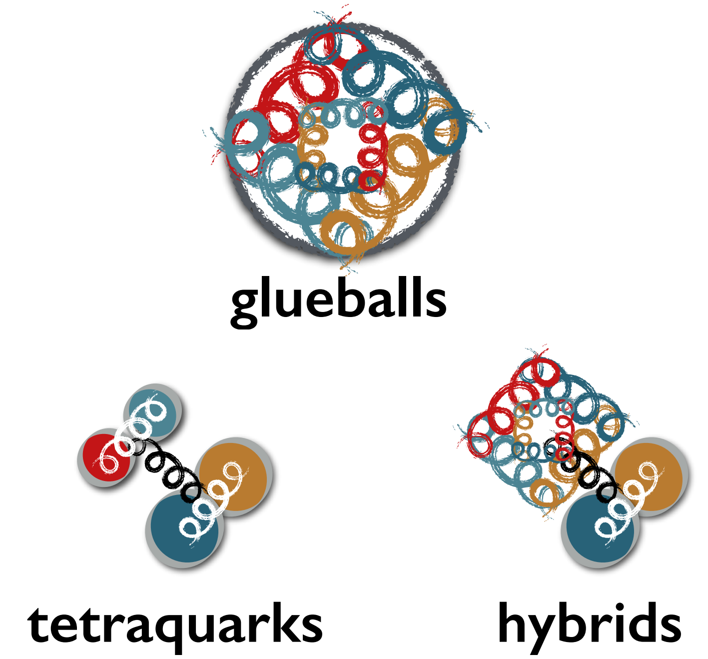
The rich resonance structure of QCD…



PRL 91, 262001 (2003)

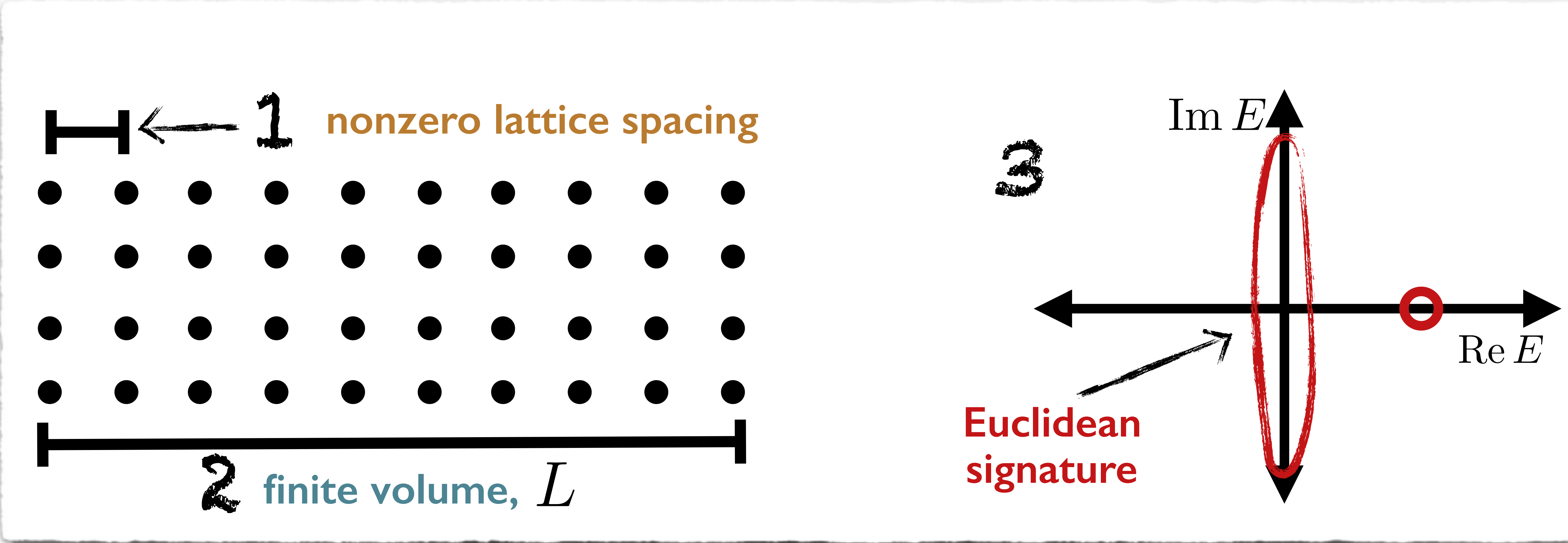


PRL95,142001 (2005)



- ❑ **Quantify how resonances modify scattering and production rates (*T = 0*)**
- ❑ **Explore how they couple to external probes / respond to extreme conditions**
- ❑ **Determine how their properties depend on QCD's fundamental parameters**

Inherently difficult:
Very far from low-energy symmetry constraints and high-energy pQCD



In real-time observables, the interplay of these is subtle…

- ❑ **Euclidean signature → inverse methods → understanding covariance → challenged by discretization**

This talk: two sides of lattice resonance physics…

- ❑ **First half: Finite *L* as a tool (→ Euclidean not a disadvantage)**
- ❑ **Second half: Finite *L* as an unwanted artifact (→ Euc. is a problem)**

Based on work with…

R. Briceño

S. Sharpe

D. Robaina

H. Meyer

- ❑ *Three-particle systems with resonant subprocesses in a finite volume*
  Briceño, MTH, Sharpe, PRD 2019, (arXiv:1810.01429)

- ❑ *Lattice QCD and Three-particle Decays of Resonances*
  MTH, Sharpe, (arXiv:1901.00483)

- ❑ *Total rates into multihadron final states from lattice QCD*
  MTH, Meyer, Robaina (arXiv:1704.08993)

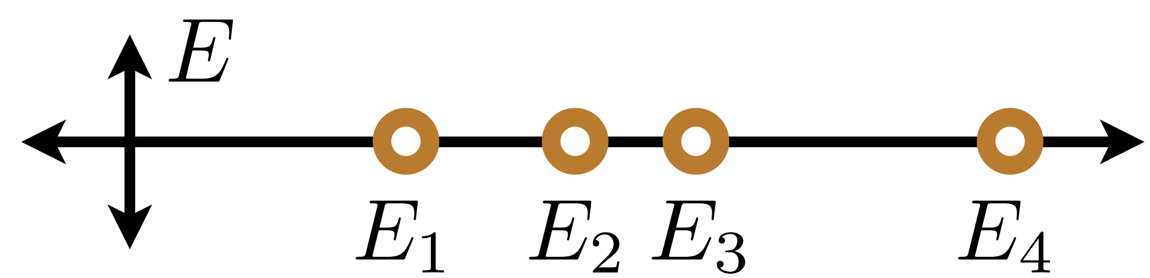
Of bound states and cuts

- **Consider a trapped particle** in non-relativistic quantum mechanics

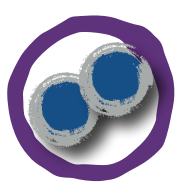
A useful tool is the correlation function…

$$\mathcal{C}(E) = \int_0^\infty dt\ e^{iEt} \langle \vec{x} | e^{-i(H-i\epsilon)t} | \vec{x} \rangle = \sum_{n=1}^{\infty} \int_0^\infty dt\ e^{i(E-E_n+i\epsilon)t} |\langle \vec{x} | E_n \rangle|^2$$

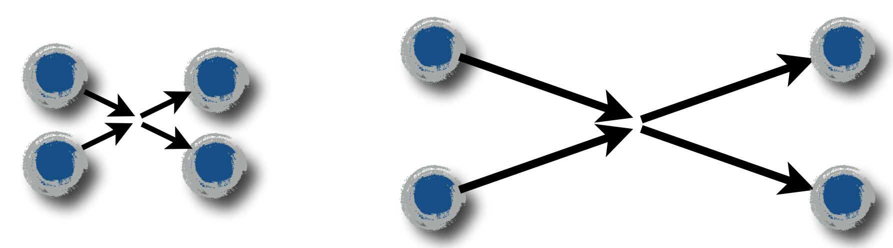
$$\mathcal{C}(E) = i \sum_{n=1}^{\infty} \frac{|\psi_n(\vec{x})|^2}{E - E_n + i\epsilon}$$



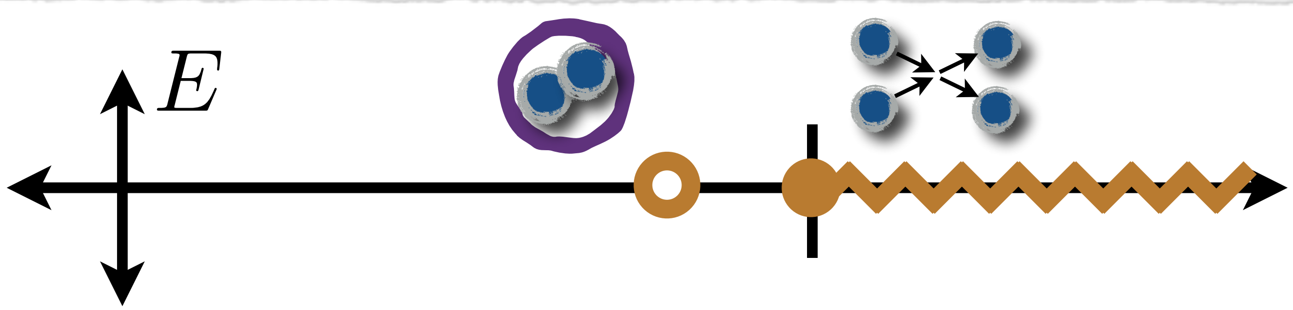
- **What is different in systems with scattering states?**



Isolated state with $E = M$



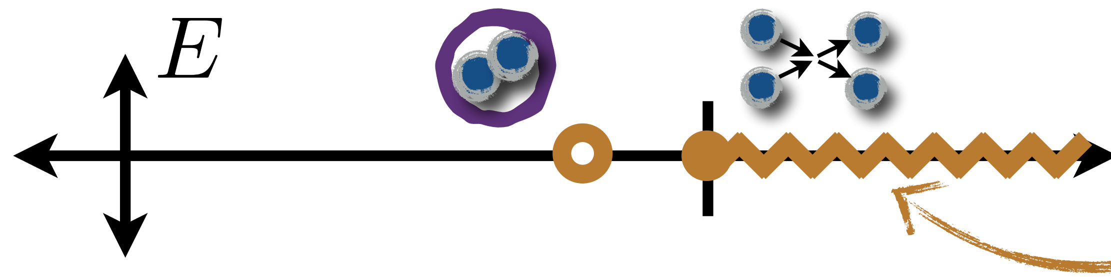
Can have any energy $E > 2m$



Poles become dense and form a branch cut

S-matrix and scattering

- **For scattering states, listing allowed energies is no longer useful**



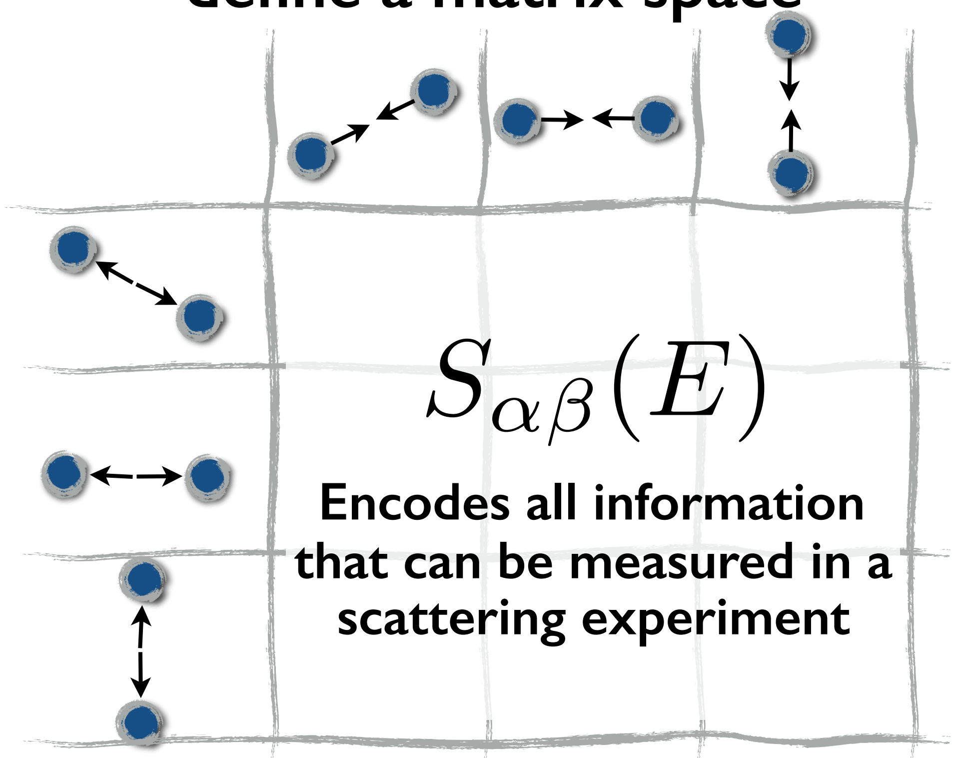
All above-threshold energies appear

- **Instead, physical information is in the matrix elements…**

$$S(E) = \langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle$$

S-matrix

Angular degrees of freedom define a matrix space

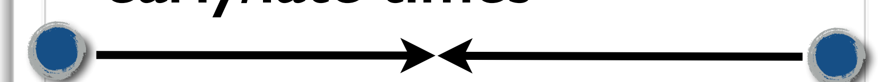


$$S_{\alpha\beta}(E)$$

Encodes all information that can be measured in a scattering experiment

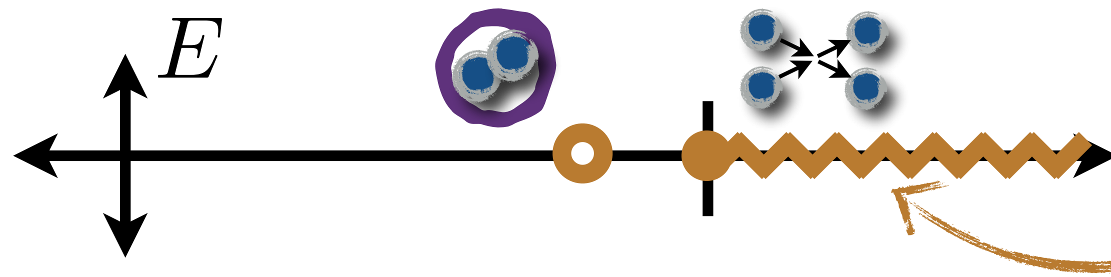
asymptotic state

- **hamiltonian eigenstate**
- **pions well separated at early/late times**



S-matrix and scattering

- For scattering states, listing allowed energies is no longer useful



All above-threshold energies appear

- Instead, physical information is in the matrix elements…

$$S(E) = \langle \pi\pi, \mathrm{out} | \pi\pi, \mathrm{in} \rangle$$

S-matrix properties

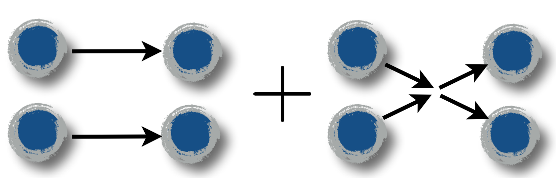
	$S_0(E)$	0	0
	0	$S_1(E)$	0
	0	0	$S_2(E)$

- Diagonal in angular momentum
- S-matrix unitarity

$$S^\dagger(E) S(E) = \sum_\alpha \langle \pi\pi, \mathrm{in} | \alpha \rangle \langle \alpha | \pi\pi, \mathrm{in} \rangle = \mathbb{I}$$

- Relation to the **scattering amplitude**

$S(E) =$

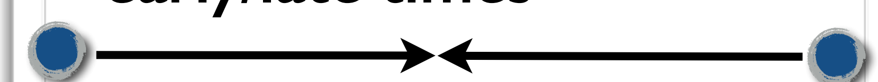


$$S_0(E) = e^{2i\delta_0(E)} \longrightarrow \mathcal{M}_0(E) \propto e^{2i\delta_0(E)} - 1$$

real function contains the scattering information

asymptotic state

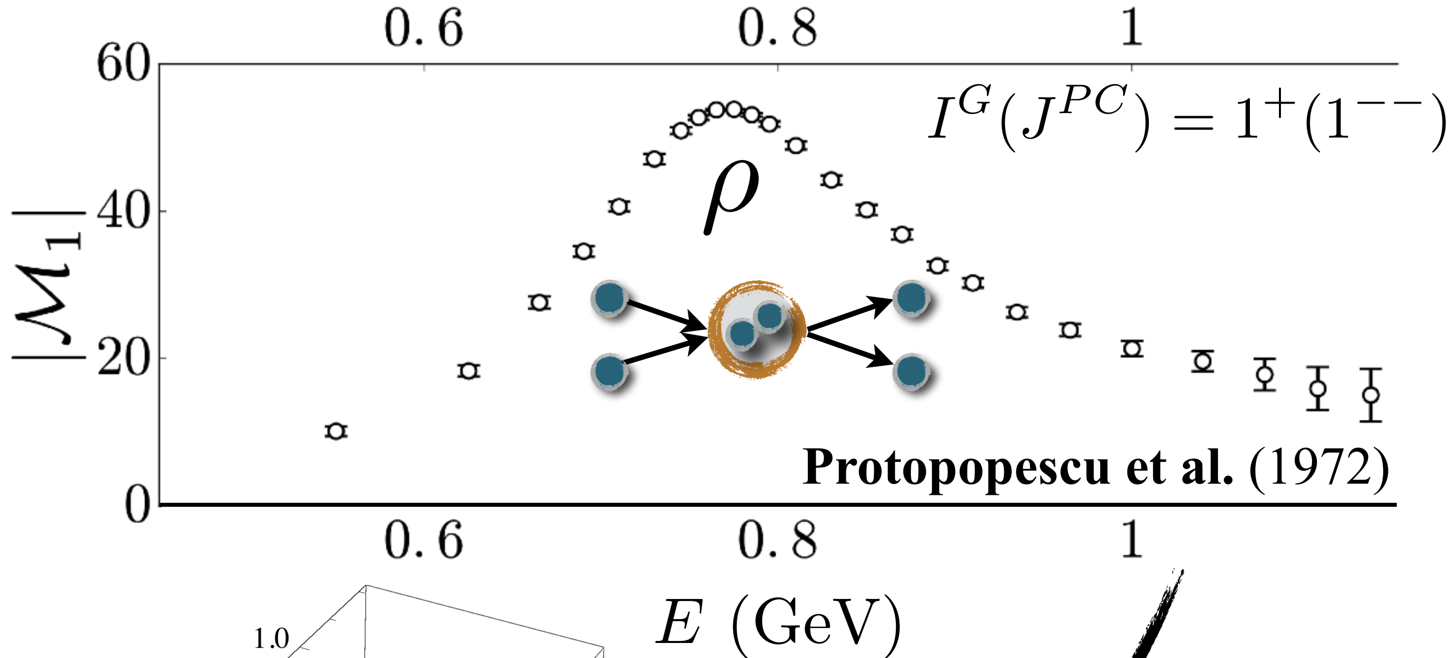
- hamiltonian eigenstate
- pions well separated at early/late times

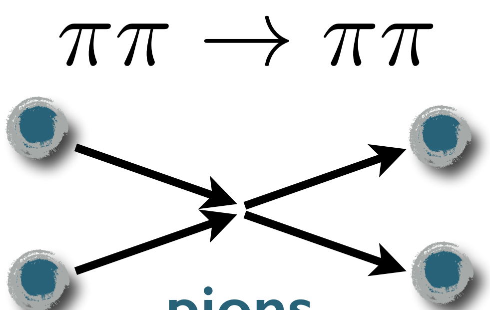


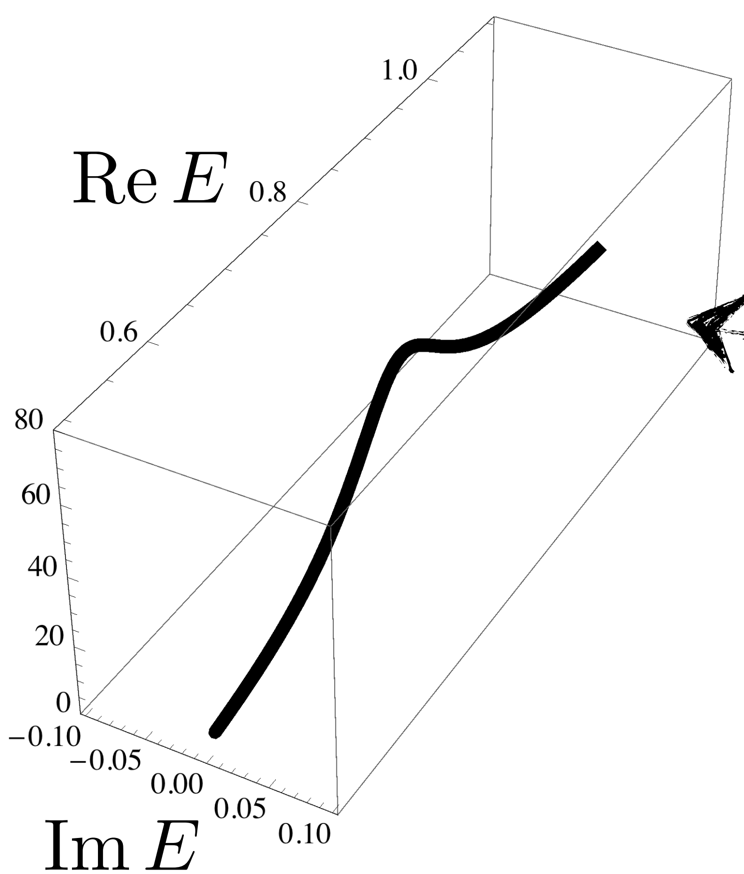
Definition of a resonance

- Roughly speaking, a bump in $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2\delta(E)$

scattering rate unitarity relation





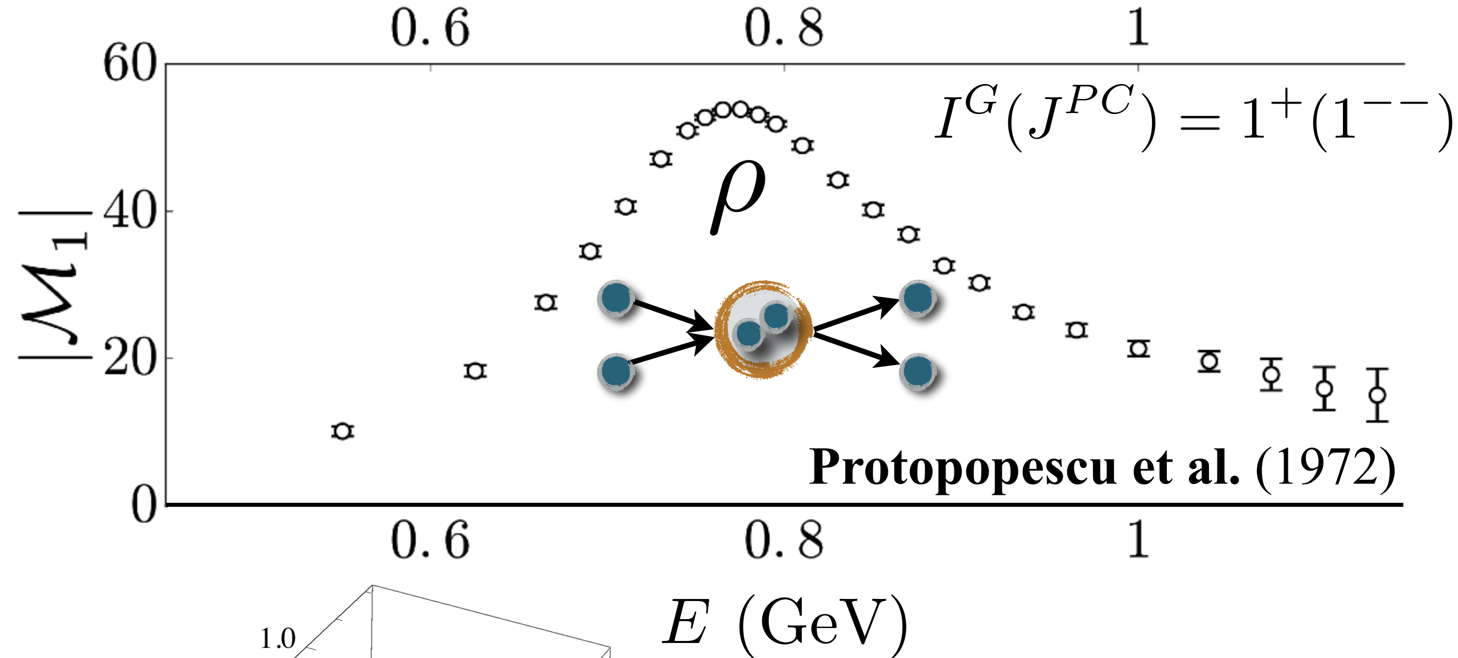


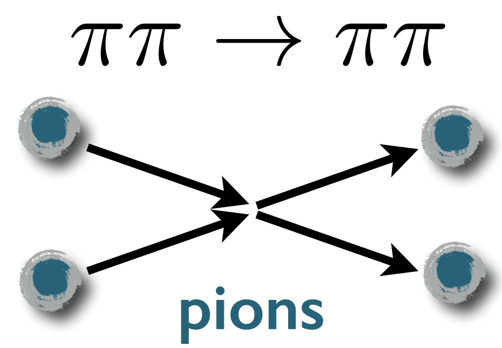
Consider this curve in the complex plane

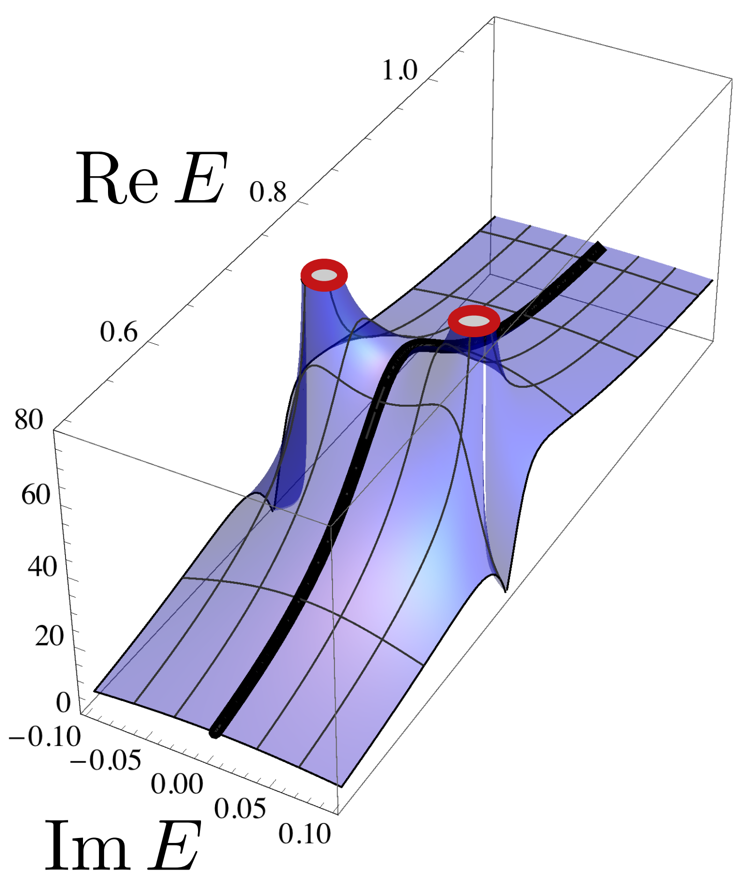
Definition of a resonance

- **Roughly speaking, a bump in** $|\mathcal{M}(E)|^2 \propto |e^{2i\delta(E)} - 1|^2 \propto \sin^2\delta(E)$

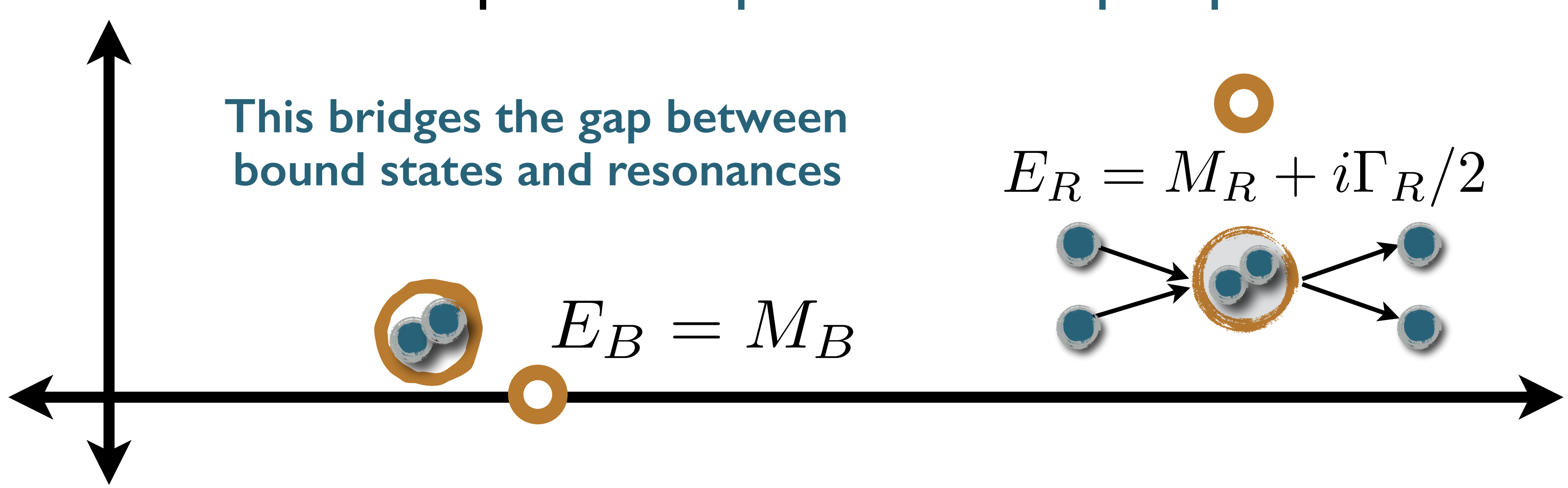
scattering rate — unitarity relation







Analytic continuation reveals that the bump corresponds to a pole in the complex plane

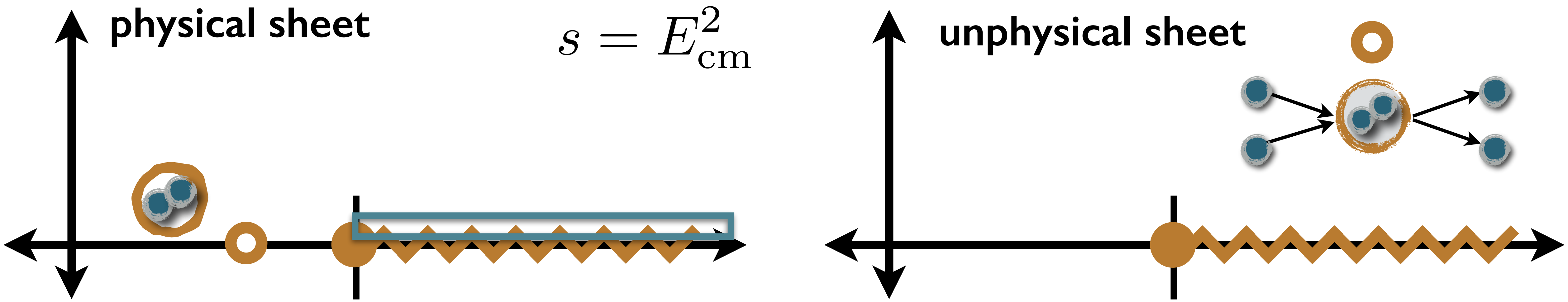


Multiple Riemann sheets

- It is most instructive to analytically continue the scattering amplitude

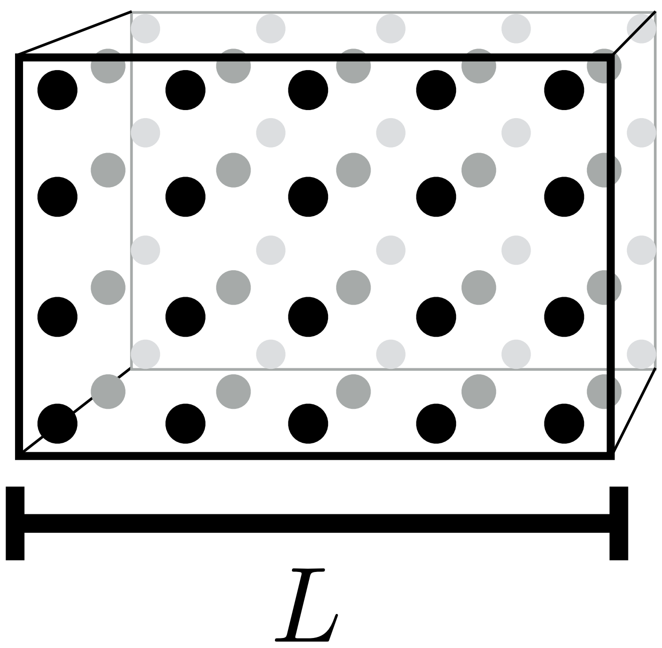
$$\mathcal{M}_\ell(s) = \frac{1}{\mathcal{K}_\ell(s)^{-1} + \rho(s)} \qquad \rho(s) \propto -i\sqrt{s-(2M_\pi)^2}$$

- Each channel generates a **branchcut** that doubles the number of sheets



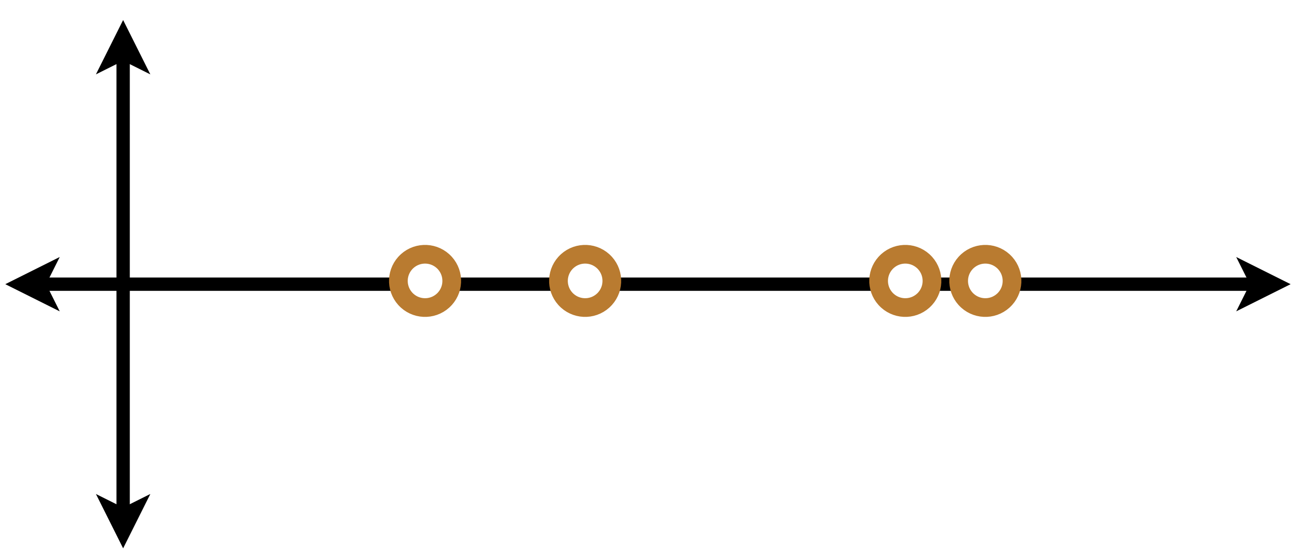
- Our aim is to map out this structure in a systematic and rigorous way

Difficulties for multi-hadron / resonance states

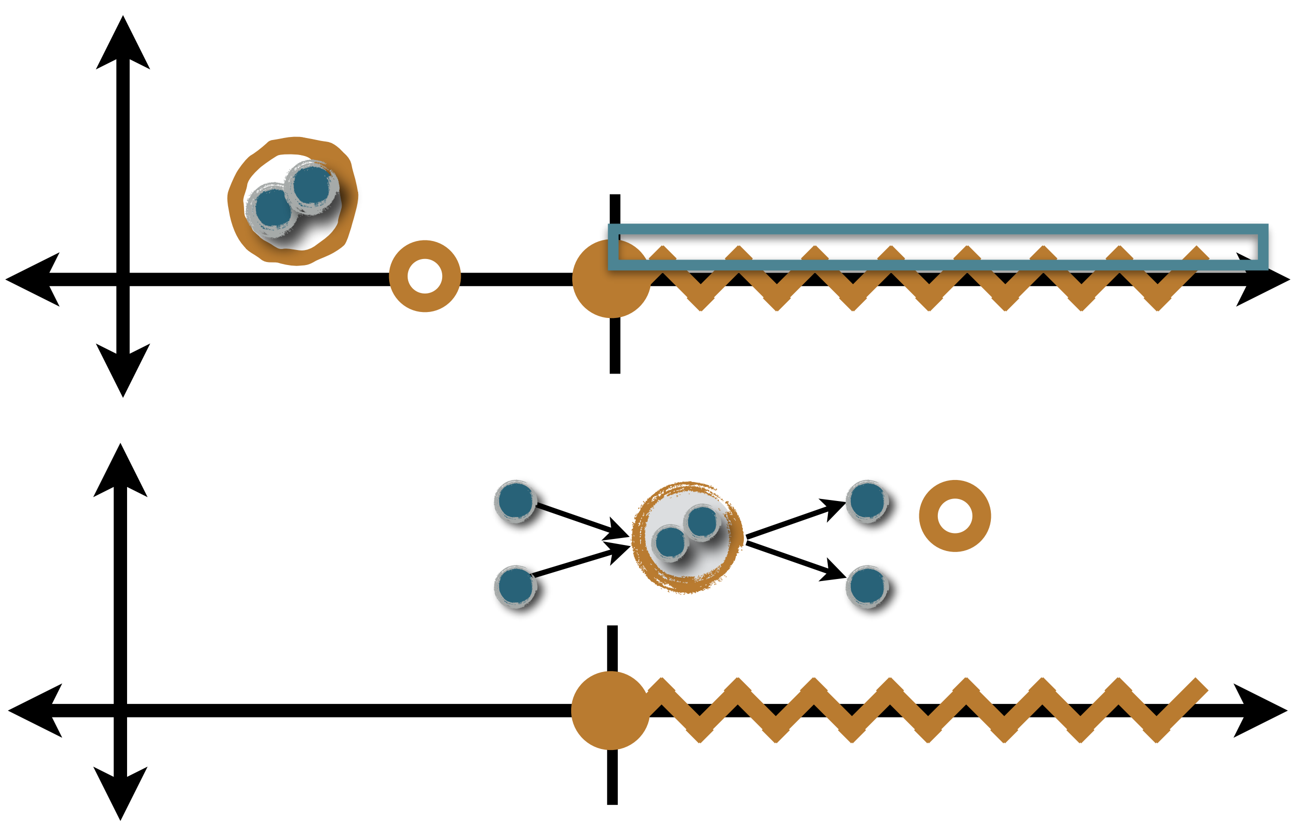


- **The analytic structure is changed by the finite volume…**
  - Discretizes the spectrum
  - Eliminates the branch cuts
  - Removes the second Riemann sheet
  - Hides the resonance poles

Finite-volume analytic structure

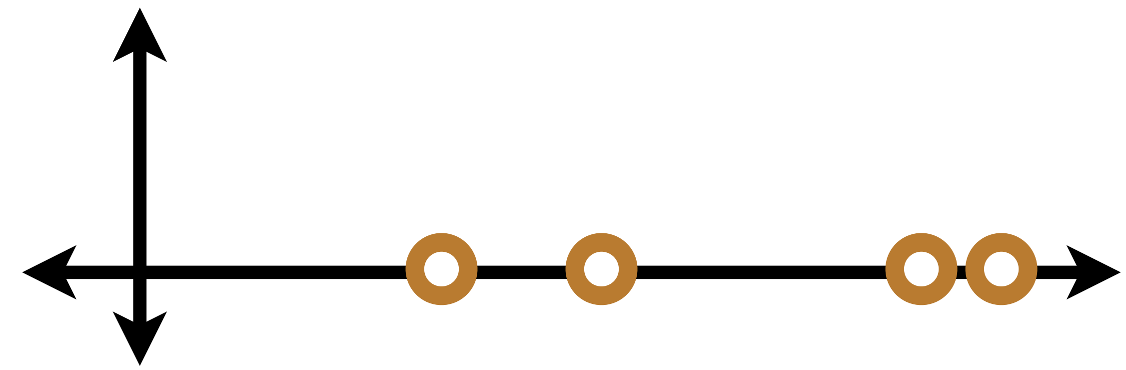


Infinite-volume analytic structure



Finite-volume observables

'On the lattice' one can calculate finite-volume energies and matrix elements



$$\langle \mathcal{O}_j(\tau)\mathcal{O}_i^\dagger(0)\rangle = \sum_n \langle 0|\mathcal{O}_j(\tau)|E_n\rangle\langle E_n|\mathcal{O}_i^\dagger(0)|0\rangle = \sum_n e^{-E_n(L)\tau} Z_{n,j} Z_{n,i}^*$$

Can determine optimized operators by 'diagonalizing' the correlator matrix (GEVP)

$$\big\langle \Omega_m(\tau)\Omega_m^\dagger(0)\big\rangle \sim e^{-E_m(L)\tau} + \cdots$$

$$\big\langle \Omega_{m'}(\tau)\ \mathcal{J}(0)\ \Omega_m^\dagger(-\tau)\big\rangle \sim e^{-E_{m'}\tau}\ e^{-E_m\tau}\ \langle E_{m'}|\mathcal{J}(0)|E_m\rangle + \cdots$$

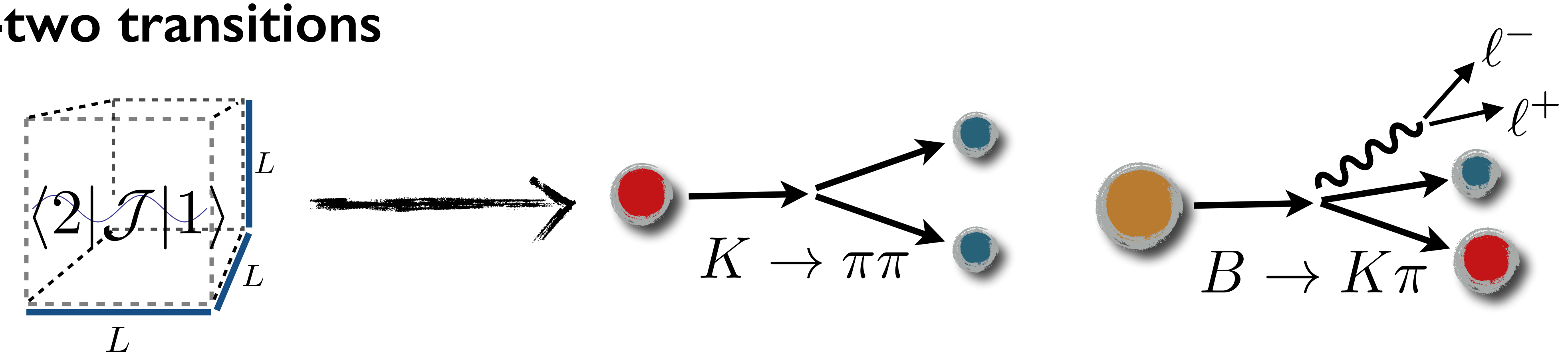
- [ ] **Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to experimental observables**

Multi-hadron processes from LQCD

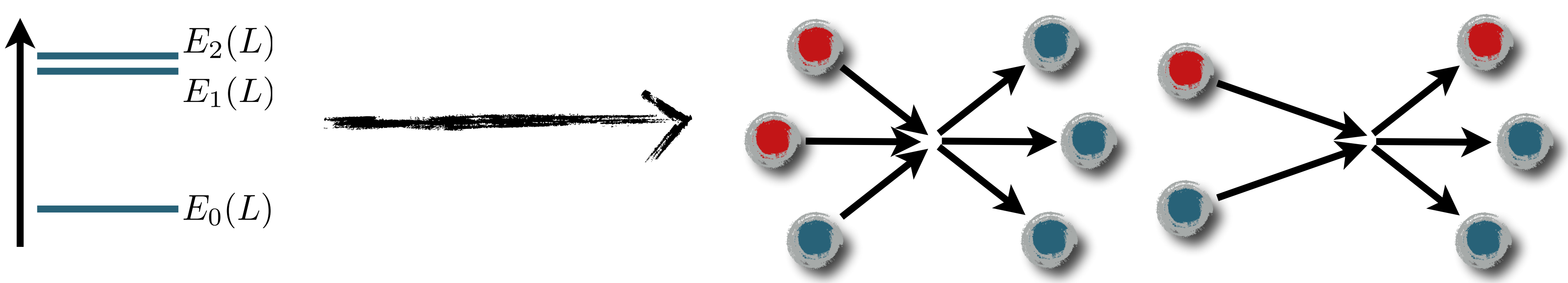
KEY IDEA: **We can use the finite volume as a tool to extract multi-hadron observables**

- **Two-to-two scattering**

- **One-to-two transitions**

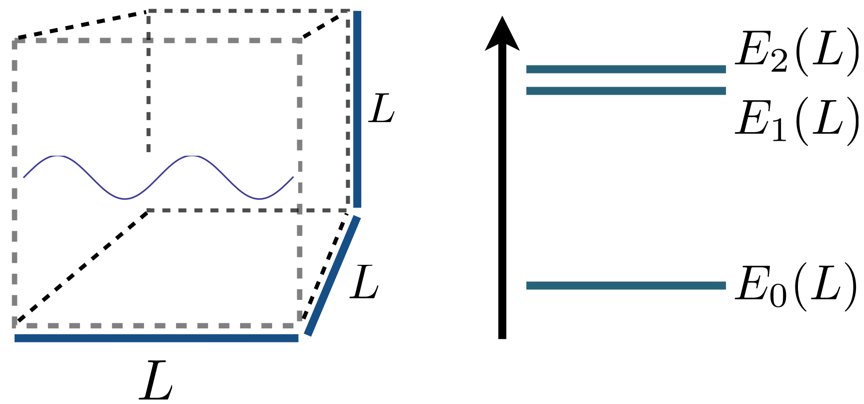


- **Two-to-three and three-to-three scattering**



The finite-volume as a tool

- **Finite-volume set-up**



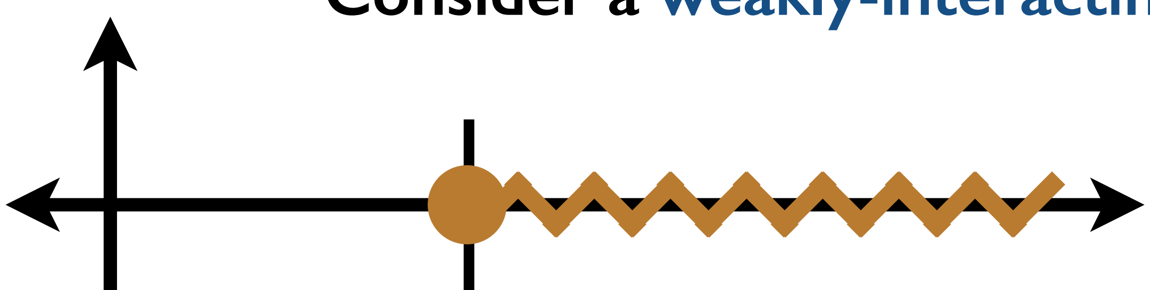
- **cubic, spatial volume (extent *L*)**
- **periodic boundary conditions**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- ***L* is large enough to neglect** $e^{-M_\pi L}$

- **Scattering observables leave an *imprint* on finite-volume quantities**

Consider a weakly-interacting, two-body system with no bound states

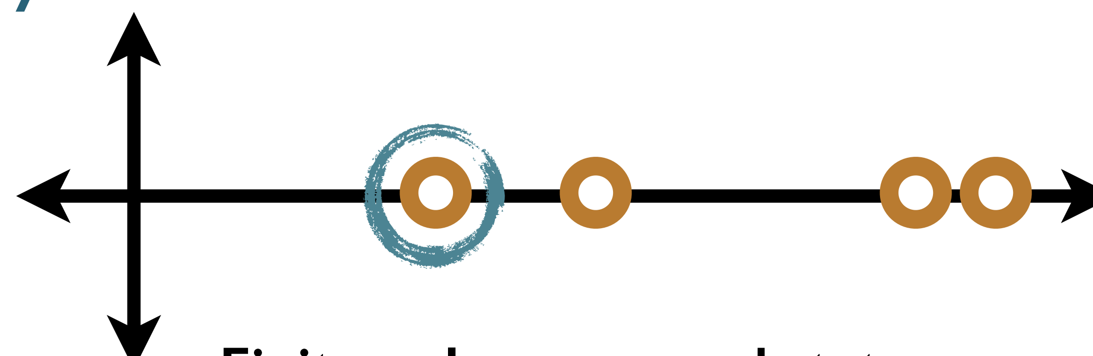


$$E_0 = 2M_\pi$$

Infinite-volume ground state

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

Information is in the scattering amplitude



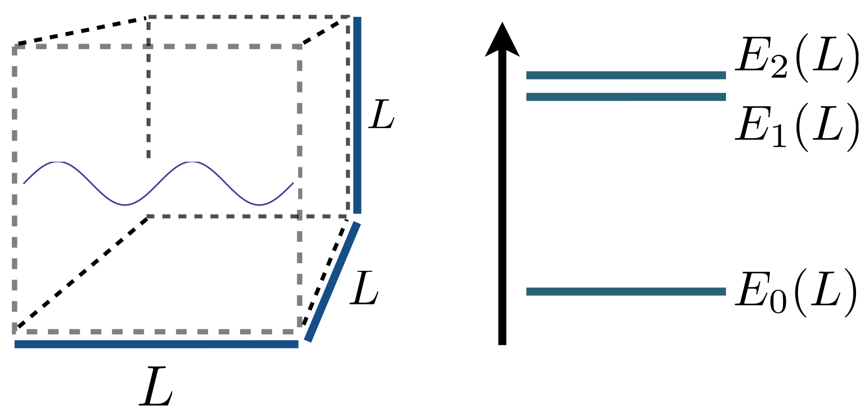
Finite-volume ground state

$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

The finite-volume as a tool

- **Finite-volume set-up**



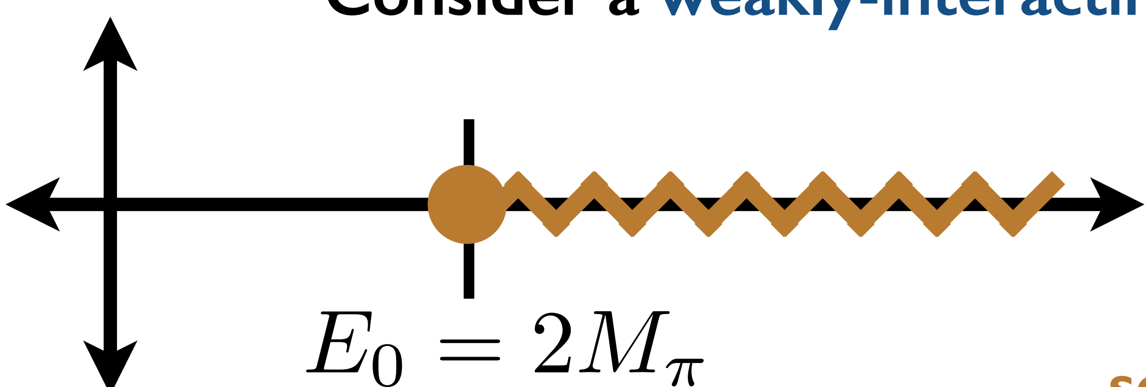
- **cubic, spatial volume (extent *L*)**
- **periodic boundary conditions**

$$\vec{p} = \frac{2\pi}{L}\vec{n}\,, \quad \vec{n} \in \mathbb{Z}^3$$

- ***L* is large enough to neglect** $e^{-M_\pi L}$

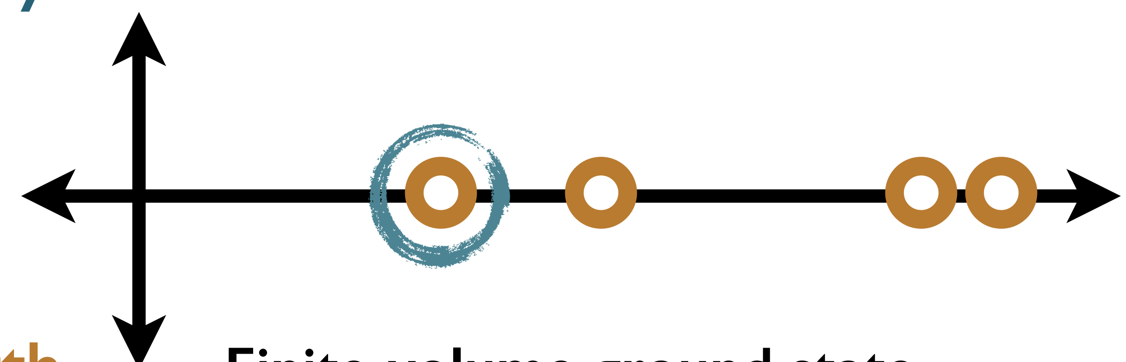
- **Scattering observables leave an *imprint* on finite-volume quantities**

Consider a weakly-interacting, two-body system with no bound states



$$E_0 = 2M_\pi$$

Infinite-volume ground state



Finite-volume ground state

scattering length

$$\mathcal{M}_{\ell=0}(2M_\pi) = -32\pi M_\pi a$$

Information is in the scattering amplitude

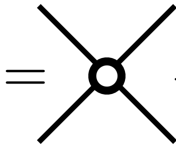
$$E_0(L) = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

Huang, Yang (1958)

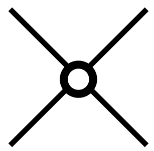
Hint of the derivation

- **In the infinite-volume world…**

scattering length

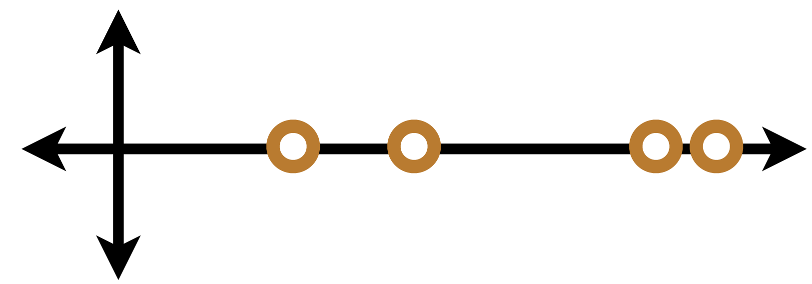
$$\mathcal{M}_{\ell=0}(E_{\text{cm}}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

- **In the finite-volume world…**

$$\mathcal{M}_L(E_{\text{cm}}) =$$  $$+ \cdots$$

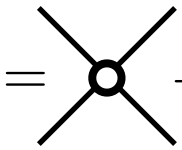
$$= -\lambda + \cdots$$

At leading order, the finite-volume amplitude has no poles

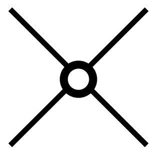
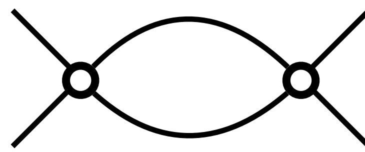


Hint of the derivation

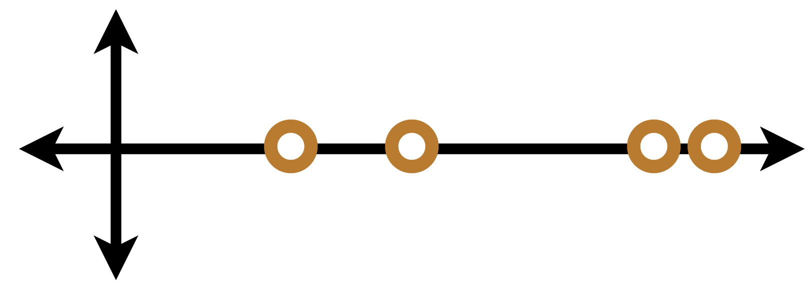
- **In the infinite-volume world…**

$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

- **In the finite-volume world…**

$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+$$  $$+ \cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$

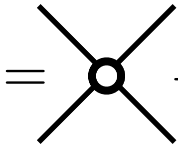


At next-to-leading order, we see poles of two non-interacting particles

$$\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + M_\pi^2} \quad \textbf{where} \quad \mathbf{k} = \frac{2\pi}{L}\mathbf{n}\,, \quad \mathbf{n} \in \mathbb{Z}^3$$

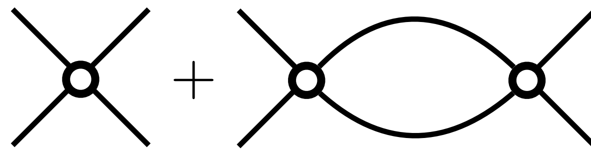
Hint of the derivation

- In the infinite-volume world…

$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+ \cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

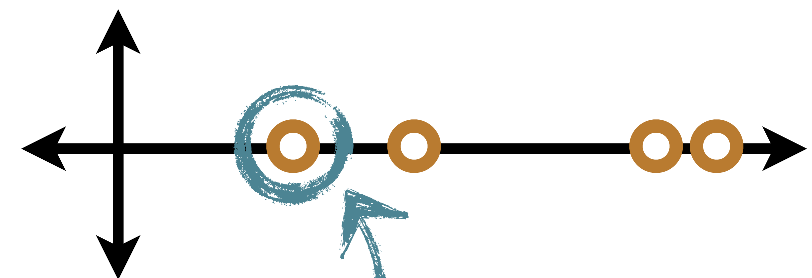
scattering length

- In the finite-volume world…

$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+ \cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\frac{1}{(2M_\pi)^2(E_{\rm cm} - 2M_\pi)}\lambda + \cdots$$



zoom-in on the lowest-lying pole

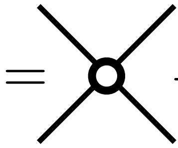
pole from two zero-momentum pions

The truncated series is failing because we are interested in $E_{\rm cm} - 2M_\pi = \mathcal{O}(\lambda)$

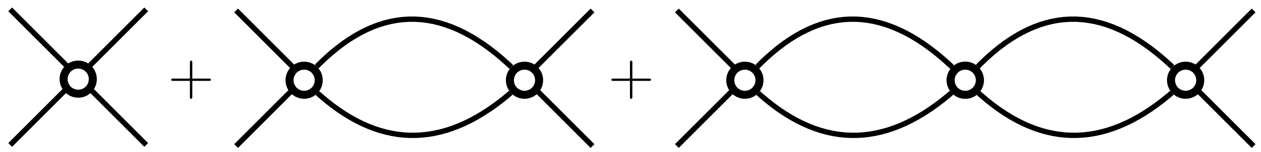
Hint of the derivation

- In the infinite-volume world…

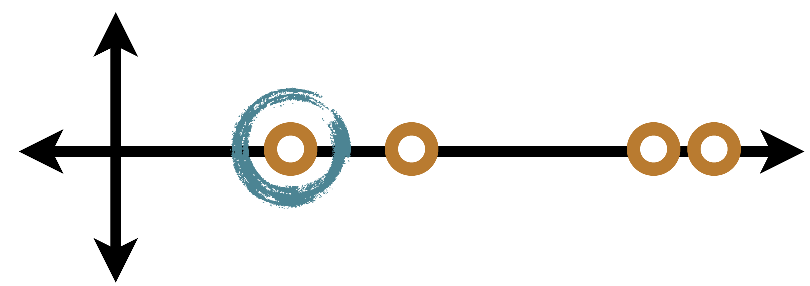
scattering length

$$\mathcal{M}_{\ell=0}(E_{\rm cm}) =$$  $$+\cdots = -\lambda + \mathcal{O}(\lambda^2) \longrightarrow a = \frac{\lambda}{32\pi M_\pi} + \mathcal{O}(\lambda^2)$$

- In the finite-volume world…

$$\mathcal{M}_L(E_{\rm cm}) =$$  $$+\cdots$$

$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\sum_{\mathbf{k}} \frac{1}{(2\omega_{\mathbf{k}})^2(E_{\rm cm} - 2\omega_{\mathbf{k}})}\lambda + \cdots$$



$$= -\lambda - \lambda \frac{1}{2}\frac{1}{L^3}\frac{1}{(2M_\pi)^2(E_{\rm cm} - 2M_\pi)}\lambda + \cdots$$

$$= -\lambda \sum_{n=0}^{\infty} [f(E_{\rm cm}, L)\ \lambda]^n = \frac{1}{-1/\lambda + f(E_{\rm cm}, L)}$$

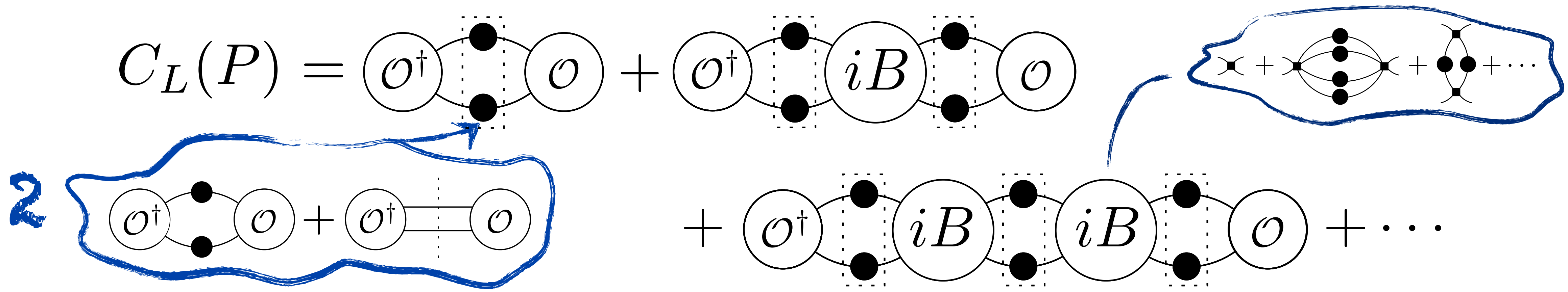
Summing this singular contribution to all orders gives the final expression…

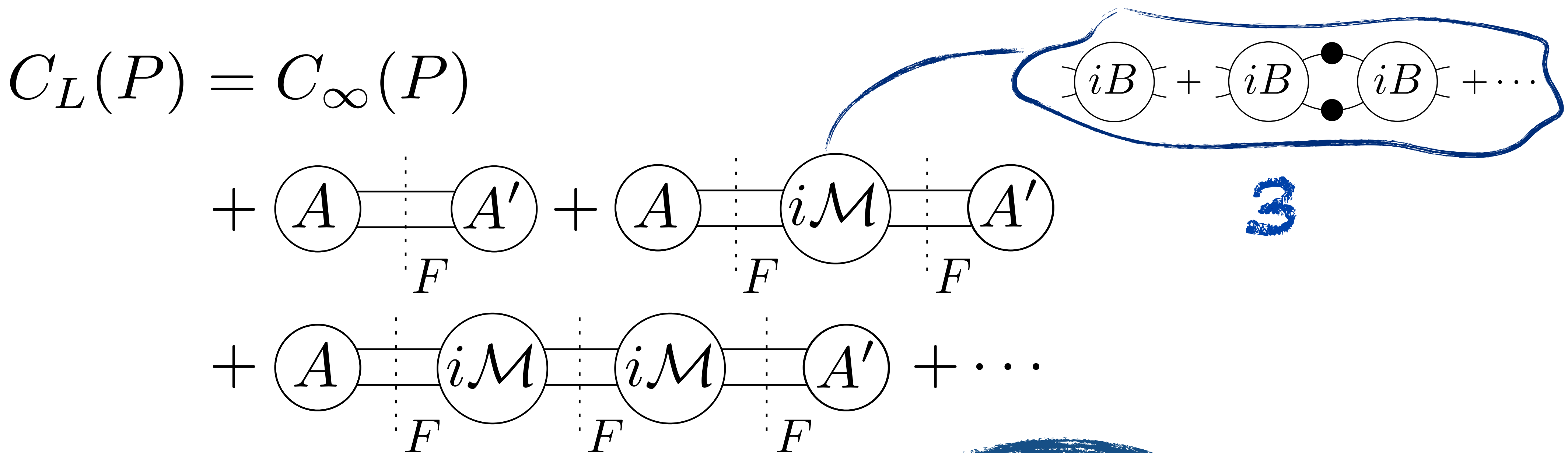
$$-1/\lambda + f(E_{\rm cm}, L) = 0 \implies E_{\rm cm} = 2M_\pi + \frac{4\pi a}{M_\pi L^3} + \mathcal{O}(1/L^4)$$

- This result can be generalized…

Lüscher quantization condition

1

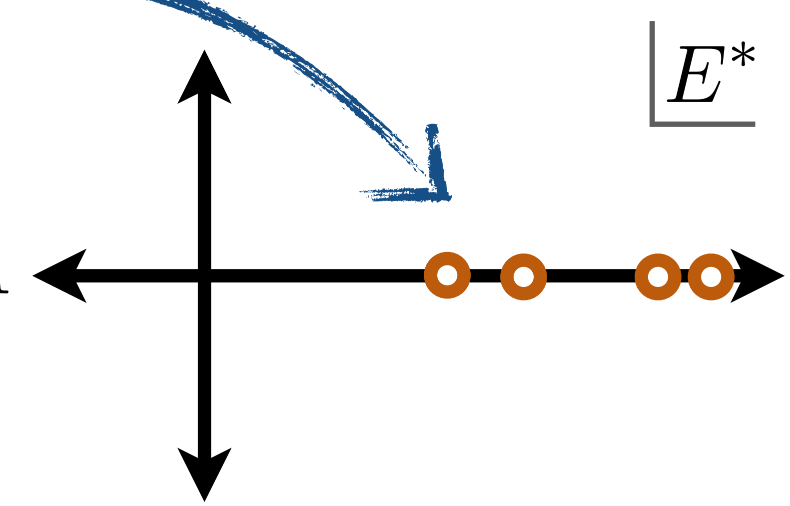




We deduce...

poles are in here

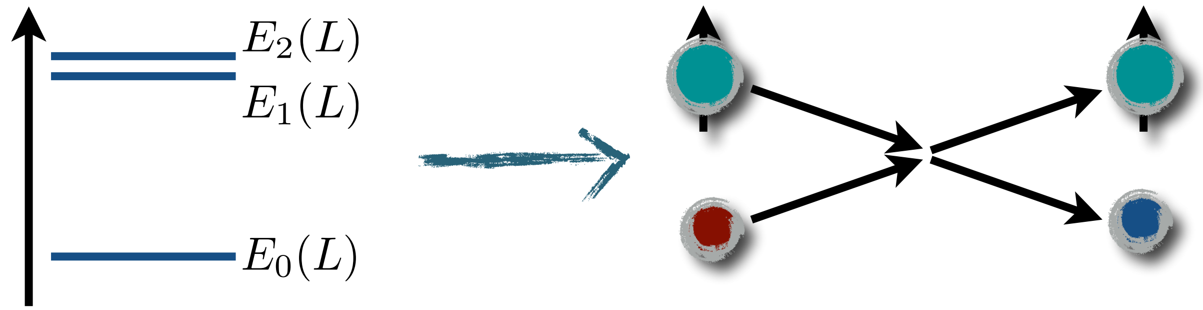
$$C_L(P) = C_\infty(P) - A' F \boxed{\frac{1}{1 + \mathcal{M}_{2\to 2} F}} A$$



Lüscher (1986, 1991) ○ Kim, Sachrajda, Sharpe (2005)

Two-to-two scattering

- Lüscher's formalism + extensions give a general mapping



- All results are contained in a generalized quantization condition

$$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$$

scattering amplitude — known geometric function

Matrices in angular momentum, spin and channel space

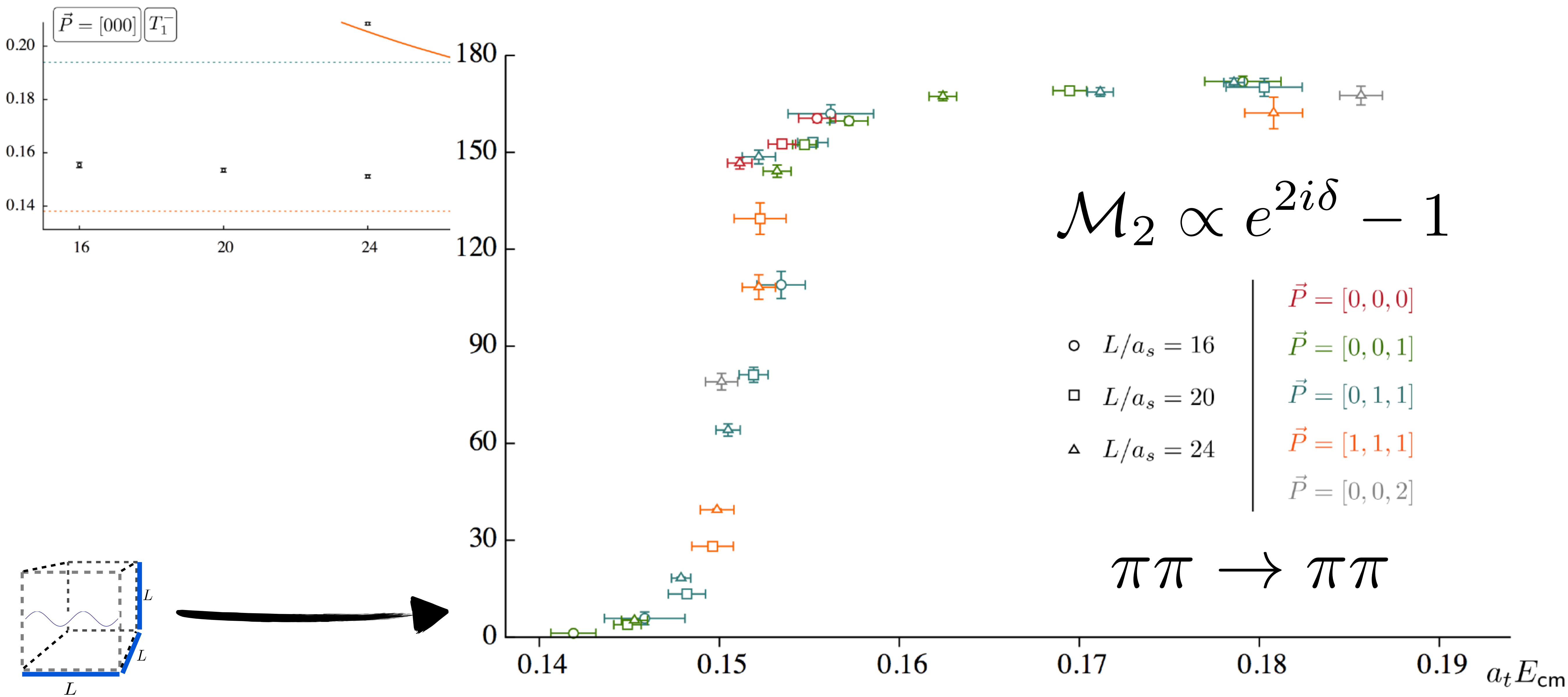
- Varying $E, \vec{P}$ gives more constraints on functions of $E^{*2} = E^2 - \vec{P}^2$
- Derivation ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)

Huang, Yang (1958) ○ **Lüscher** (1986, 1991) ○ **Rummukainen, Gottlieb** (1995)
Kim, Sachrajda, Sharpe (2005) ○ **Christ, Kim, Yamazaki** (2005) ○ **He, Feng, Liu** (2005)
Beane, Detmold, Savage (2007) ○ **Tan** (2008) ○ **Leskovec, Prelovsek** (2012) ○ **Bernard *et. al.*** (2012)
MTH, Sharpe (2012) ○ **Briceño, Davoudi** (2012) ○ **Li, Liu** (2013) ○ **Briceño** (2014)

Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

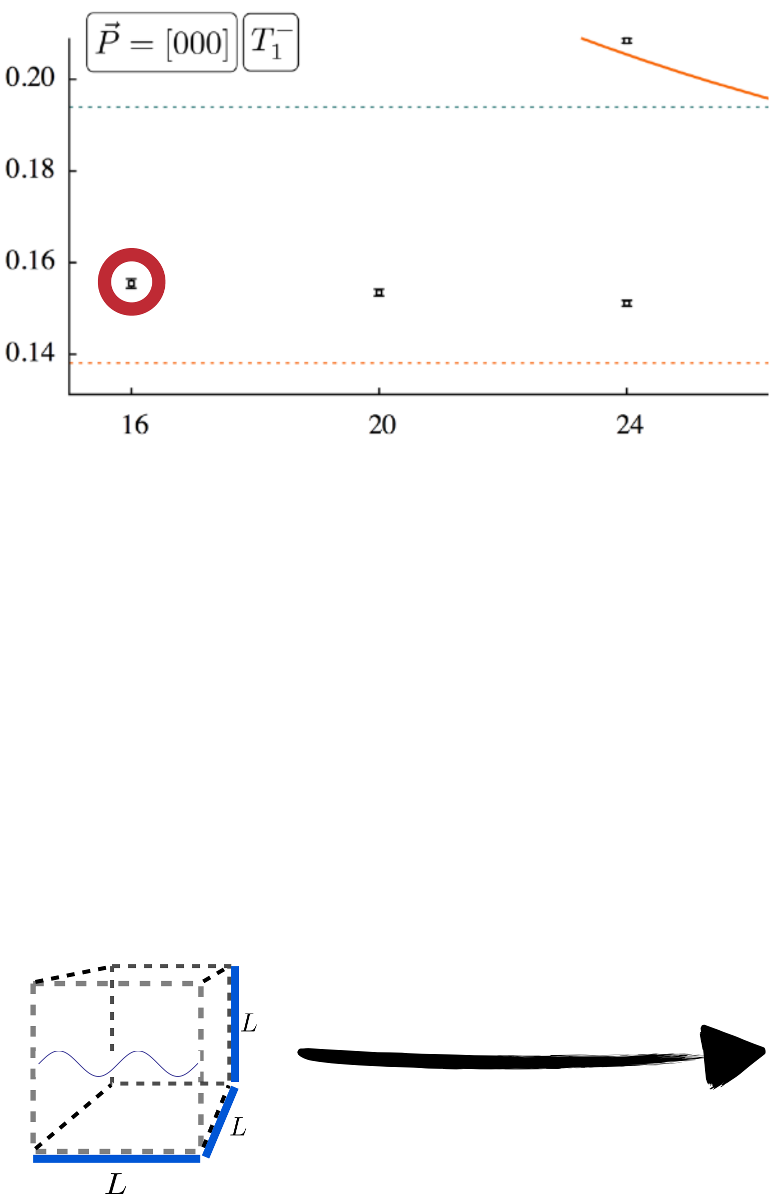


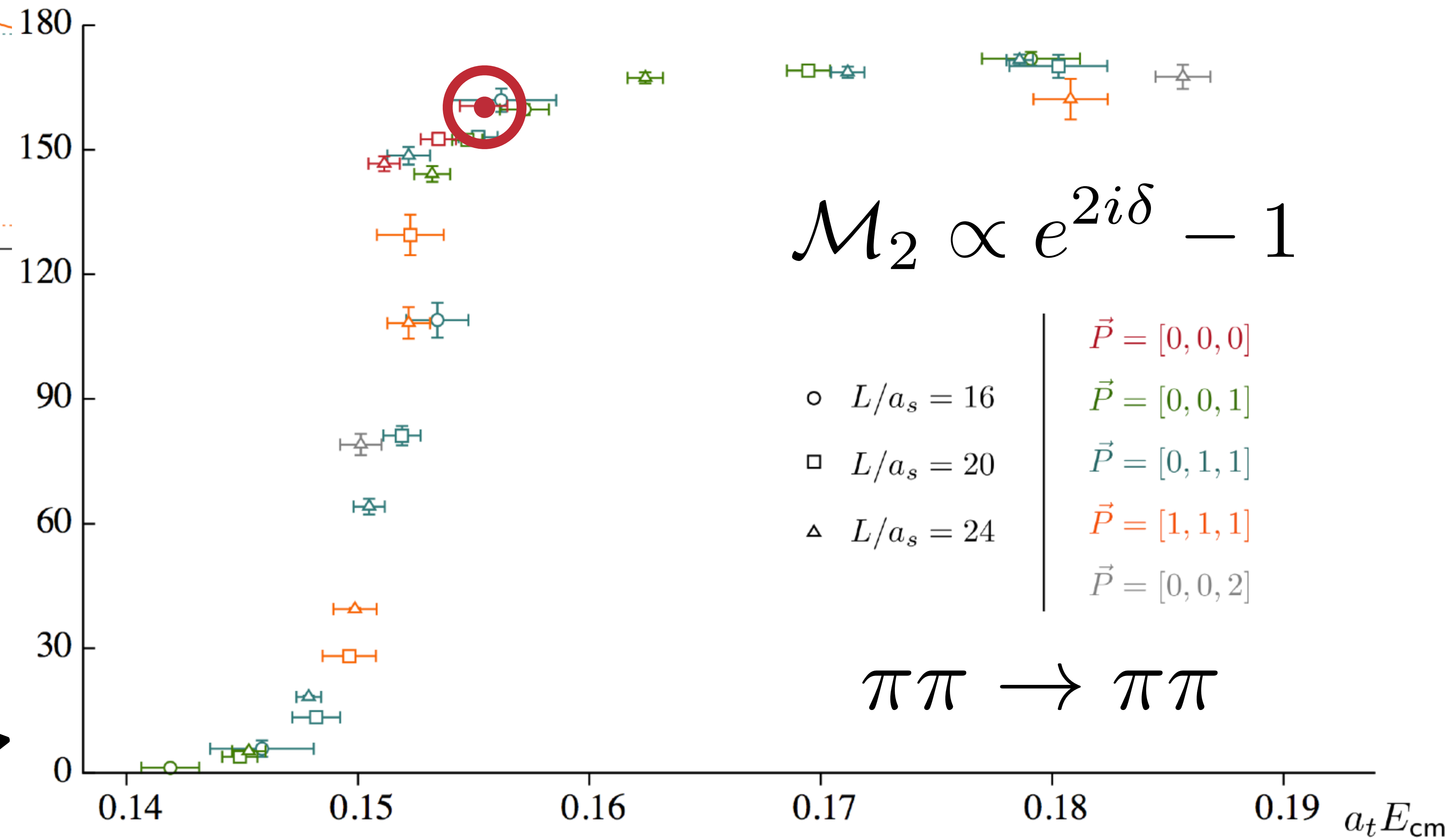
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Using the result

- **Simplest case is a single channel**
  (e.g. for pions in a p-wave the relation reduces to)

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$





$$\mathcal{M}_2 \propto e^{2i\delta} - 1$$

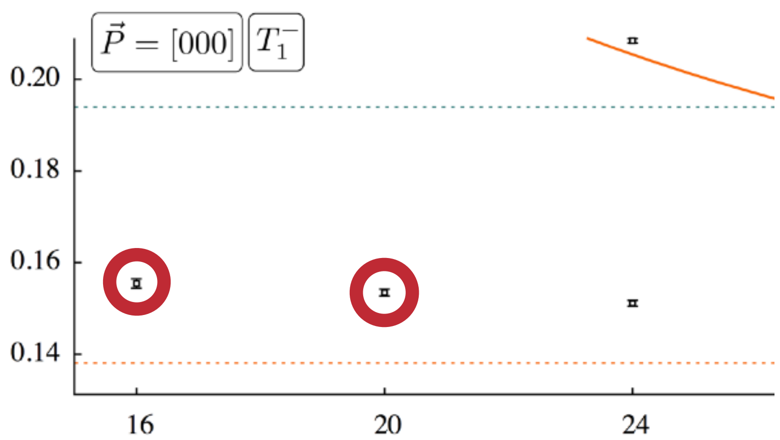
$$\pi\pi \to \pi\pi$$

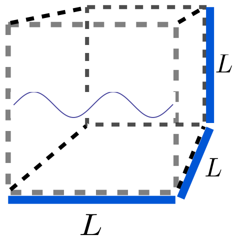
from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

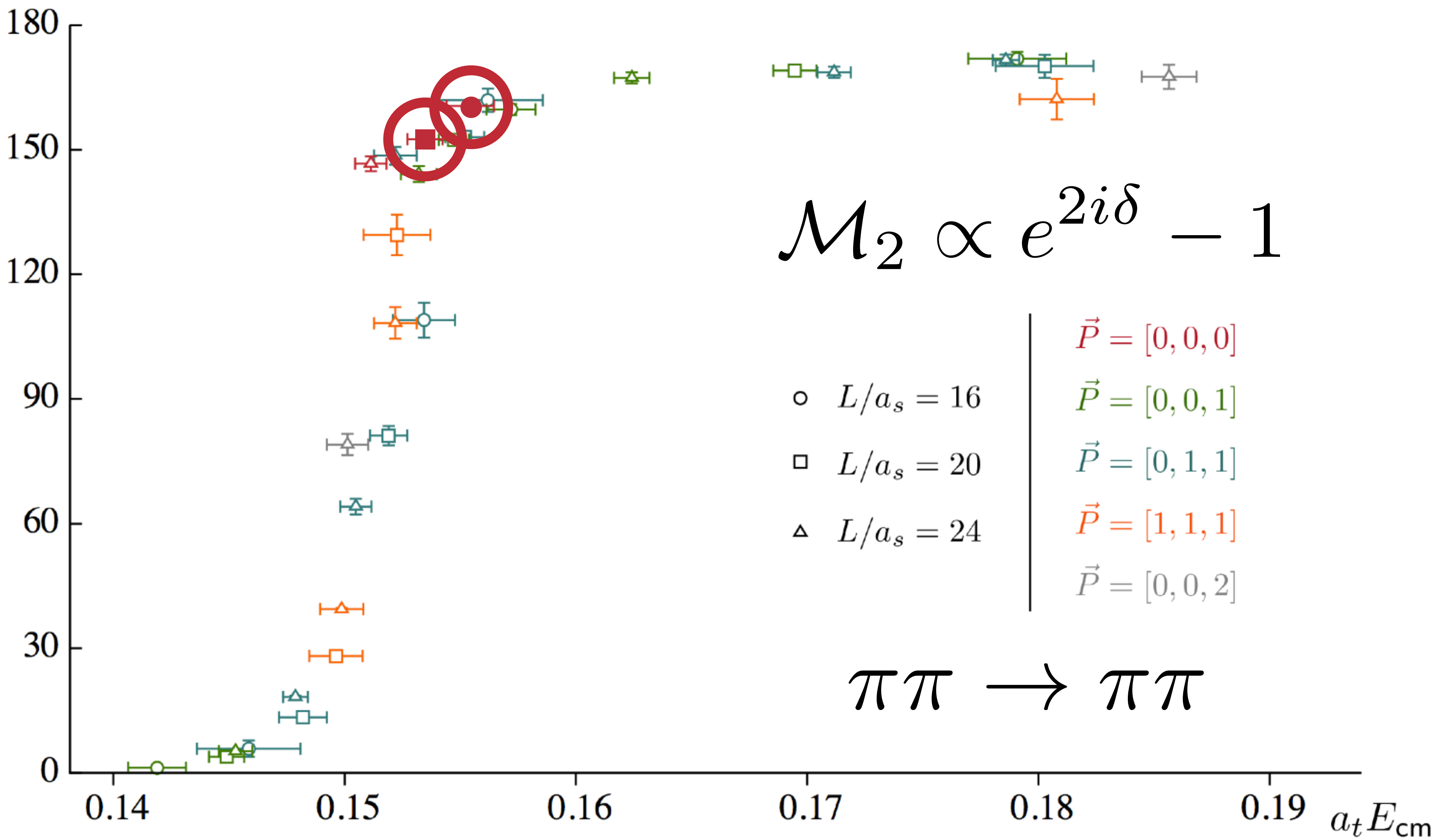
Using the result

- Simplest case is a single channel
  (e.g. for pions in a p-wave the relation reduces to)

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$





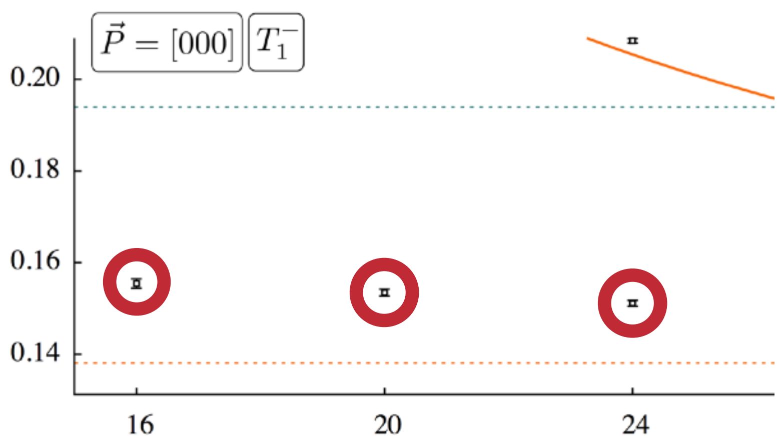


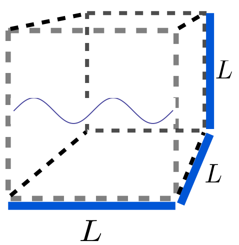
from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

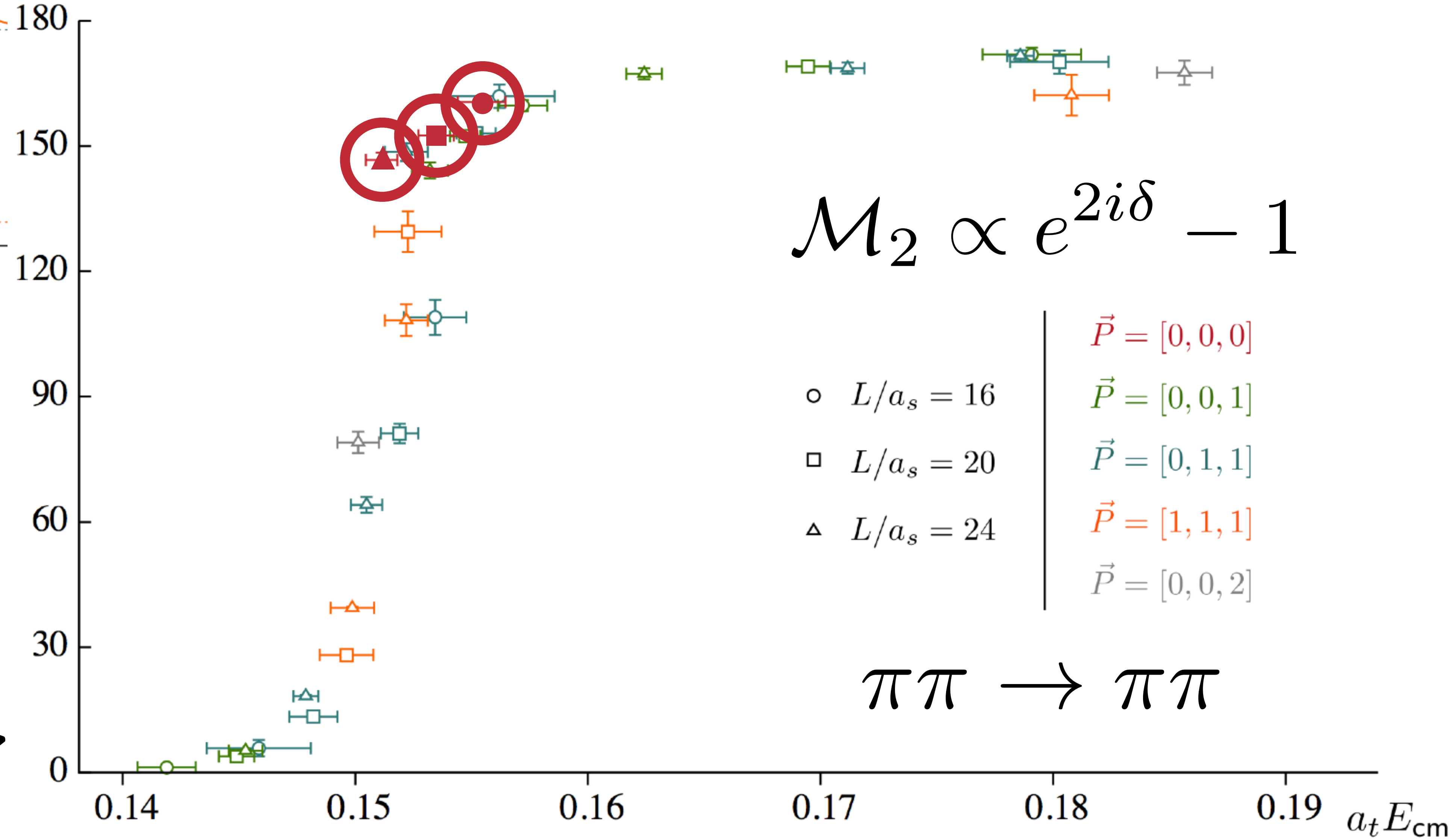
Using the result

- **Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)**

$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$







$$\mathcal{M}_2 \propto e^{2i\delta} - 1$$

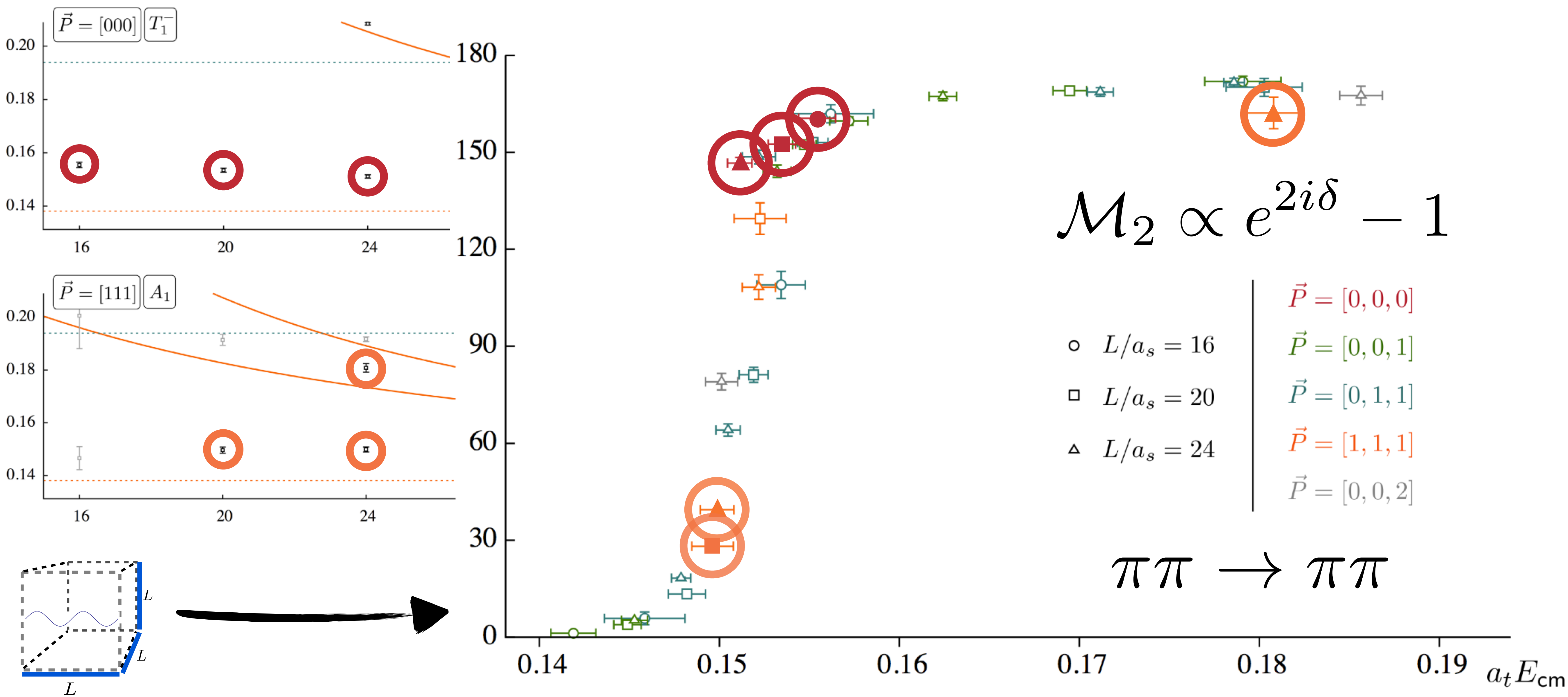
$$\pi\pi \to \pi\pi$$

from Dudek, Edwards, Thomas in *Phys.Rev*. D87 (2013) 034505

Using the result

- **Simplest case is a single channel (e.g. for pions in a p-wave the relation reduces to)**

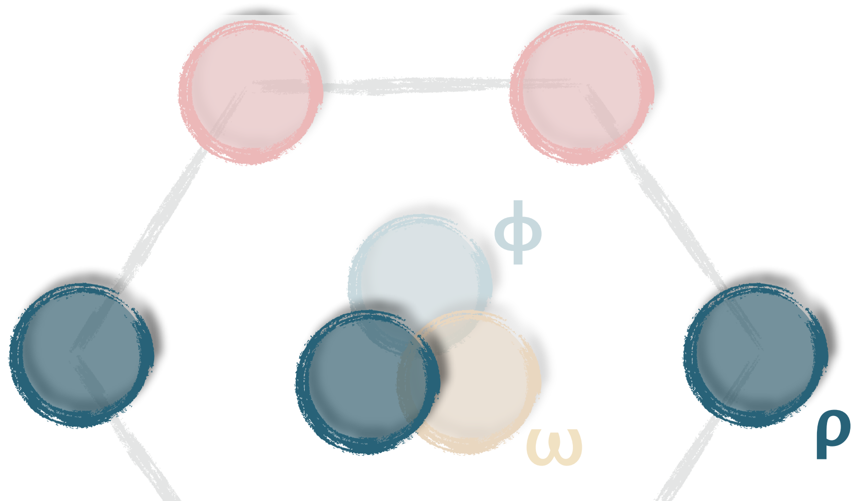
$$\mathcal{M}_2(E_n^*) = -1/F(E_n, \vec{P}, L)$$

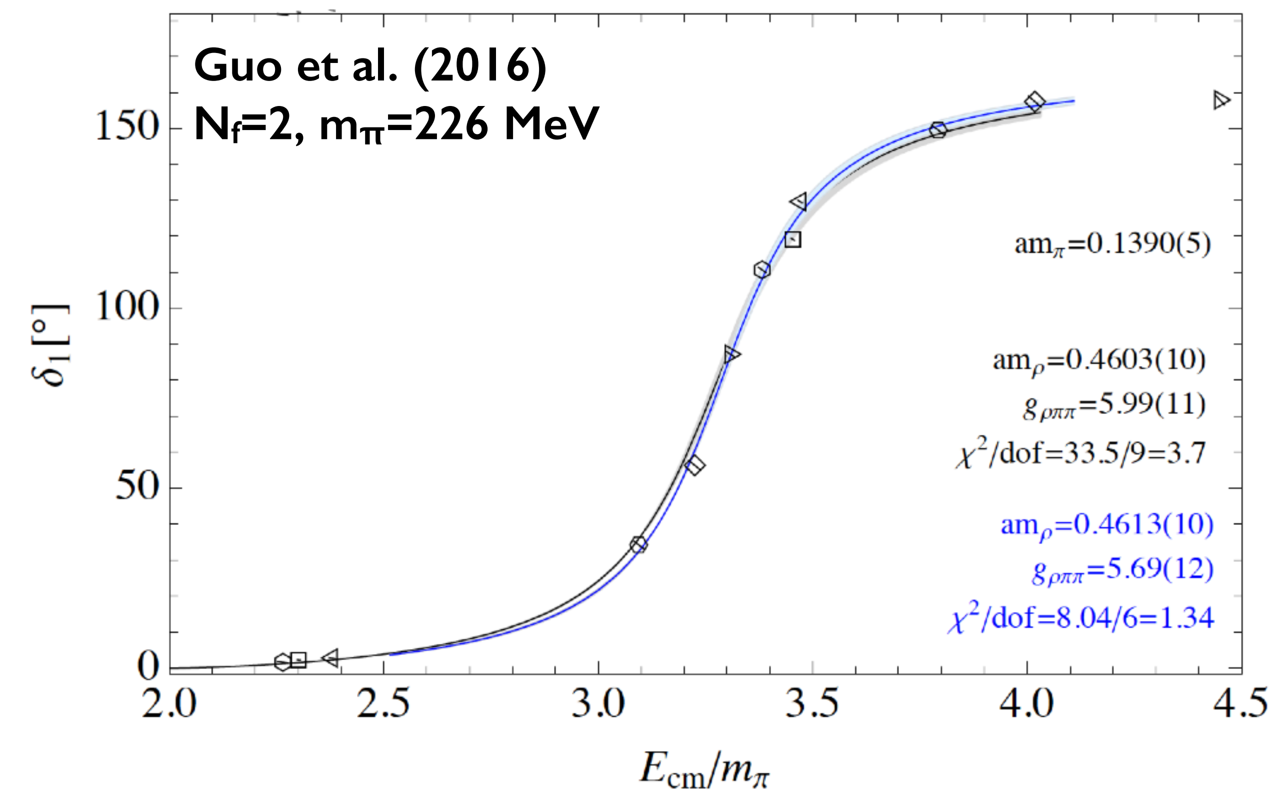


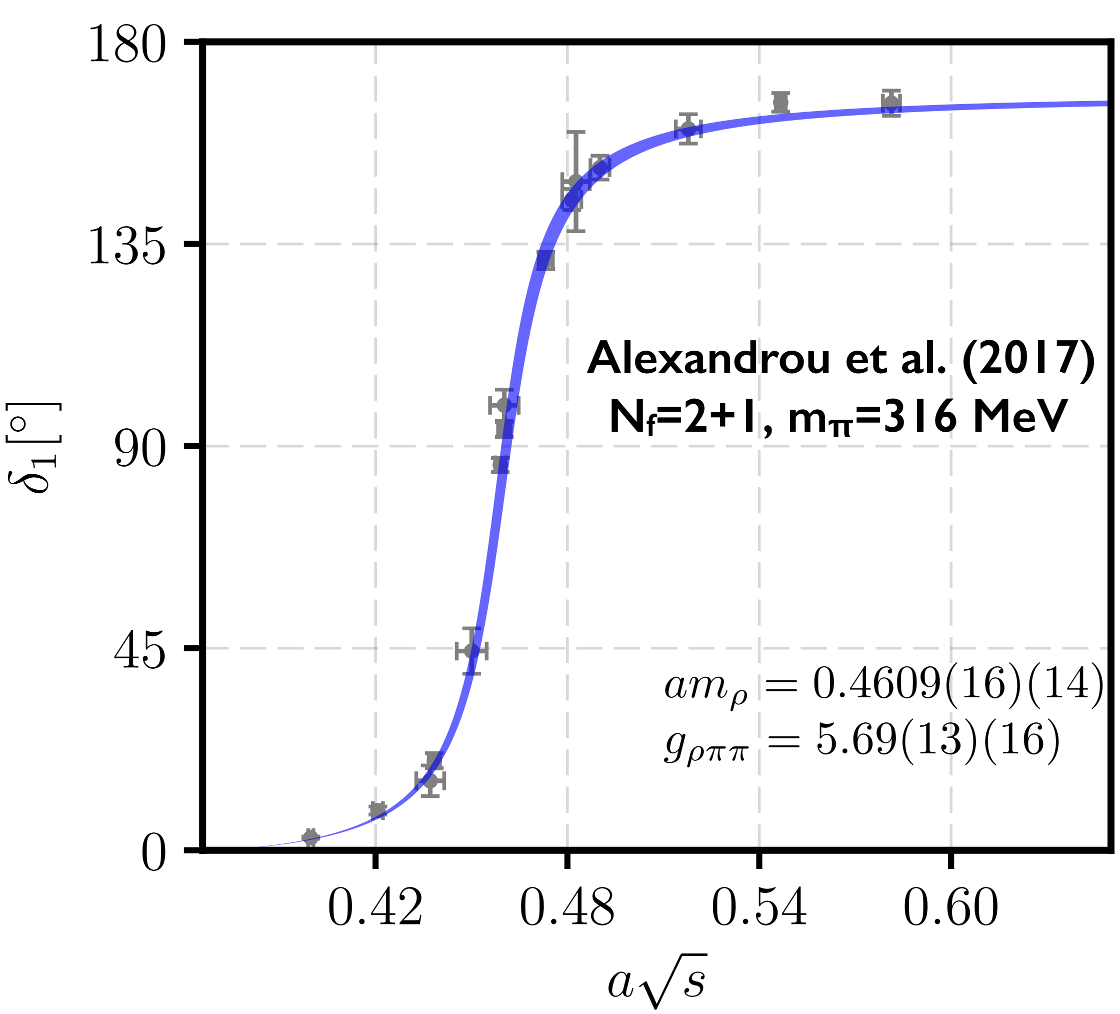
from Dudek, Edwards, Thomas in *Phys.Rev.* D87 (2013) 034505

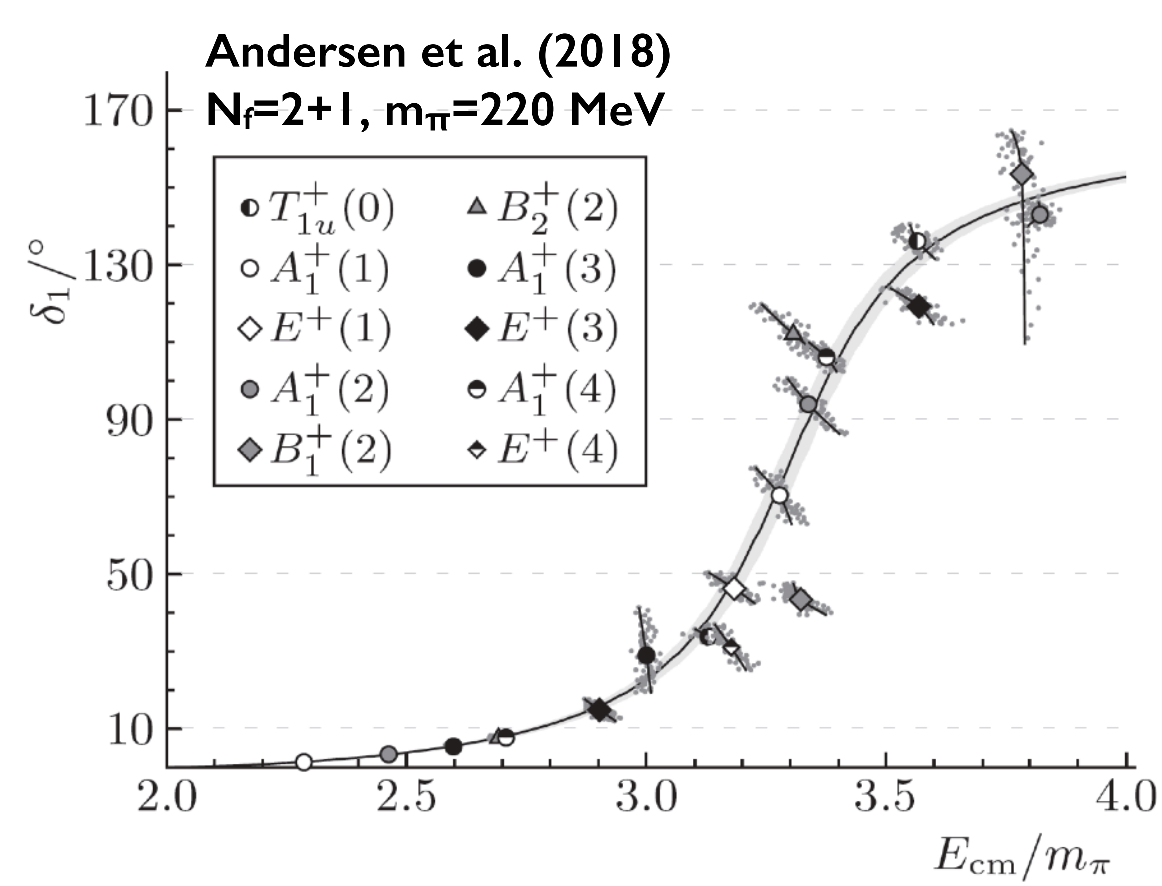
$$\rho \to \pi\pi$$

$$I^G(J^{PC}) = 1^+(1^{--})$$





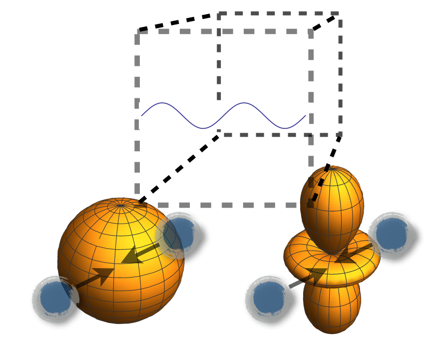




Coupled channels

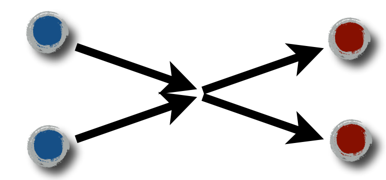
- The cubic volume mixes different partial waves…

e.g. $K\pi \to K\pi$, $\vec{P} \neq 0$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_s^{-1} & 0 \\ 0 & \mathcal{M}_p^{-1} \end{pmatrix} + \begin{pmatrix} F_{ss} & F_{sp} \\ F_{ps} & F_{pp} \end{pmatrix}\right] = 0$



…as well as different flavor channels…

e.g. $a = \pi\pi$, $b = K\overline{K}$ $\longrightarrow$ $\det\left[\begin{pmatrix} \mathcal{M}_{a\to a} & \mathcal{M}_{a\to b} \\ \mathcal{M}_{b\to a} & \mathcal{M}_{b\to b} \end{pmatrix}^{-1} + \begin{pmatrix} F_a & 0 \\ 0 & F_b \end{pmatrix}\right] = 0$



- The road to physics…

Calculate a matrix of correlators with a large & varied operator basis

$\langle \mathcal{O}_a(\tau)\mathcal{O}_b^\dagger(0)\rangle$

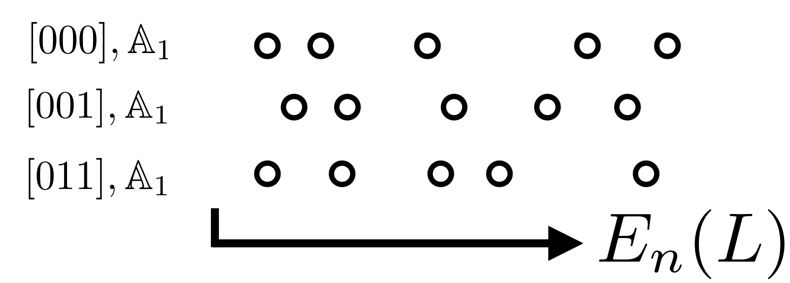
→

Diagonalize (GEVP) to reliably extract finite-volume energies

$\langle \Omega_m(\tau)\Omega_m^\dagger(0)\rangle \sim e^{-E_m(L)\tau}$

→

Vary *L* and *P* to recover a dense set of energies



↓

Identify a broad list of K-matrix parametrizations

polynomials and poles

EFT based

dispersion theory based

→

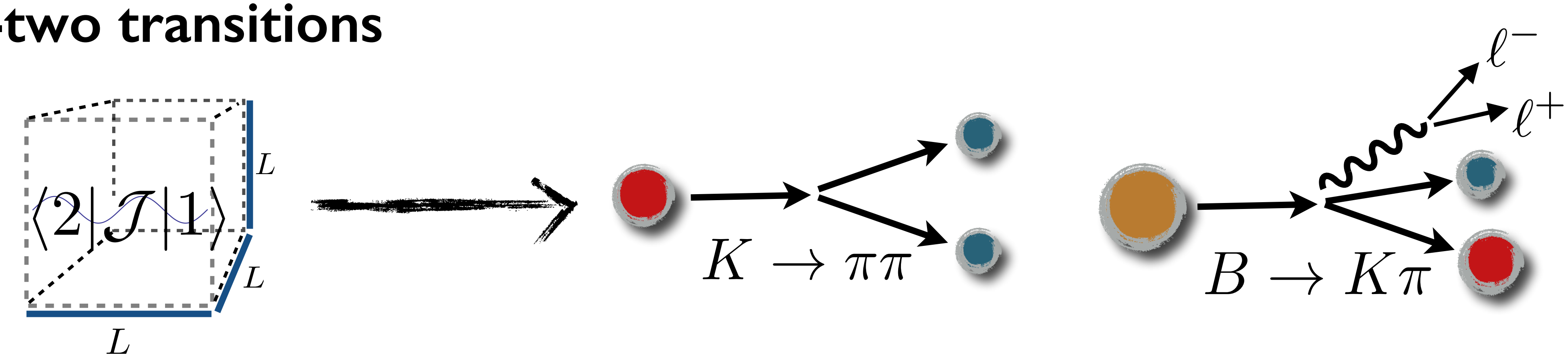
Perform global fits to the finite-volume spectrum

Multi-hadron processes from LQCD

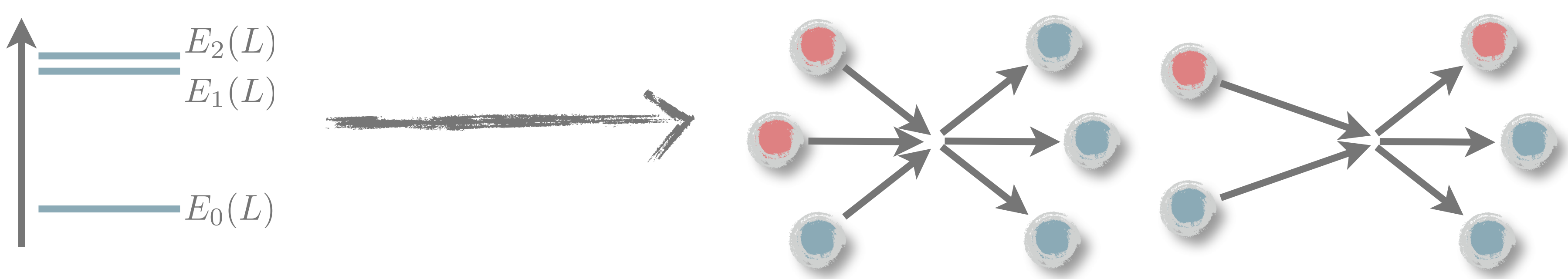
KEY IDEA: We can use the finite volume as a **tool** to extract multi-hadron observables

- [x] Two-to-two scattering

- [ ] **One-to-two transitions**

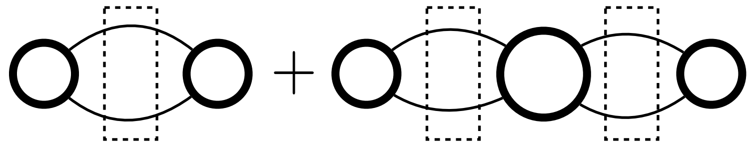


- [ ] Two-to-three and three-to-three scattering



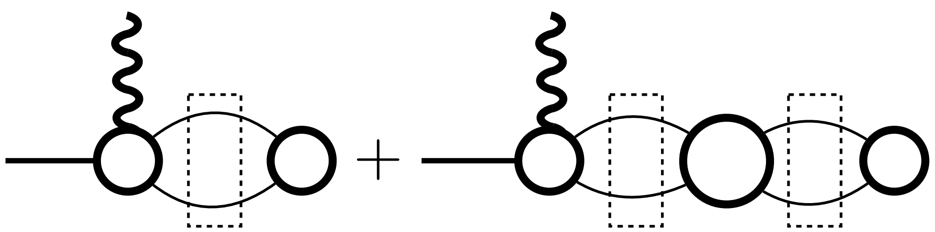
Transition matrix elements…

- Recall how we derived the relation for extracting scattering

$C_L(P) \equiv$  $+ \cdots$

$\det\left[\mathcal{M}_2^{-1}(E_n^*) + F(E_n, \vec{P}, L)\right] = 0$

- One can use a similar analysis to derive a relation on matrix elements

$C_L^{1+\mathcal{J}\to 2}(P) \equiv$  $+ \cdots$

get this from the lattice

experimental observable

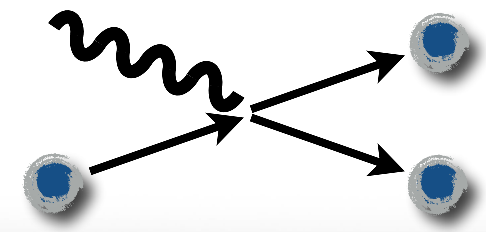
$$|\langle n, L|\mathcal{J}_\mu|B\rangle|^2 = \langle B|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, L) \langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|B\rangle$$

depends on scattering phase

Briceño, MTH, Walker-Loud (2015)

- Important caveat: The relation holds only for $s_{\pi\pi} < (4M_\pi)^2$
- Again, the results ignores (drops) suppressed volume effects ($e^{-M_\pi L}$)

Pion photo-production

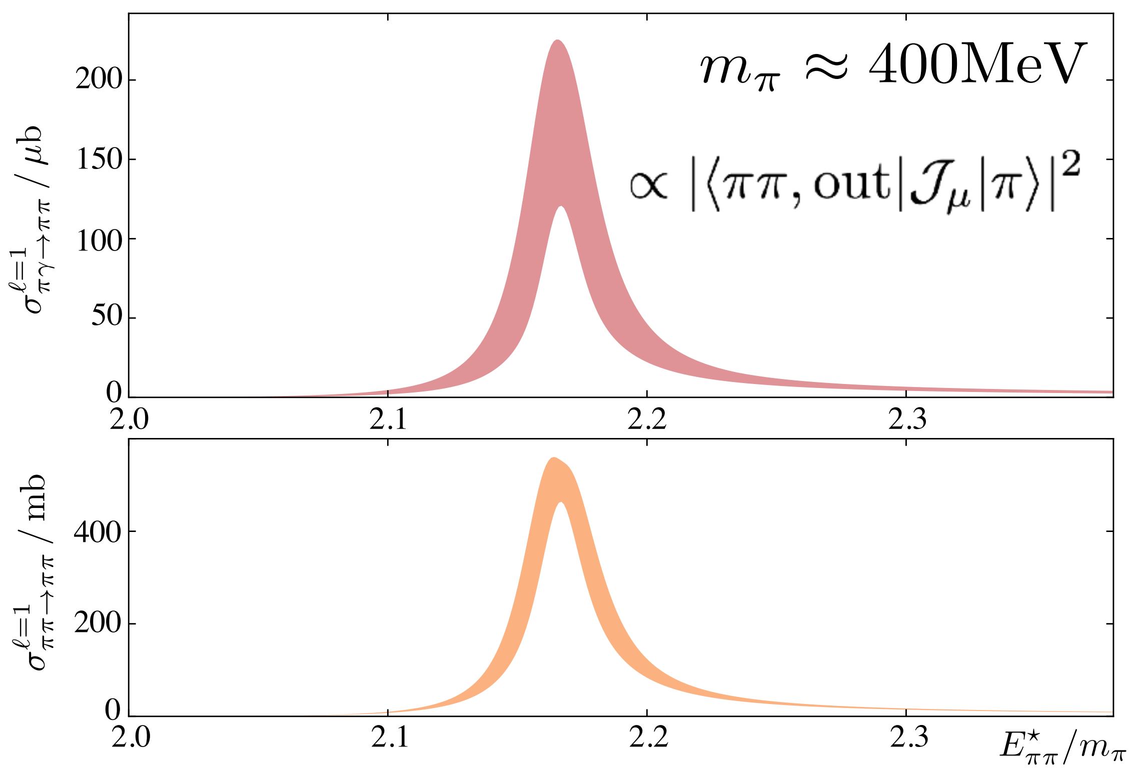
$\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle \equiv$ 

get this from the lattice **experimental observable**

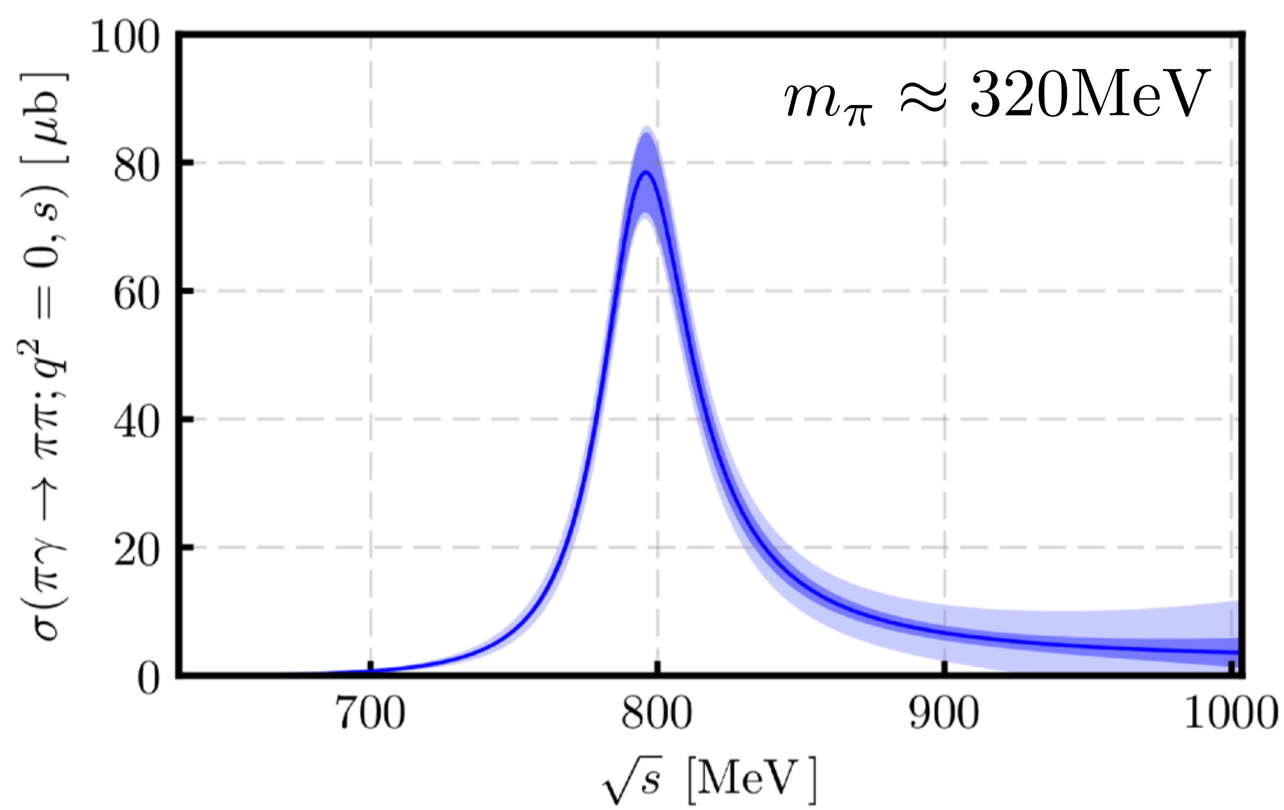
$$|\langle n, L|\mathcal{J}_\mu|\pi\rangle|^2 = \langle \pi|\mathcal{J}_\mu|\pi\pi, \mathrm{in}\rangle \mathcal{R}(E_n, L)\langle \pi\pi, \mathrm{out}|\mathcal{J}_\mu|\pi\rangle$$

depends on scattering phase

- **The formalism has now been applied in two LQCD calculations**

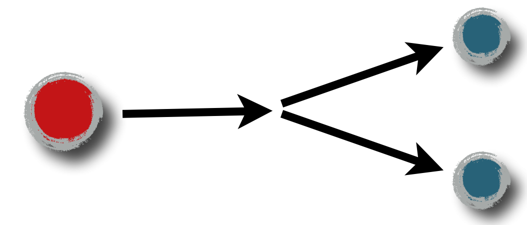


Briceño et. al., Phys. Rev. D93, 114508 (2016)

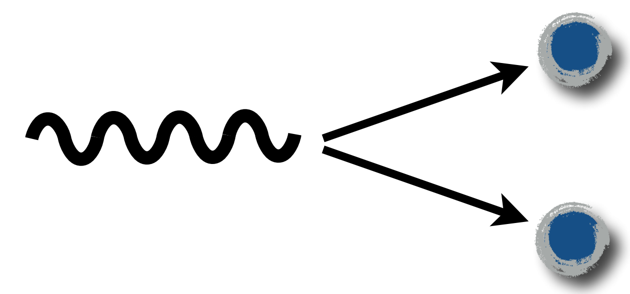


Alexandrou et. al., Phys. Rev. D98, 074502 (2018)

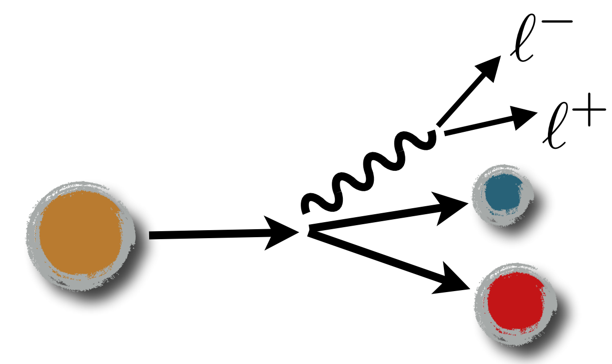
Same basic idea in many different contexts…

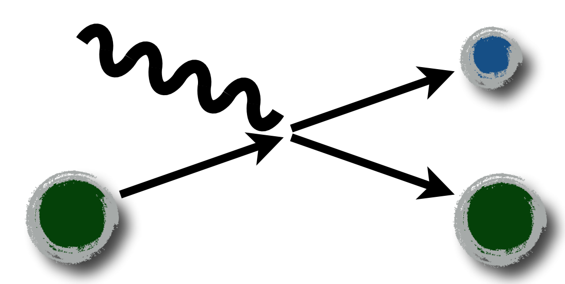
Kaon decay $\langle \pi\pi, \mathrm{out}|\mathcal{H}|K\rangle \equiv$ 

Lellouch, Lüscher (2001) ○ Kim, Sachrajda, Sharpe (2005) ○ Christ, Kim, Yamazaki (2005)

Time-like form factors $\langle \pi\pi,\ \mathrm{out}|\mathcal{J}_\mu|0\rangle \equiv$ 

Meyer (2011)

Resonance transition amplitudes $\langle K\pi, \mathrm{out}|\mathcal{J}_{\alpha\beta}|B\rangle \equiv$ 

Particles with spin $\langle N\pi, \mathrm{out}|\mathcal{J}_\mu|N\rangle \equiv$ 

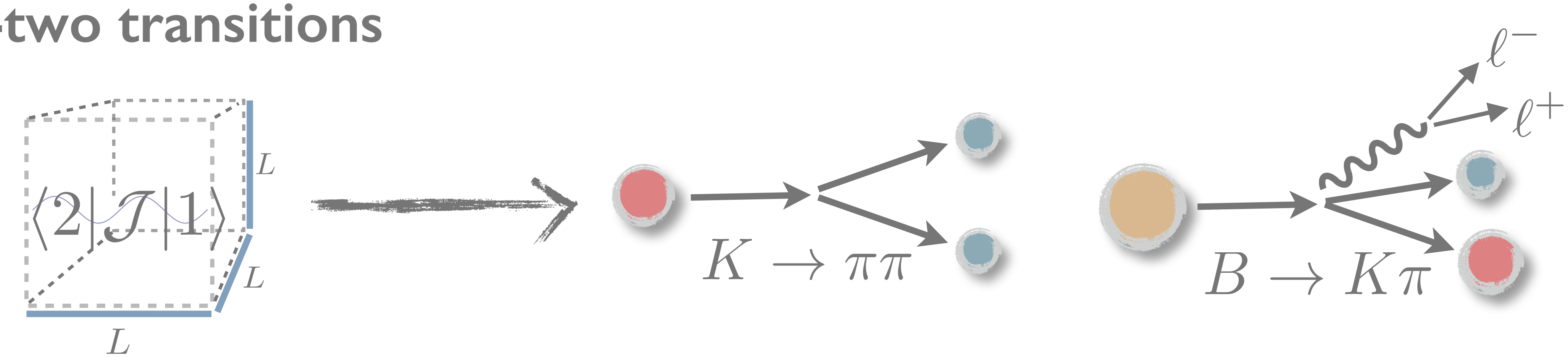
Agadjanov *et al.* (2014) ○ Briceño, MTH, Walker-Loud (2015) ○ Briceño, MTH (2016)

Multi-hadron processes from LQCD

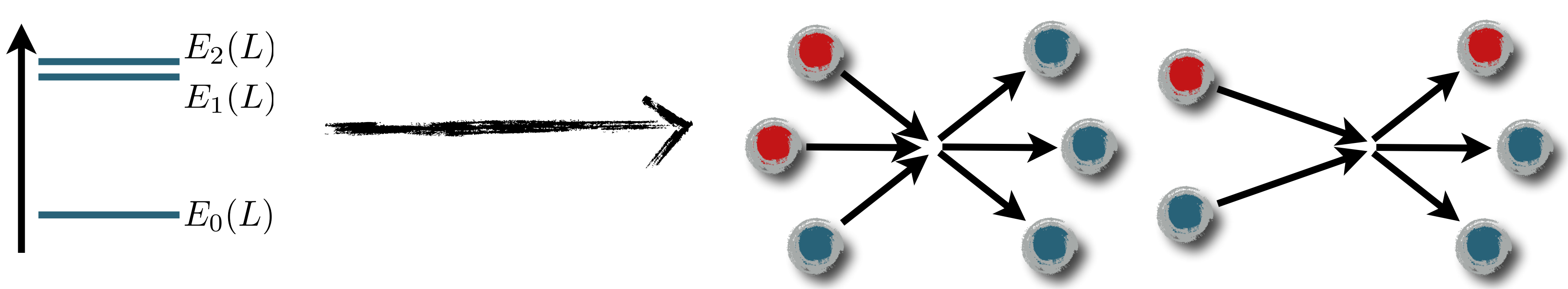
Key Idea: We can use the finite volume as a **tool** to extract multi-hadron observables

☑ Two-to-two scattering

☑ One-to-two transitions

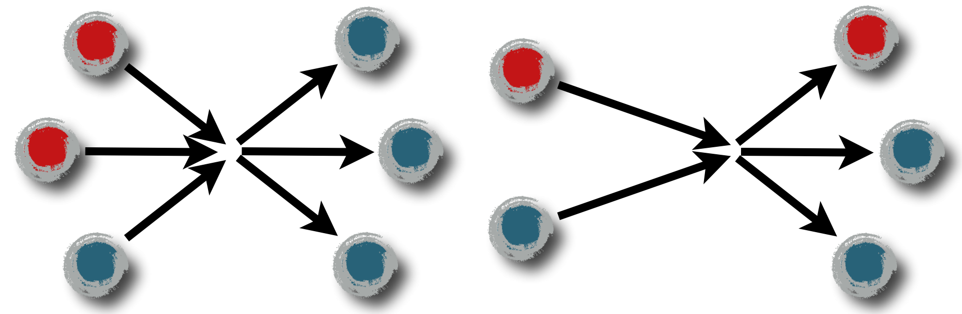


☐ Two-to-three and three-to-three scattering



Three-particle scattering

The aim is to derive a formalism for studying relativistic two- and three-particle systems from lattice QCD



Potential applications...

- **Studying three-particle resonances**

$$\omega(782),\ a_1(1420) \to \pi\pi\pi \qquad \eta(1405) \to a_0(980)\pi$$

$$N(1440) \to N\pi, N\pi\pi \qquad \eta(1475) \to K^*(892)\overline{K}$$

- **Calculating weak decays, form factors and transitions**

$$K \to \pi\pi\pi \qquad N\gamma^* \to N\pi\pi$$

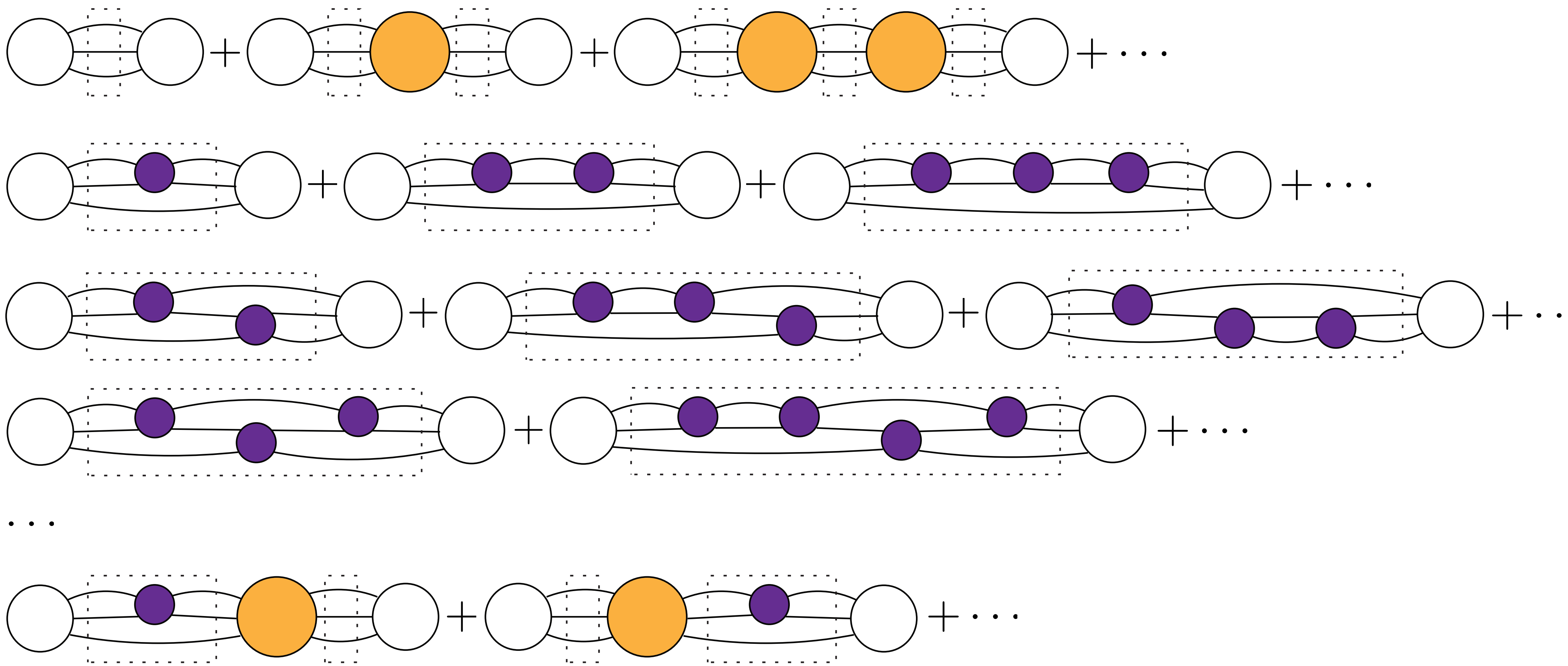
Current status

- **Formalism is complete for two and three (identical) scalars**

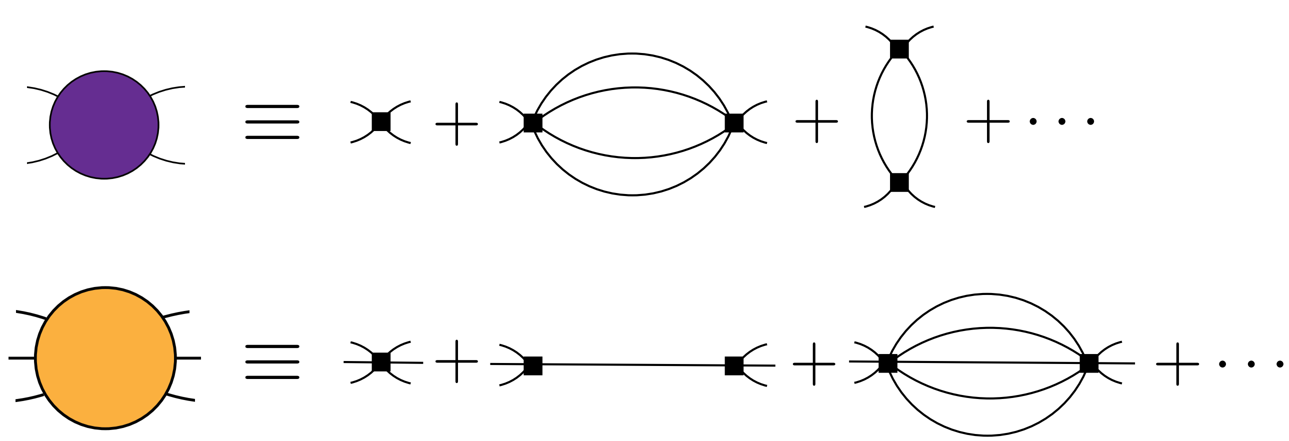
MTH, Sharpe (2014-2016) ○ Briceño, MTH, Sharpe (2017, 2018)

Skeleton expansion

$C_L(E, \vec{P}) =$



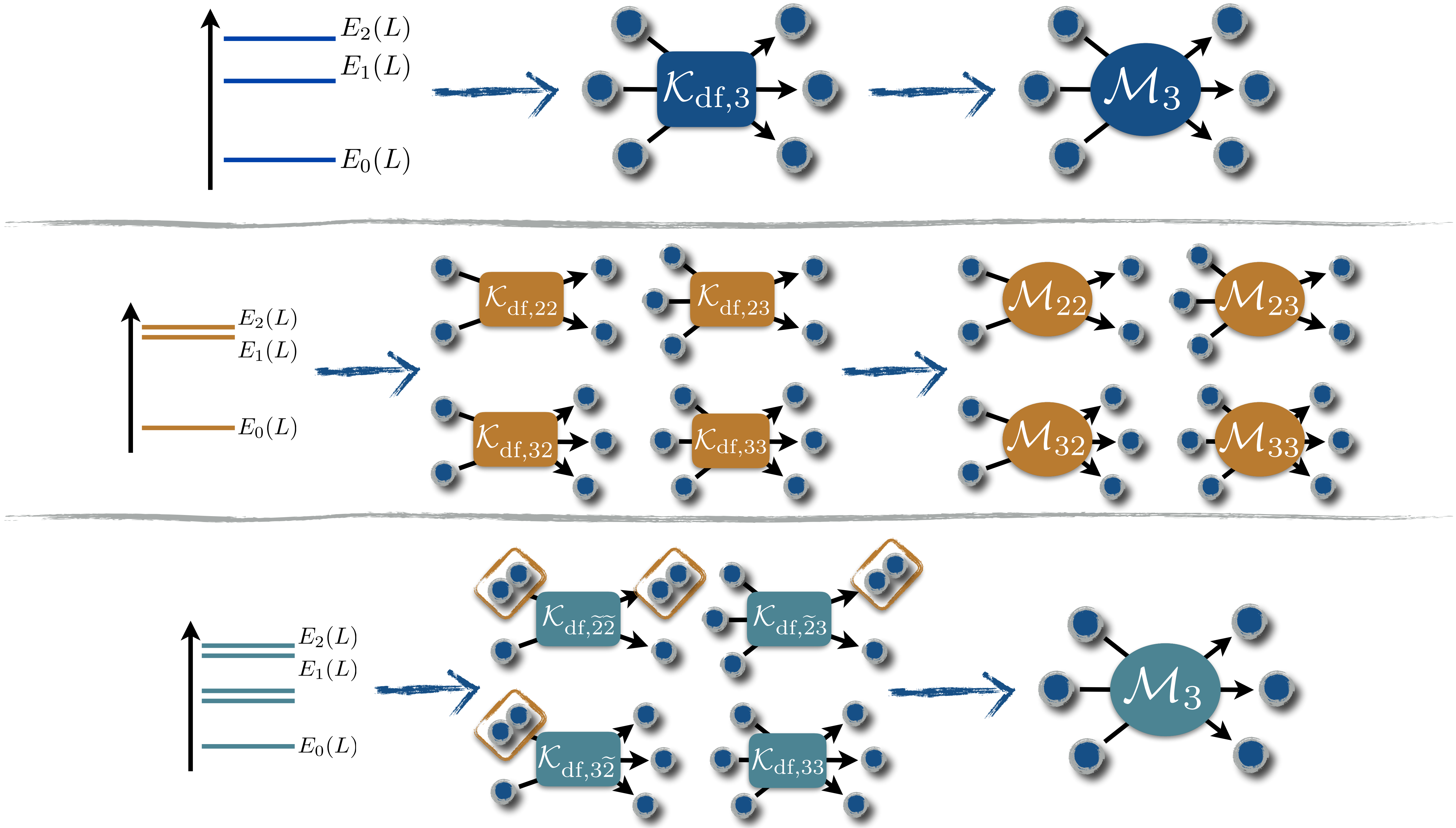
Kernel definitions:



- All lines represent pions
- Boxes represent sums over finite-volume momenta

Quantization conditions

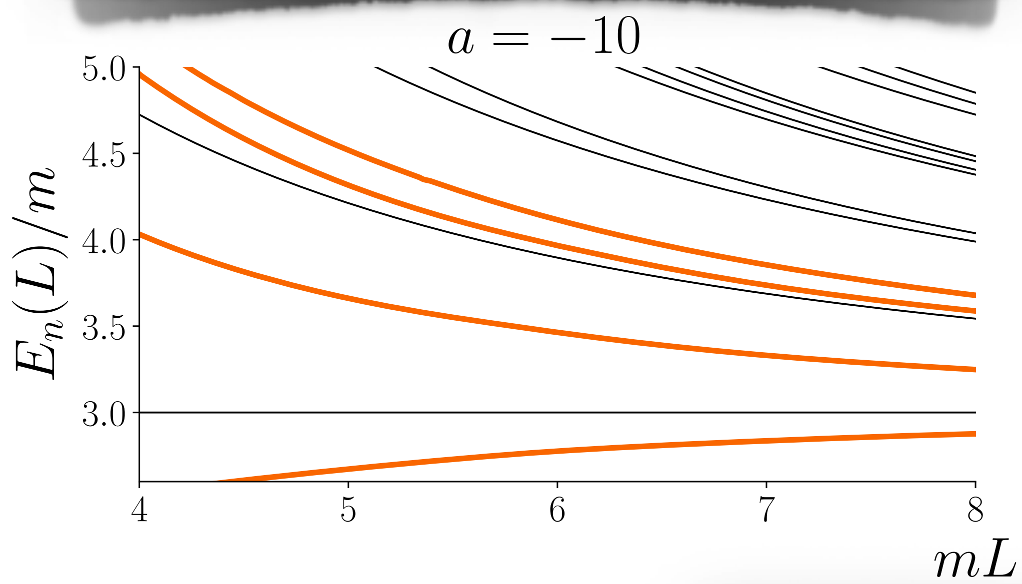
- **Scattering observables via an intermediate quantity**



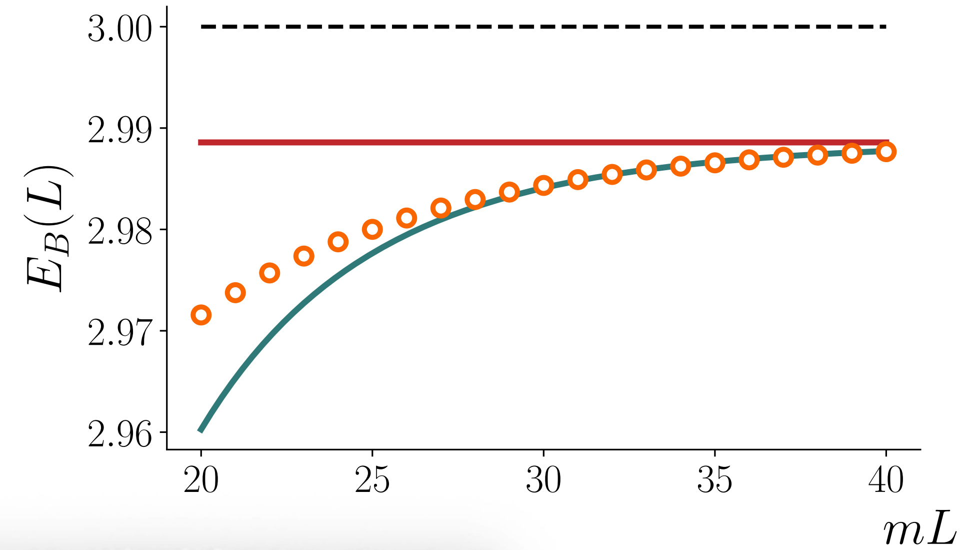
MTH, Sharpe (2014-2016) ○ Briceño, MTH, Sharpe (2017, 2018)

Toy numerics

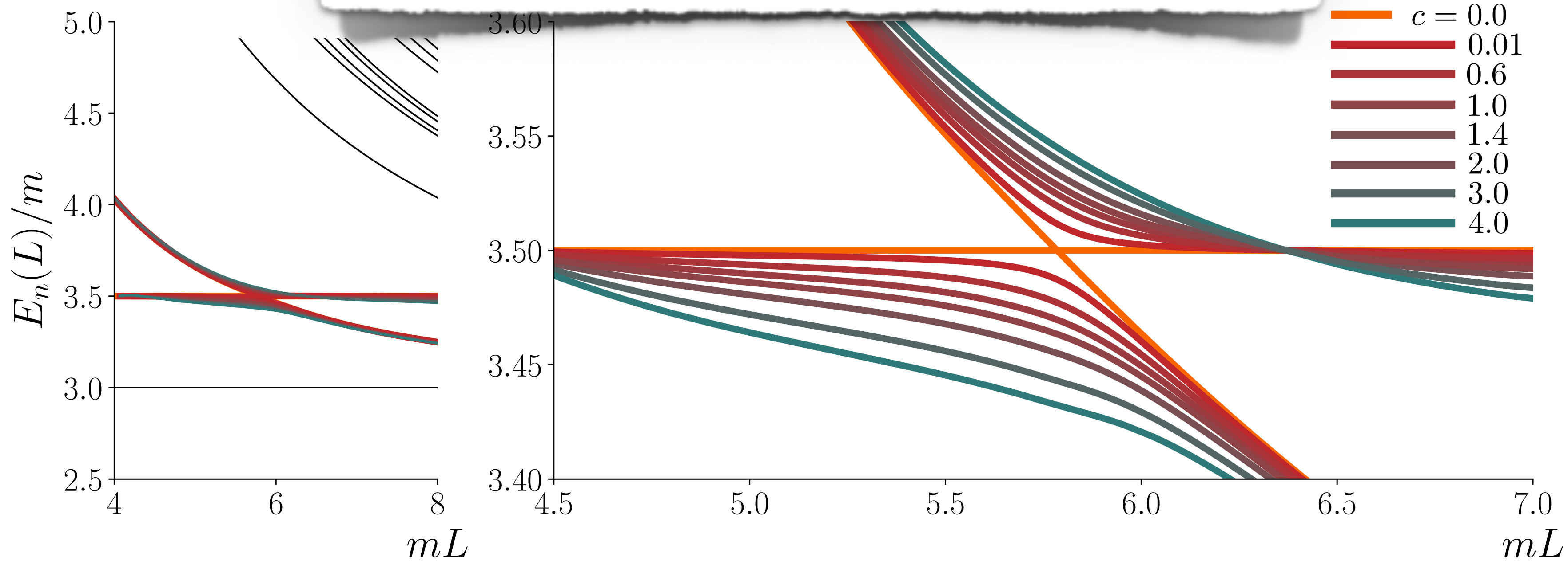
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



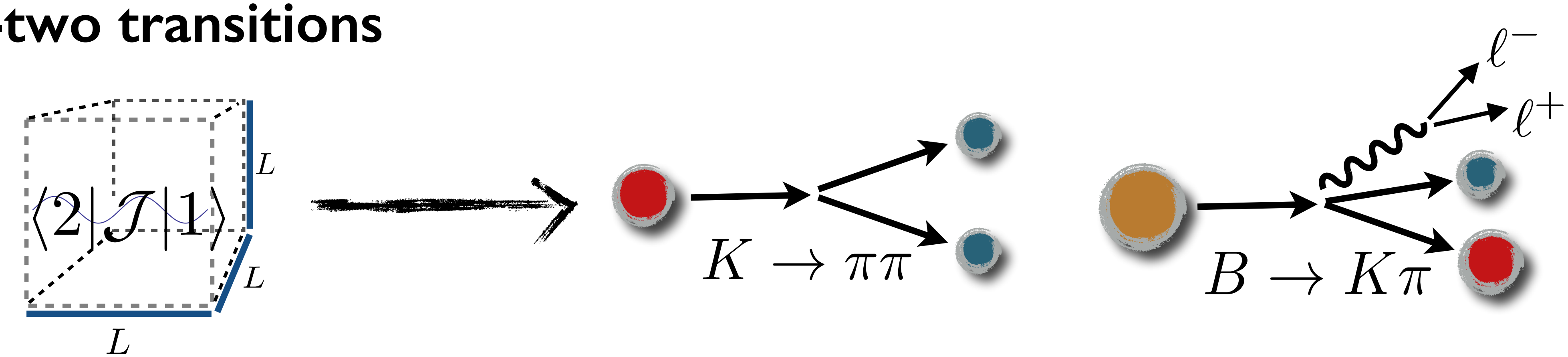
Briceño, MTH, Sharpe (2017)

Multi-hadron processes from LQCD

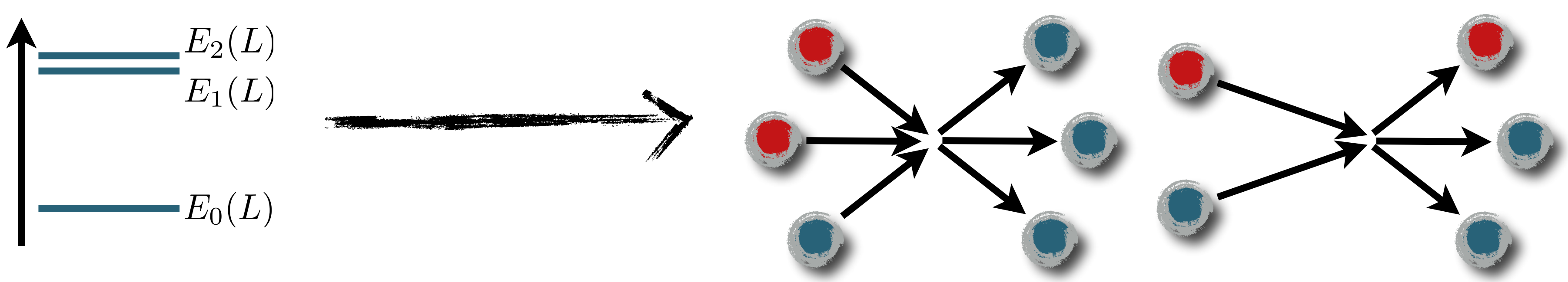
KEY IDEA: **We can use the finite volume as a tool to extract multi-hadron observables**

Two-to-two scattering

One-to-two transitions



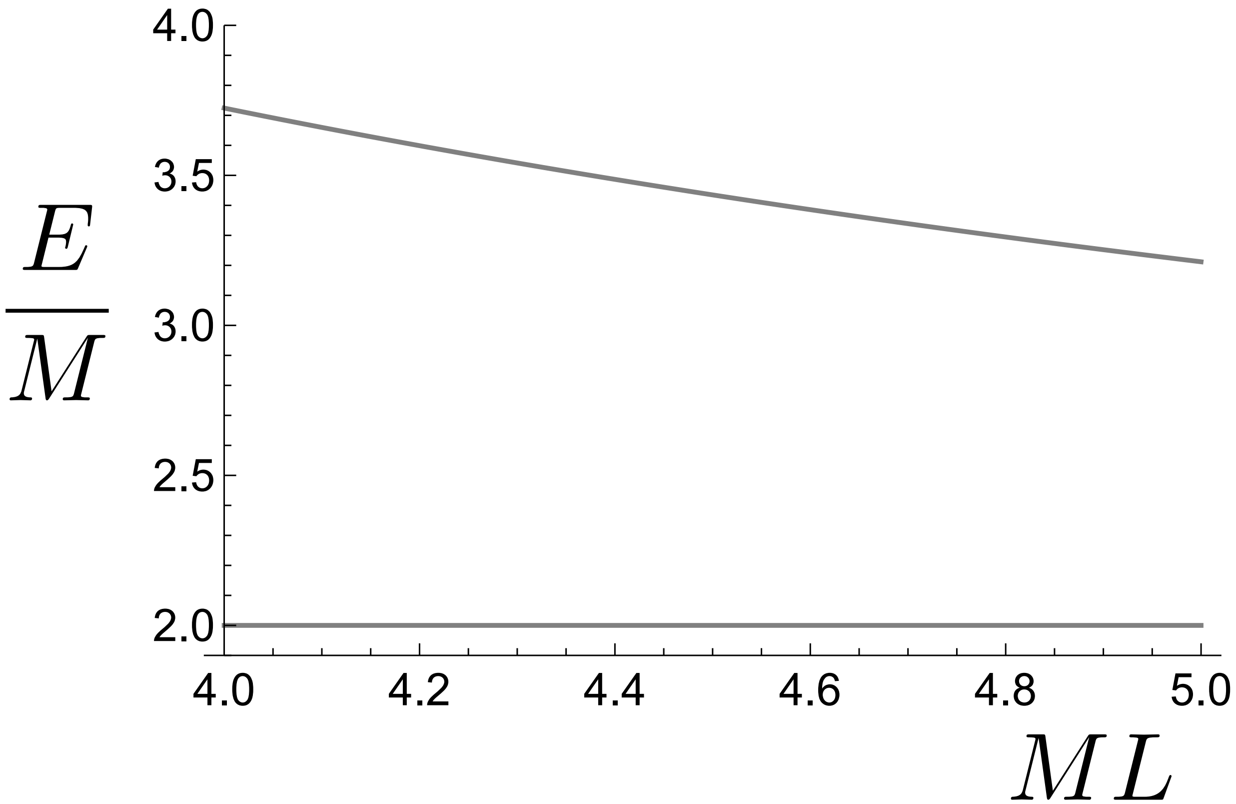
Two-to-three and three-to-three scattering



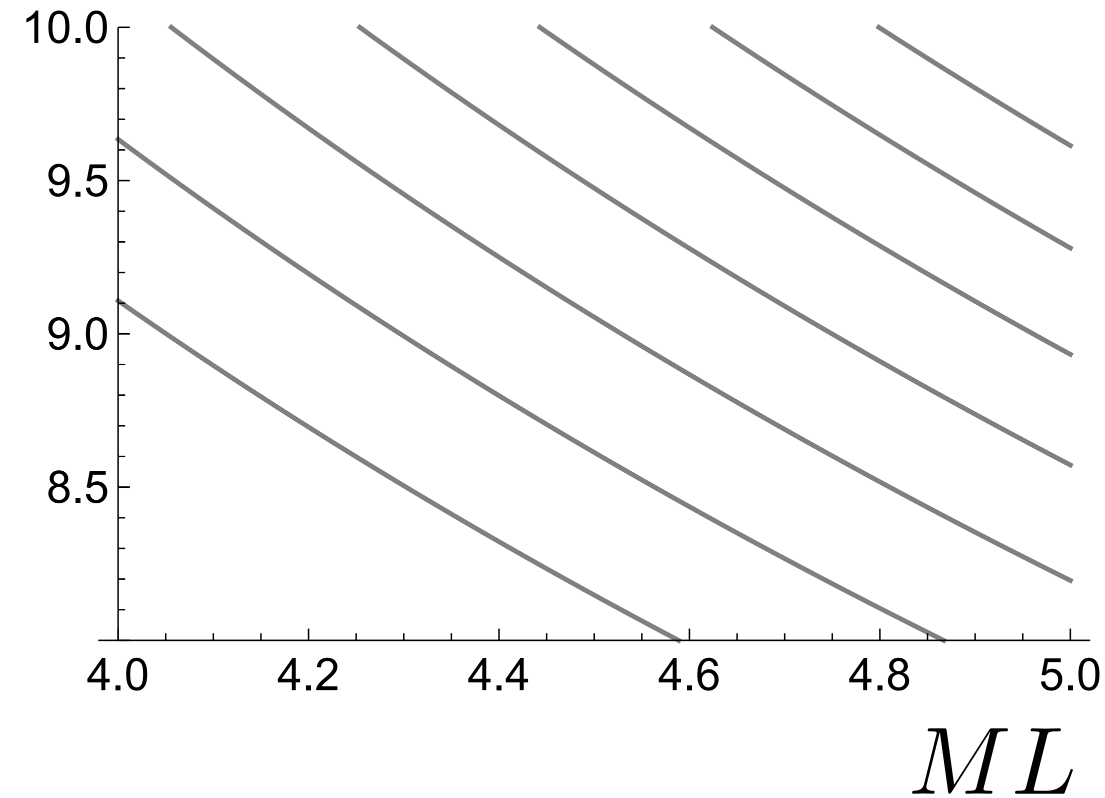
Taking a step back…

- **How far can we take this?**
- **My *speculation*: This method could be generalized to an *n*-particle quantization condition depending on a multi-channel $K_{df,3}$**
- **But using it will be challenging due to the rapid growth of d.o.s.**

2-particle states



2-particle states

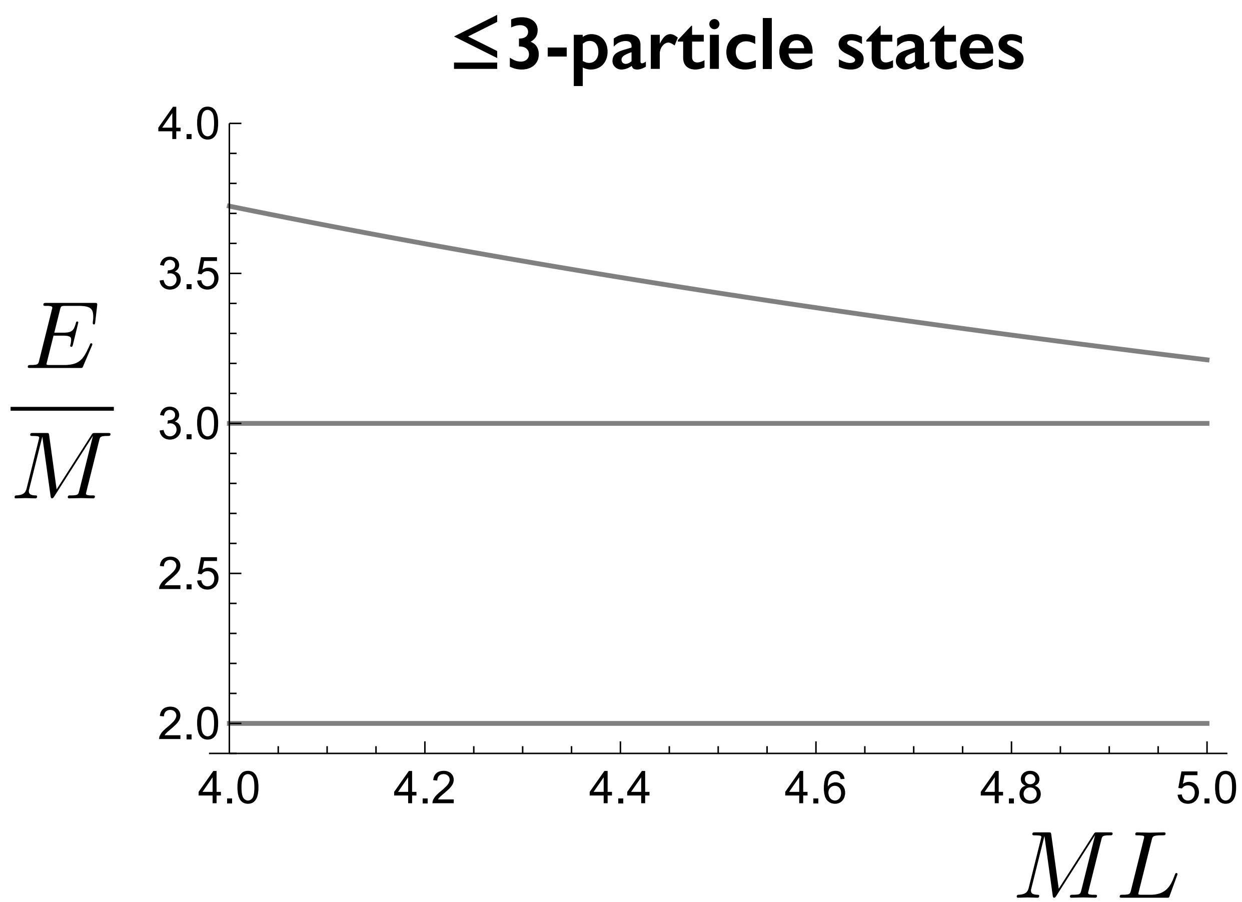


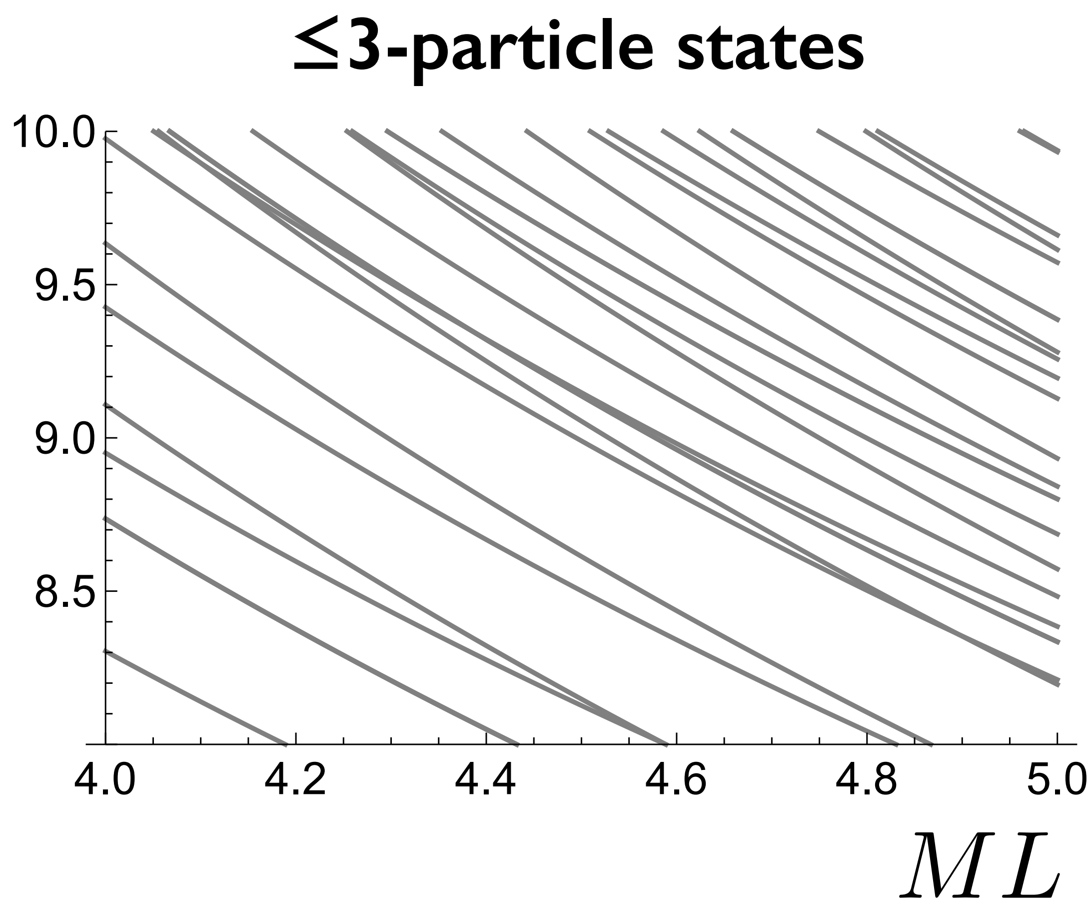
Taking a step back…

- How far can we take this?

- My *speculation*: This method could be generalized to an n-particle quantization condition depending on a multi-channel $K_{df,3}$

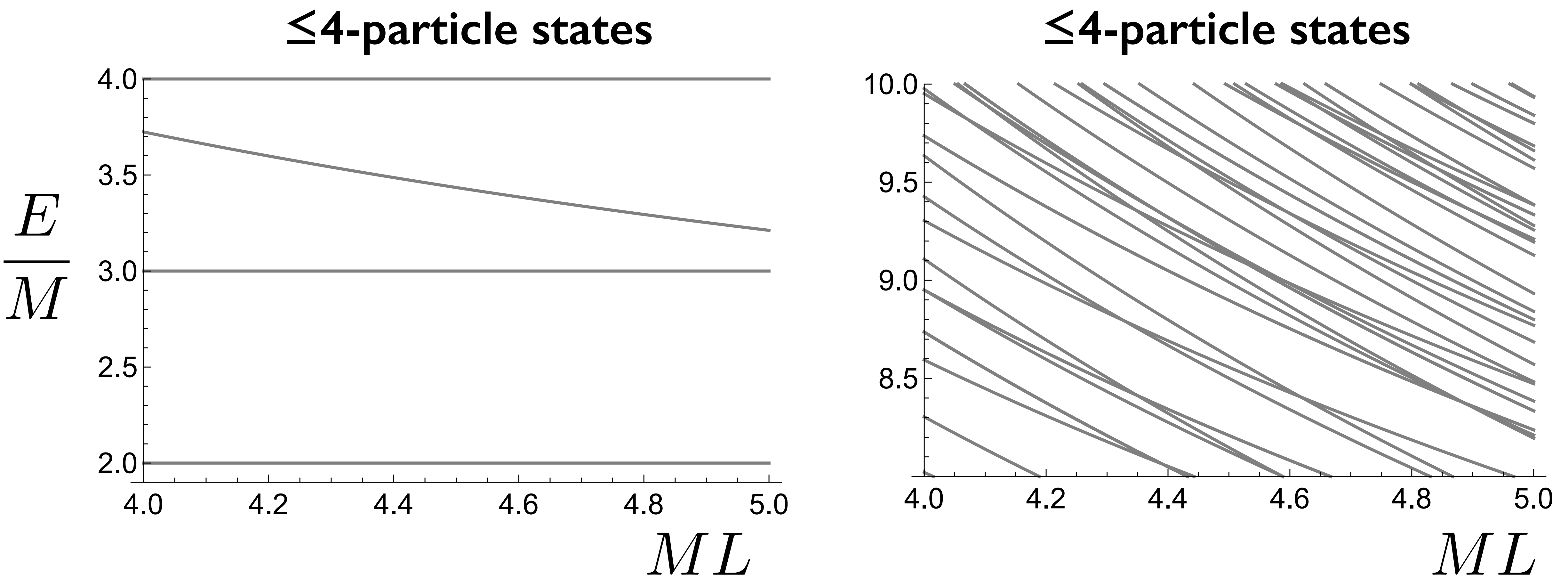
- But using it will be challenging due to the rapid growth of d.o.s.





Taking a step back…

- **How far can we take this?**
- **My *speculation*: This method could be generalized to an *n*-particle quantization condition depending on a multi-channel $K_{df,3}$**
- **But using it will be challenging due to the rapid growth of d.o.s.**



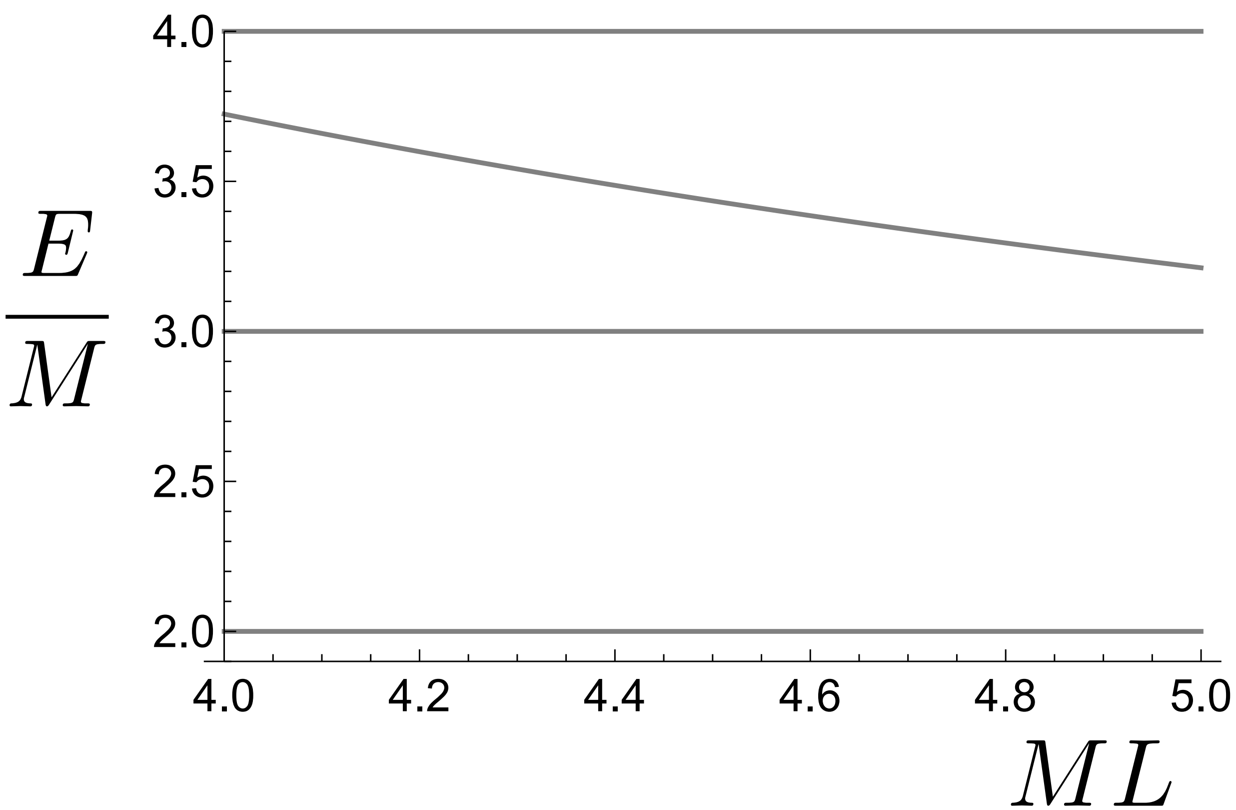
Taking a step back…

- **How far can we take this?**

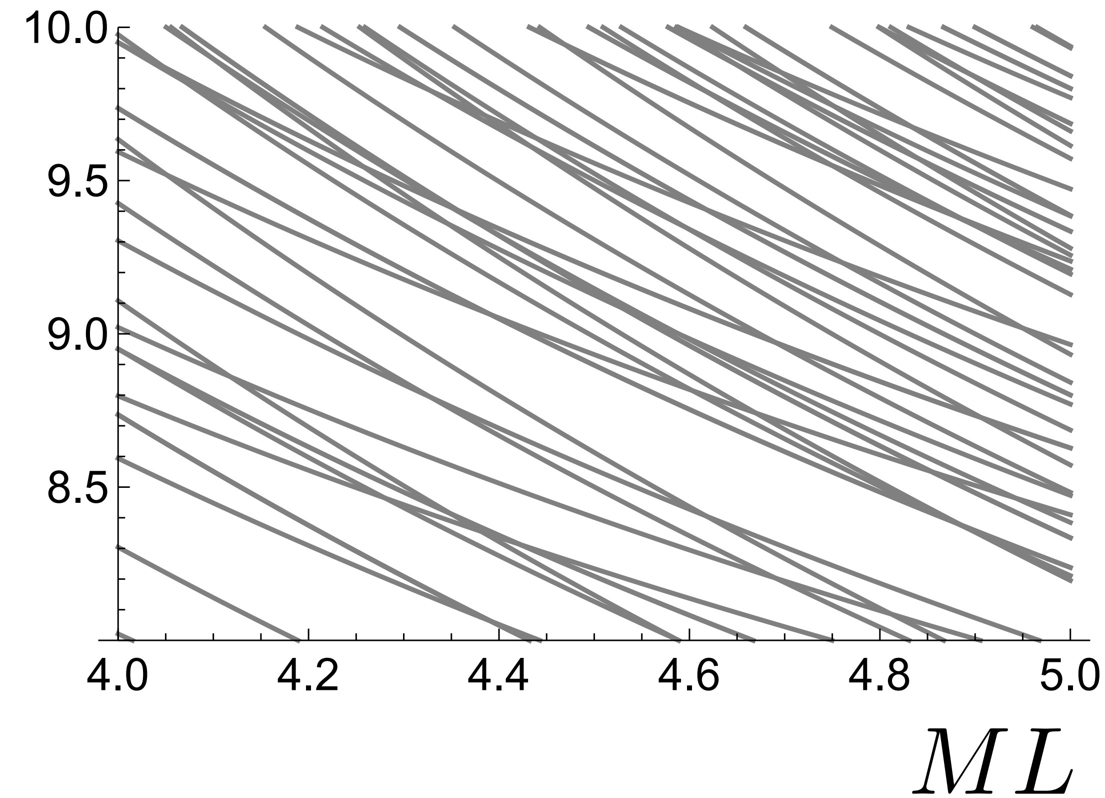
- **My *speculation*: This method could be generalized to an *n*-particle quantization condition depending on a multi-channel $K_{df,3}$**

- **But using it will be challenging due to the rapid growth of d.o.s.**

≤5-particle states



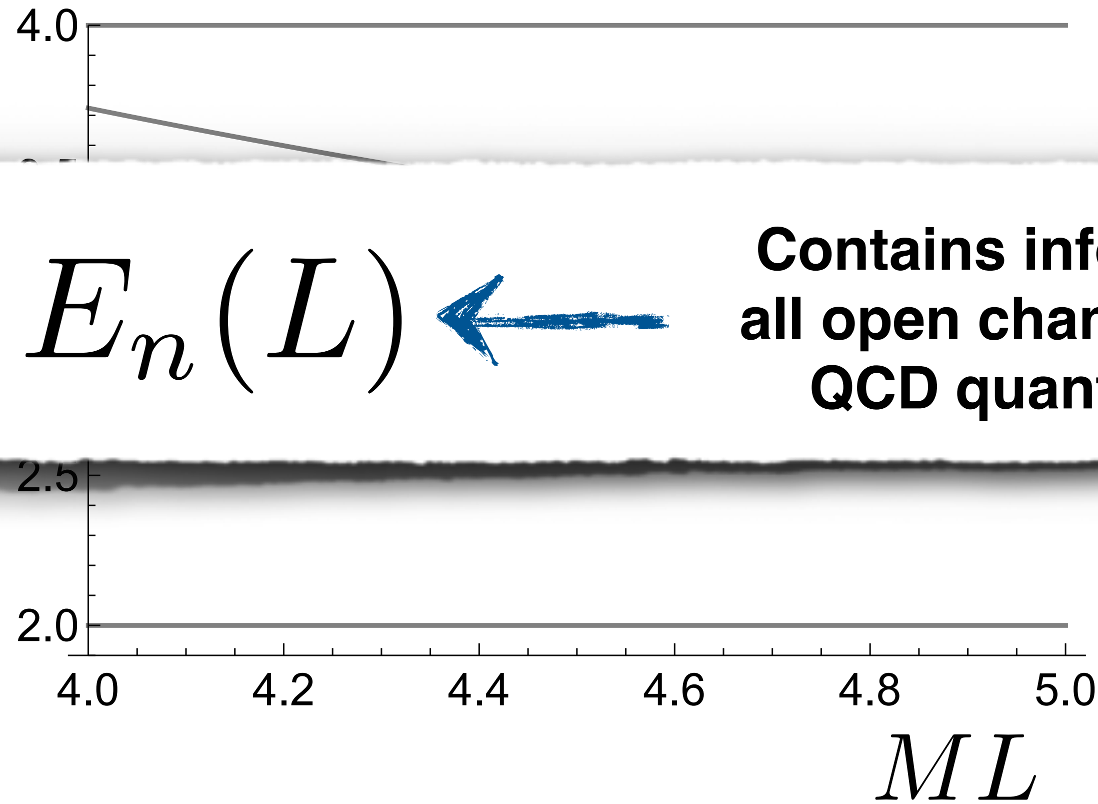
≤5-particle states



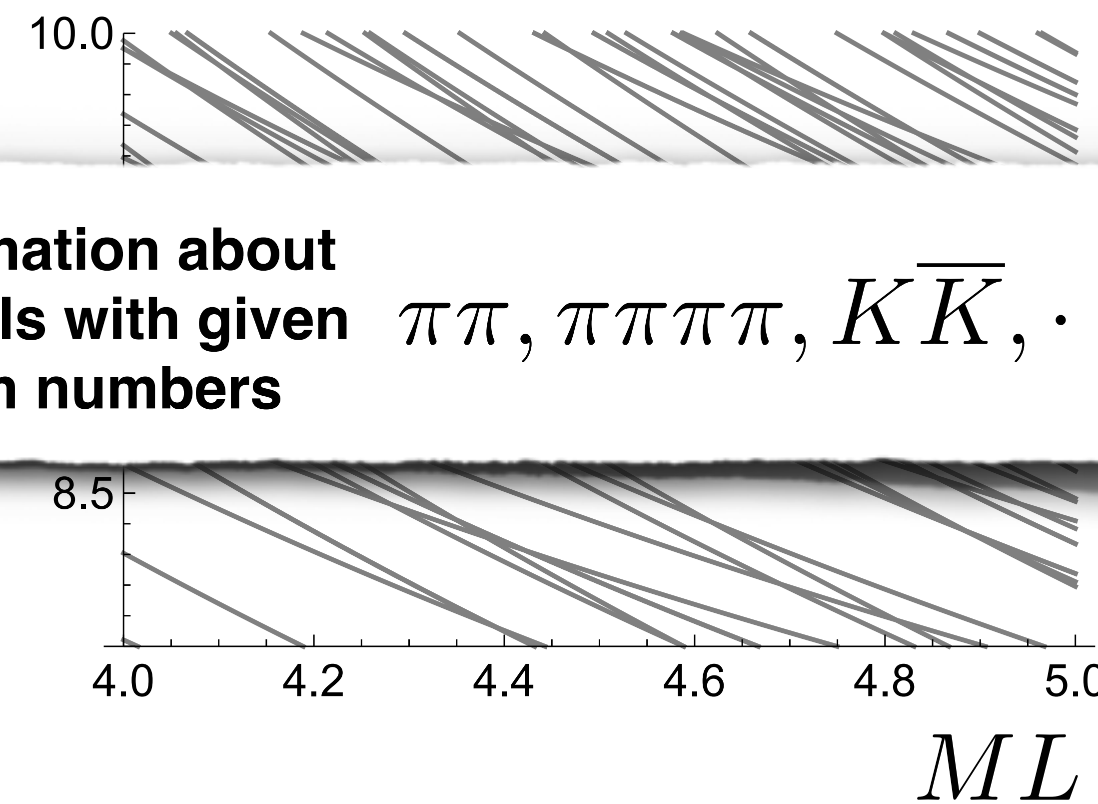
Taking a step back…

- **How far can we take this?**
- **My *speculation*: This method could be generalized to an *n*-particle quantization condition depending on a multi-channel $K_{df,3}$**
- **But using it will be challenging due to the rapid growth of d.o.s.**

≤5-particle states

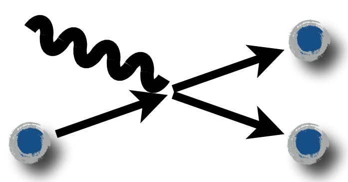
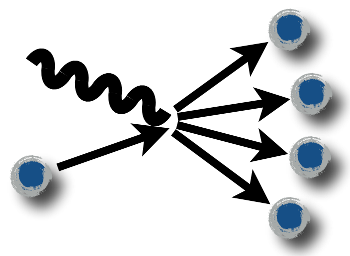
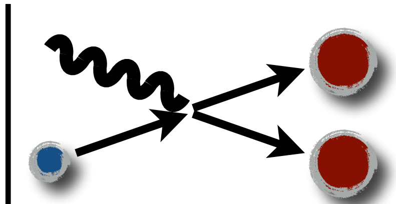


≤5-particle states



$E_n(L) \longleftarrow$ **Contains information about all open channels with given QCD quantum numbers** $\pi\pi, \pi\pi\pi\pi, K\overline{K}, \cdots$

Instead try something inclusive…

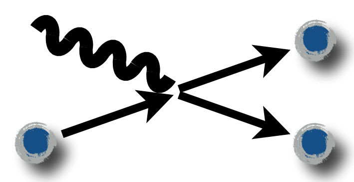
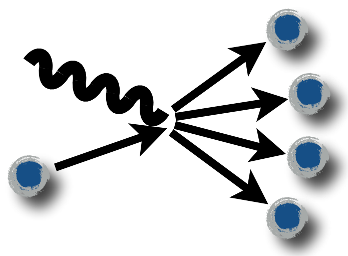
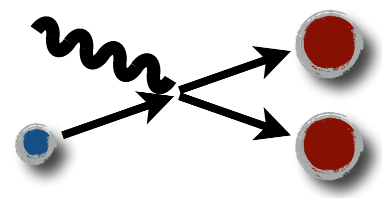
$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

$$\sigma_{\pi\gamma^*\to X} \propto \sum_\alpha \int d\Phi_\alpha |\langle E, \mathbf{p}, \alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle|^2$$

$\pi\pi, \ \pi\pi\pi\pi, \ K\overline{K}, \cdots$

n-particle phase space integral

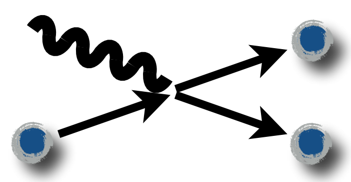
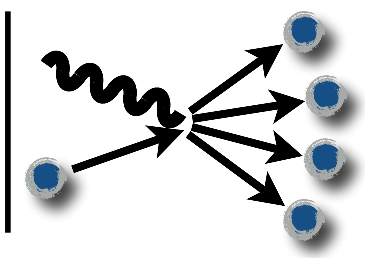
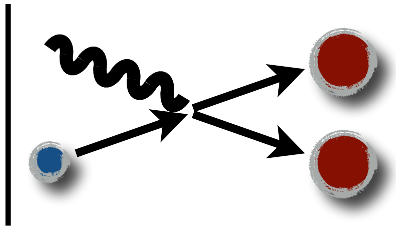
Instead try something inclusive…

$\sigma_{\pi\gamma^* \to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

$$\sigma_{\pi\gamma^* \to X} \propto \sum_\alpha \int d\Phi_\alpha |\langle E, \mathbf{p}, \alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle|^2$$

$$\propto \int_{\text{all states, } (P', \alpha')} (2\pi)^4 \delta^4(P' - P) \; \langle \pi | \mathcal{J}^\dagger(0) | E', \mathbf{p}', \alpha' \rangle \langle E', \mathbf{p}', \alpha' | \mathcal{J}(0) | \pi \rangle$$

Instead try something inclusive…

$\sigma_{\pi\gamma^* \to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

$$\sigma_{\pi\gamma^* \to X} \propto \sum_{\alpha} \int d\Phi_{\alpha} |\langle E, \mathbf{p}, \alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle|^2$$

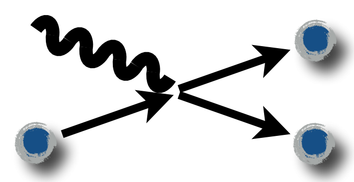
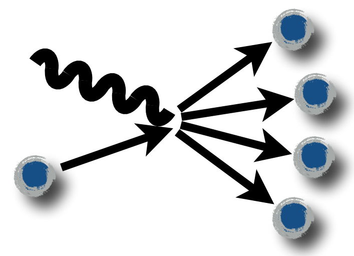
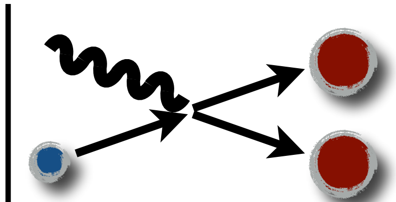
$$\propto \int_{\text{all states, } (P', \alpha')} (2\pi)^4 \delta^4(P' - P) \, \langle \pi|\mathcal{J}^\dagger(0)|E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

$$\propto \int d^4x \; e^{iqx} \langle \pi|e^{i\hat{P}\cdot x}\mathcal{J}^\dagger(0)e^{-i\hat{P}\cdot x} \int_{\text{all states, } (P', \alpha')} |E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

the delta is rewritten using

$$\int dx e^{ipx} = 2\pi\delta(p)$$

Instead try something inclusive…

$$\sigma_{\pi\gamma^*\to X} \equiv \int d\Phi \left| \vphantom{\int} \right.$$

$$\left. \vphantom{\int} \right|^2 + \int d\Phi \left| \vphantom{\int} \right.$$

$$\left. \vphantom{\int} \right|^2 + \int d\Phi \left| \vphantom{\int} \right.$$

$$\left. \vphantom{\int} \right|^2 + \cdots$$

$$\sigma_{\pi\gamma^*\to X} \propto \sum_\alpha \int d\Phi_\alpha |\langle E, \mathbf{p}, \alpha, \mathrm{out}|\mathcal{J}(0)|\pi\rangle|^2$$

$$\propto \int_{\text{all states, } (P',\alpha')} (2\pi)^4 \delta^4(P'-P)\ \langle\pi|\mathcal{J}^\dagger(0)|E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

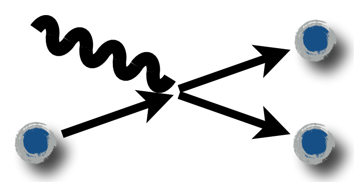
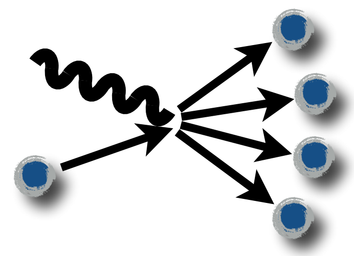
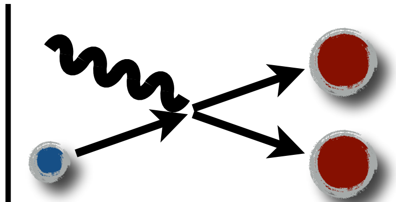
$$\propto \int d^4x\ e^{iqx} \langle\pi| e^{i\hat{P}\cdot x}\mathcal{J}^\dagger(0)e^{-i\hat{P}\cdot x} \int_{\text{all states, } (P',\alpha')} |E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

the sum over states defines the identity

$$\propto \int d^4x\ e^{iqx} \langle\pi|\mathcal{J}^\dagger(x)\mathcal{J}(0)|\pi\rangle$$

the current is translated

Instead try something inclusive…

$\sigma_{\pi\gamma^* \to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

$$\sigma_{\pi\gamma^* \to X} \propto \sum_\alpha \int d\Phi_\alpha |\langle E, \mathbf{p}, \alpha, \text{out}|\mathcal{J}(0)|\pi\rangle|^2$$

$$\propto \int_{\text{all states, }(P',\alpha')} (2\pi)^4 \delta^4(P' - P) \ \langle \pi|\mathcal{J}^\dagger(0)|E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

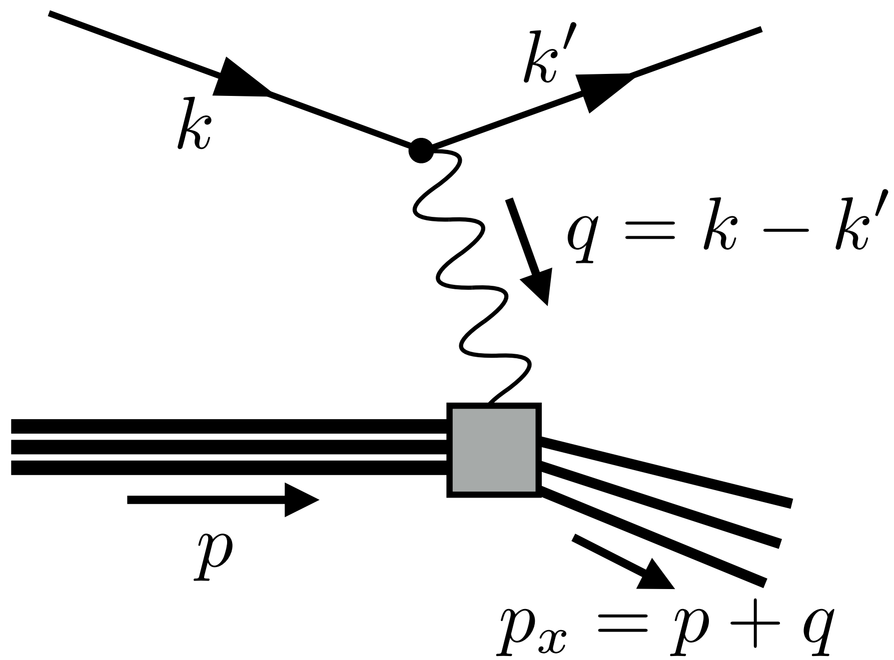
$$\propto \int d^4x \ e^{iqx} \langle \pi| e^{i\hat{P}\cdot x} \mathcal{J}^\dagger(0) e^{-i\hat{P}\cdot x} \int_{\text{all states, }(P',\alpha')} |E', \mathbf{p}', \alpha'\rangle\langle E', \mathbf{p}', \alpha'|\mathcal{J}(0)|\pi\rangle$$

$$\propto \int d^4x \ e^{iqx} \langle \pi|\mathcal{J}^\dagger(x)\mathcal{J}(0)|\pi\rangle$$

This motivates the hadronic tensor…

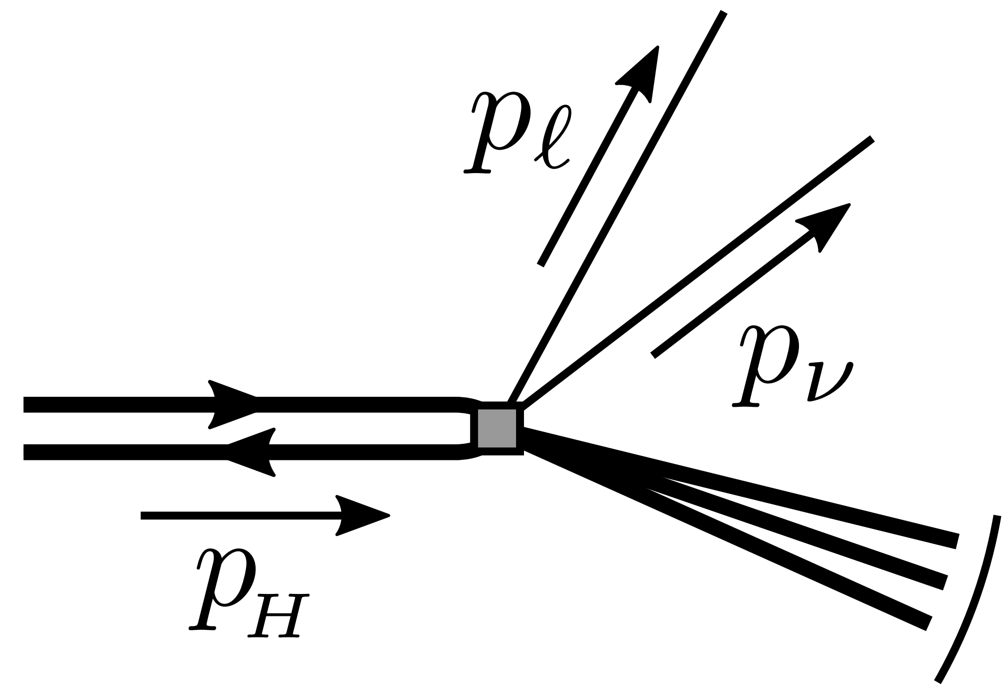
$$W_{\mu\nu}(p, q) \equiv \int d^4x \ e^{iqx} \langle \pi, p|\mathcal{J}^\dagger_\mu(x)\mathcal{J}_\nu(0)|\pi, p\rangle$$

Onset of deep-inelastic scattering



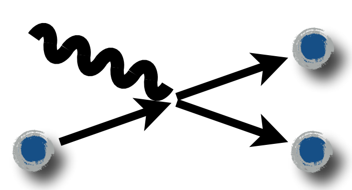
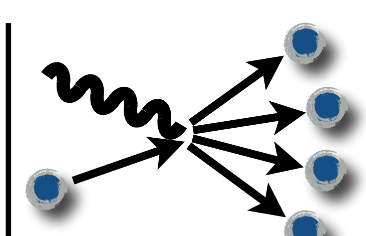
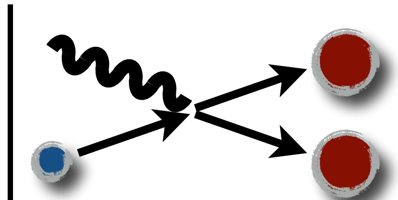
$$\langle N | J_\mu(q) J_\nu(0) | N \rangle$$

Heavy-flavor decays



$$\langle D | J_\mu(q) J_\nu(0) | D \rangle$$

Describe systems where many hadrons are produced and they are not individually detected

$\sigma_{\pi\gamma^* \to X} \equiv \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \int d\Phi \Big|$  $\Big|^2 + \cdots$

See… Liu, Dong (1993, 2017), Hashimoto (2017), Chambers et al. (2017)

A new spectral function

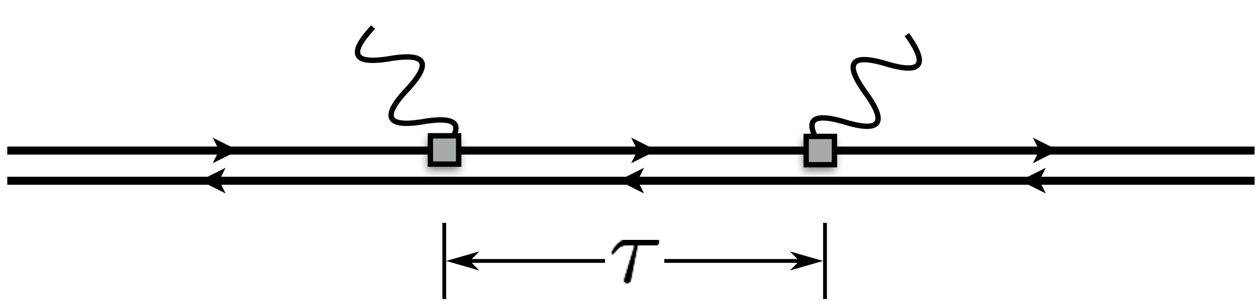
$$\rho(\omega, \mathbf{p}, \mathbf{q}) \equiv \int d^4x \; e^{iqx} \langle \pi, \mathbf{p} | \mathcal{J}_\mu^\dagger(x) \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle$$

- **Calculating this in LQCD requires solving an inverse problem…**

$$G(\tau, \mathbf{p}, \mathbf{q}) \equiv \int_{2M_\pi}^{\infty} \frac{d\omega}{2\pi} \, e^{-\omega\tau} \rho(\omega, \mathbf{p}, \mathbf{q})$$

- **with the correlator**

$$G(\tau, \mathbf{p}, \mathbf{q}) \equiv e^{-\omega_{\mathbf{p}}\tau} \langle \pi, \mathbf{p} | \mathcal{J}_\mu(\tau, \mathbf{q}) \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle$$



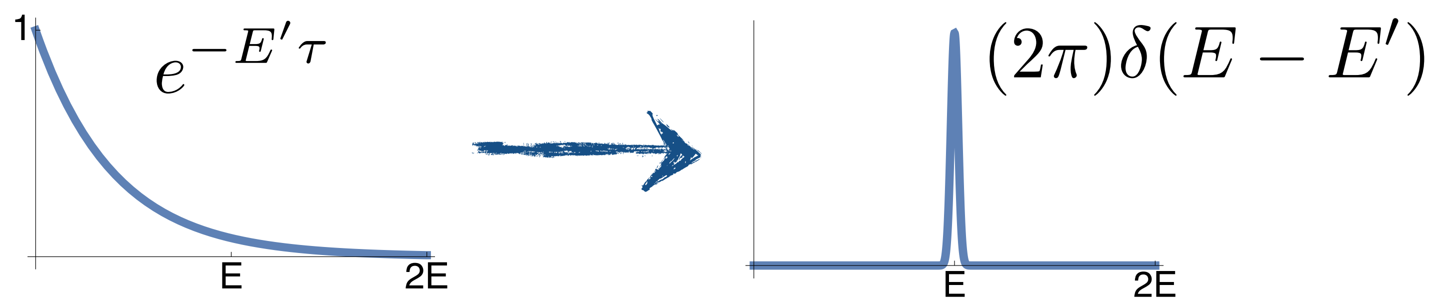
- **But don't forget about finite *L*…**

Role of the volume

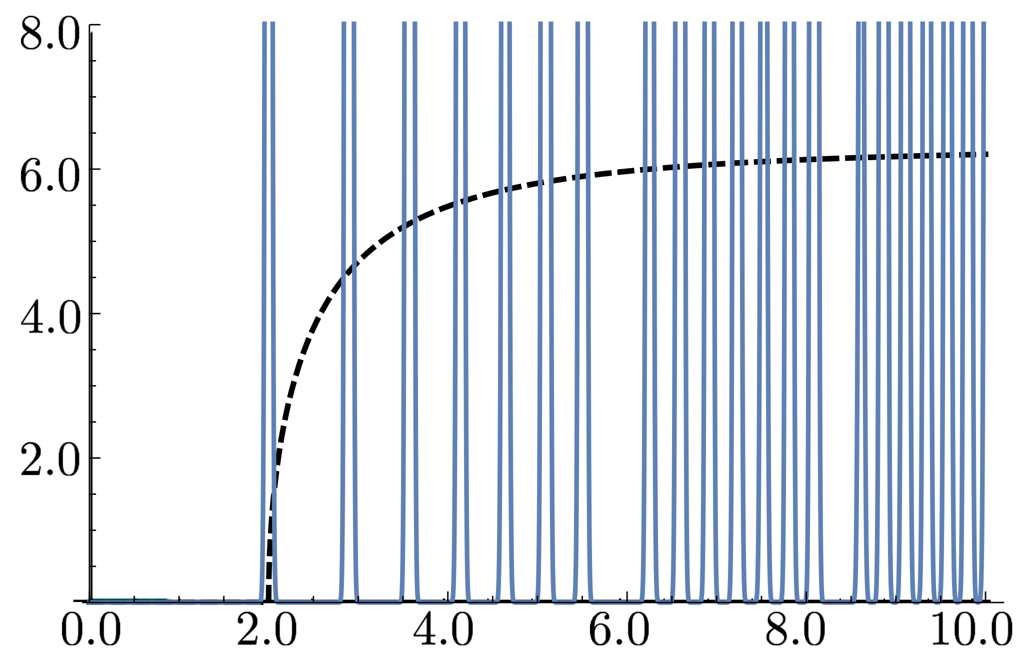
- If we calculate in a fixed finite volume…

$$G(\tau, \mathbf{p}, \mathbf{q}, L) = \sum_n e^{-E_n(L)\tau} |\langle E_n(L), \mathbf{p}_x | \mathcal{J}_\nu(0) | \pi, \mathbf{p} \rangle_L|^2$$

- and achieve a very-high-resolution inverse



- this does not give a useful estimate



Statement of the goal

- **The task is thus to identify an optimal linear combination…**

$$q_1(E)e^{-E' a_\tau} + q_2(E)e^{-E' 2a_\tau} + q_3(E)e^{-E' 3a_\tau} + \cdots = \widehat{\delta}_\Delta(E', E)$$

- **leading to a smeared out version of the spectral function**

$$2\pi \sum_n q_n(E)\, G(na_\tau, L) = \int dE'\, \widehat{\delta}_\Delta(E', E)\, \rho(E', L) \equiv \widehat{\rho}_\Delta(E, L)$$

Optimal choice depends on target precision and competition of scales

$$1/L \ll \Delta \ll M_{\mathrm{QCD}}$$

Statement of the goal

- **The task is thus to identify an optimal linear combination…**

$$q_1(E)e^{-E'a_\tau} + q_2(E)e^{-E'2a_\tau} + q_3(E)e^{-E'3a_\tau} + \cdots = \widehat{\delta}_\Delta(E', E)$$

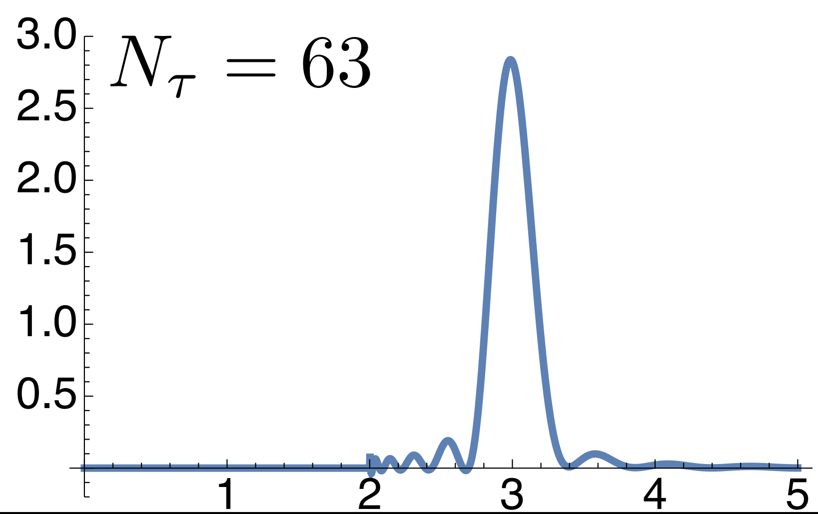
- **leading to a smeared out version of the spectral function**

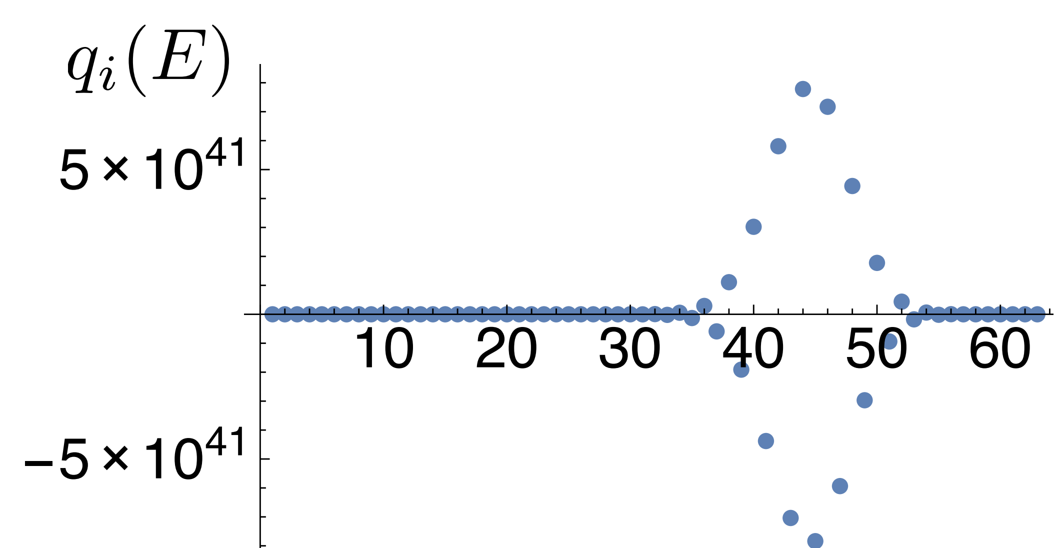
$$2\pi \sum_n q_n(E)\, G(na_\tau, L) = \int dE'\, \widehat{\delta}_\Delta(E', E)\, \rho(E', L) \equiv \widehat{\rho}_\Delta(E, L)$$

Optimal choice depends on target precision and competition of scales

$$1/L \ll \Delta \ll M_{\mathrm{QCD}}$$

- **An example of what *not* to do…**





Backus-Gilbert method

- **Developed by geophysicists in 1967**
- **Linear, model-independent approach**
- **Spectral function smeared with a known resolution function**
- **The covariance matrix is used to stabilize the inverse**

optimal coefficients

$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$

Backus-Gilbert method


- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

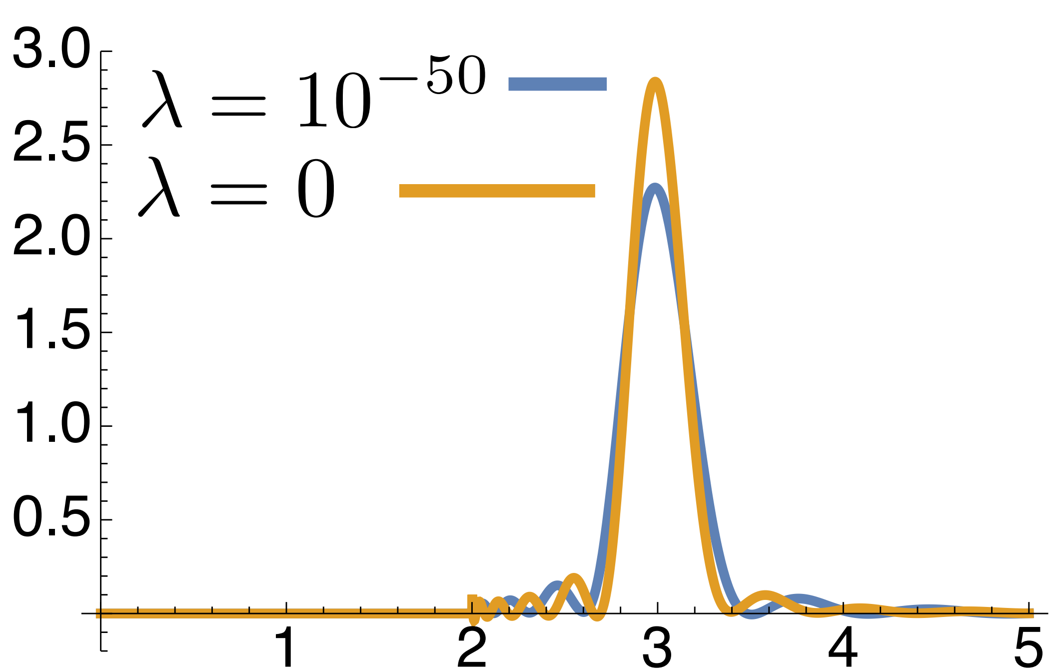
optimal coefficients

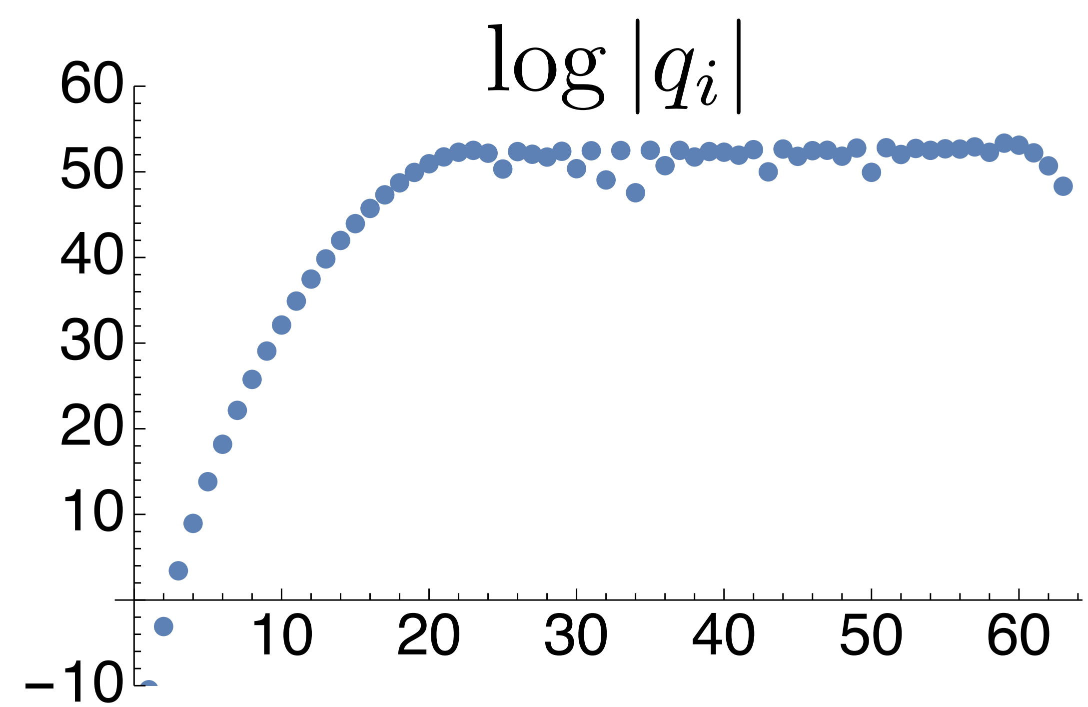
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

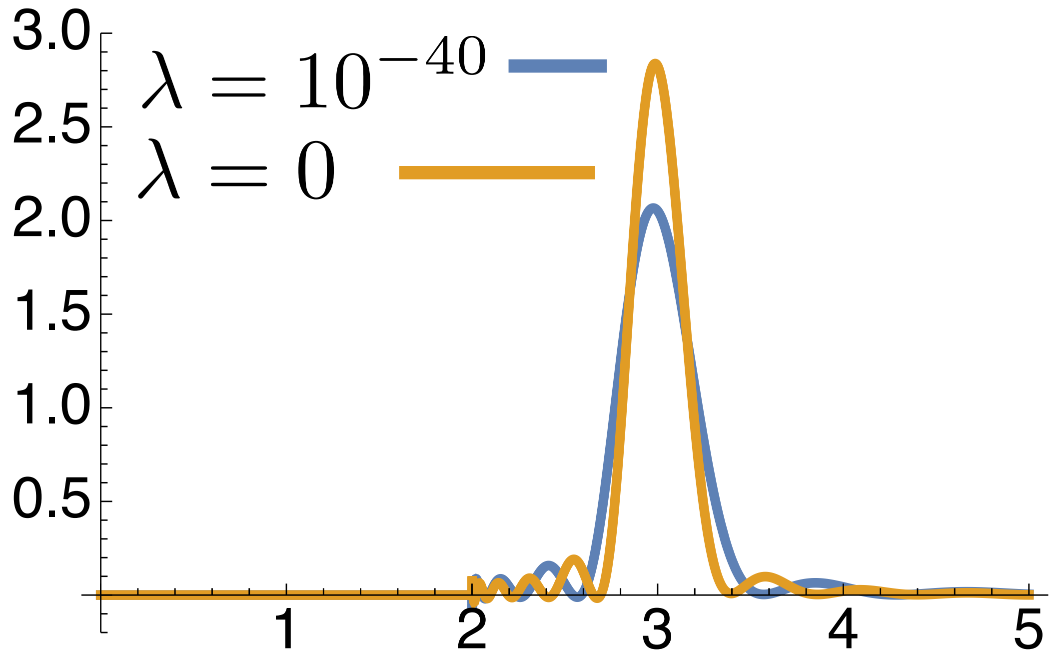
optimal coefficients

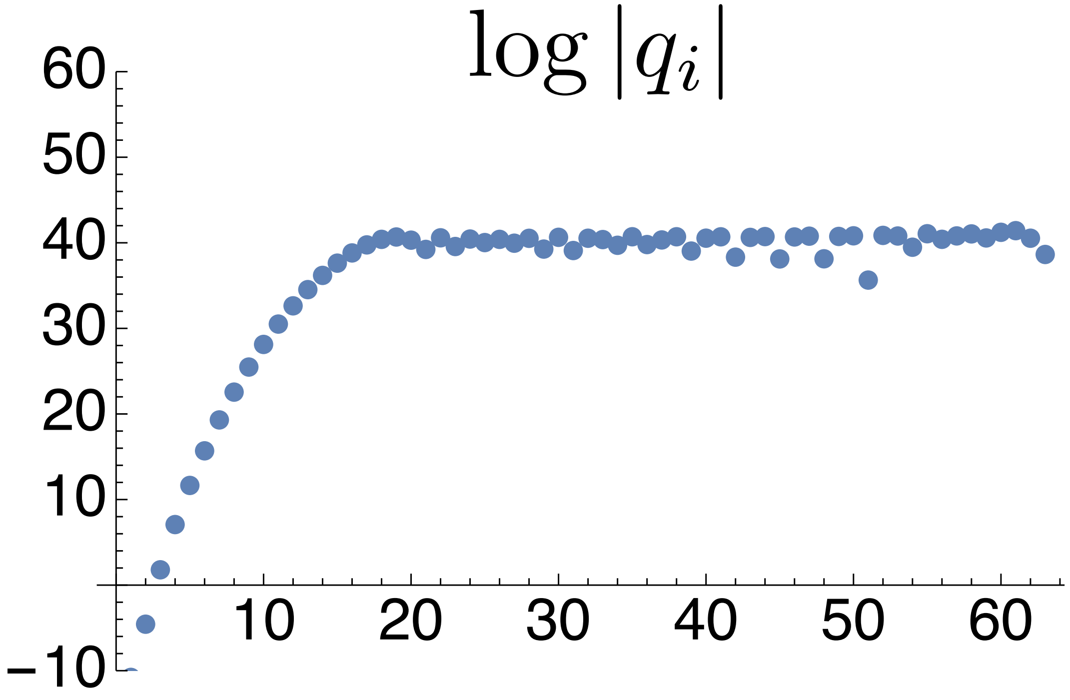
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

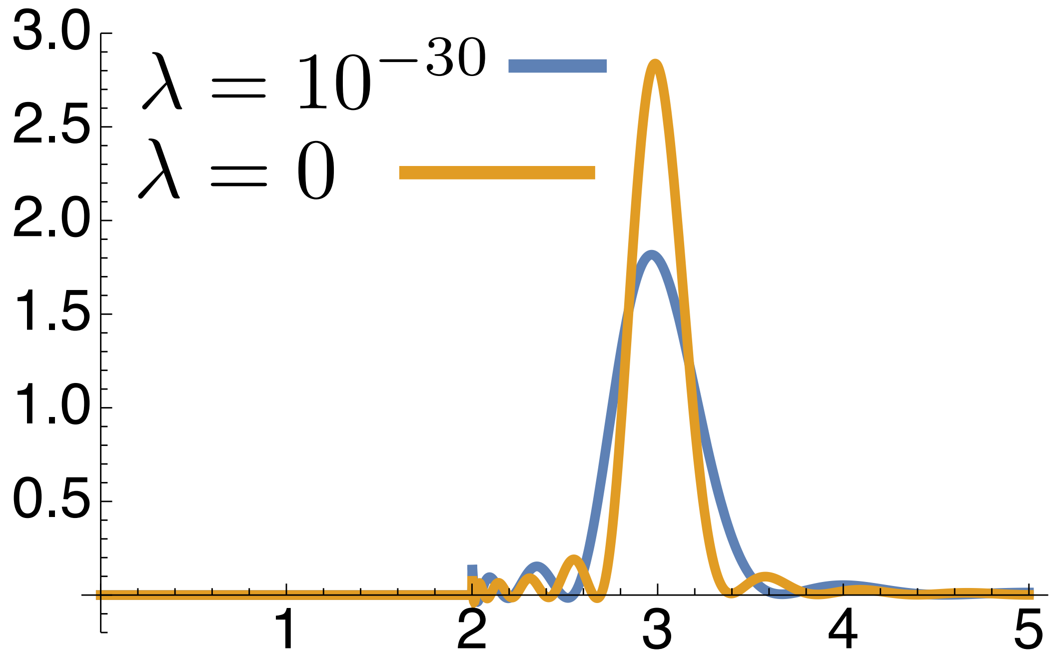
optimal coefficients

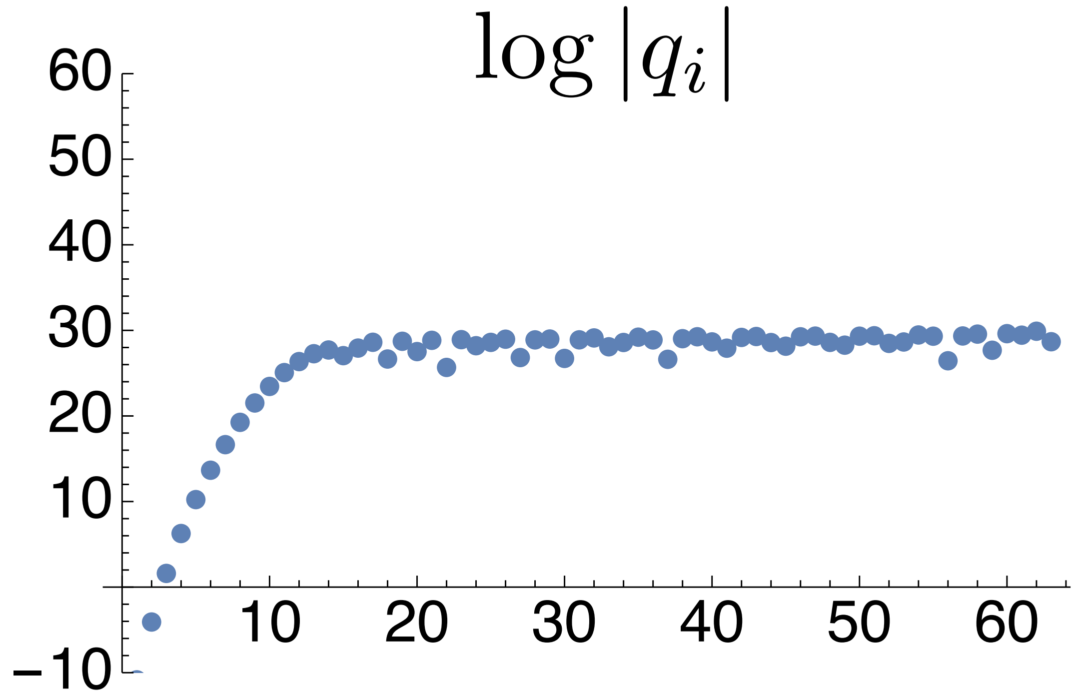
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

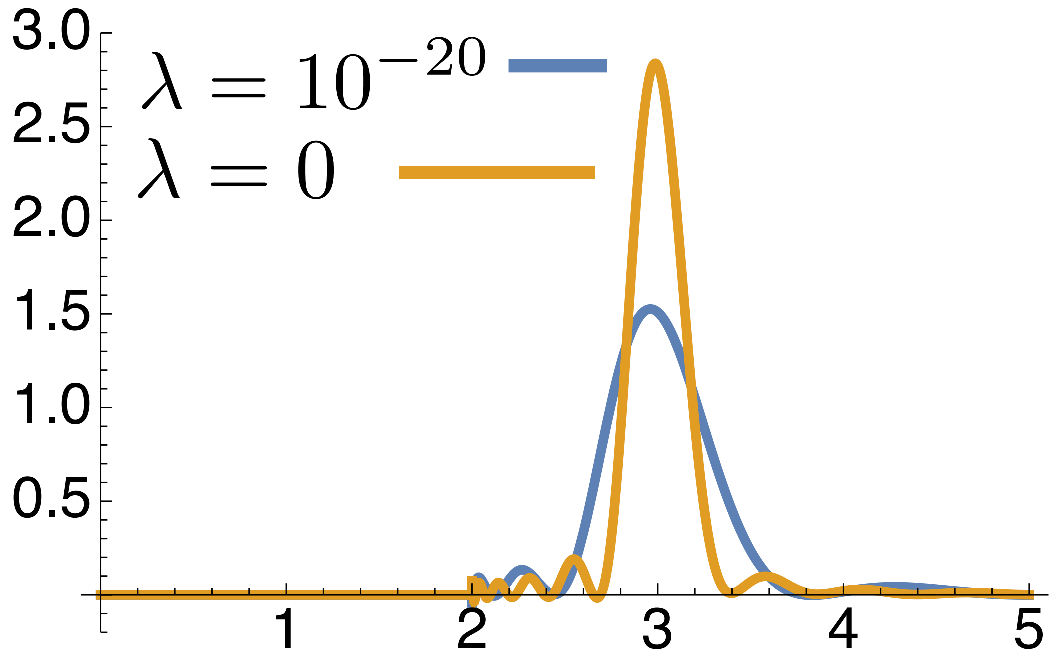
optimal coefficients

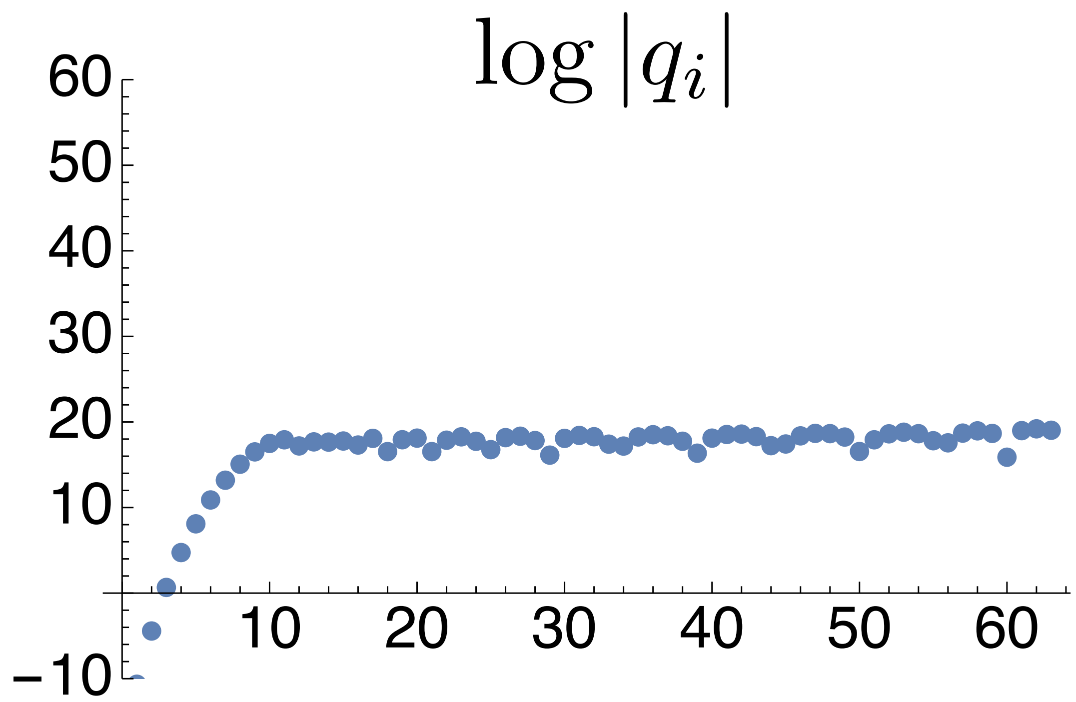
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

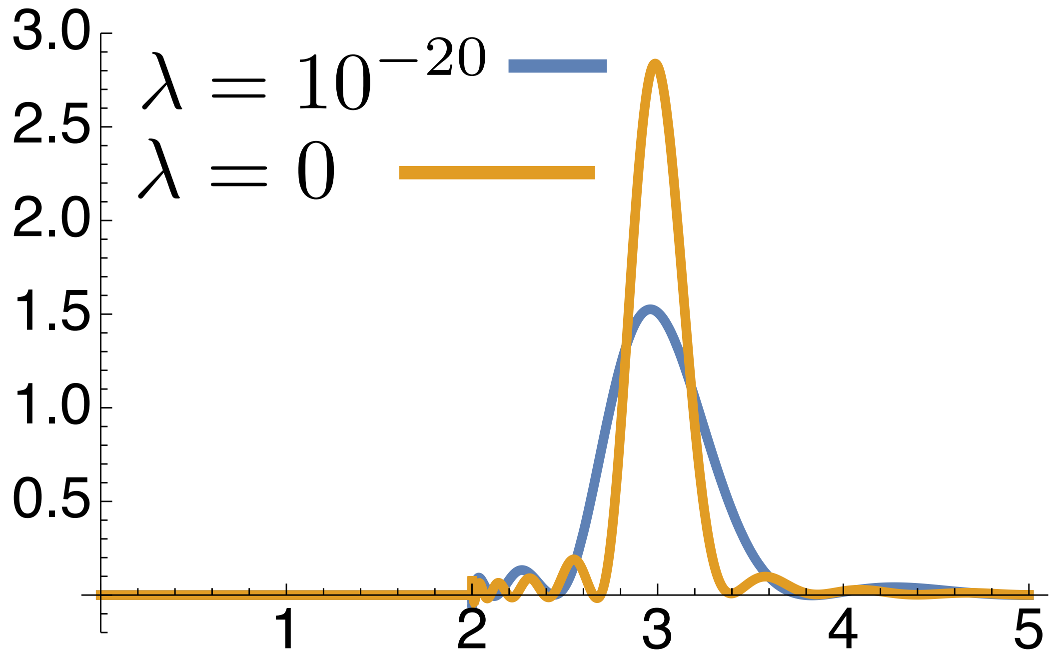
optimal coefficients

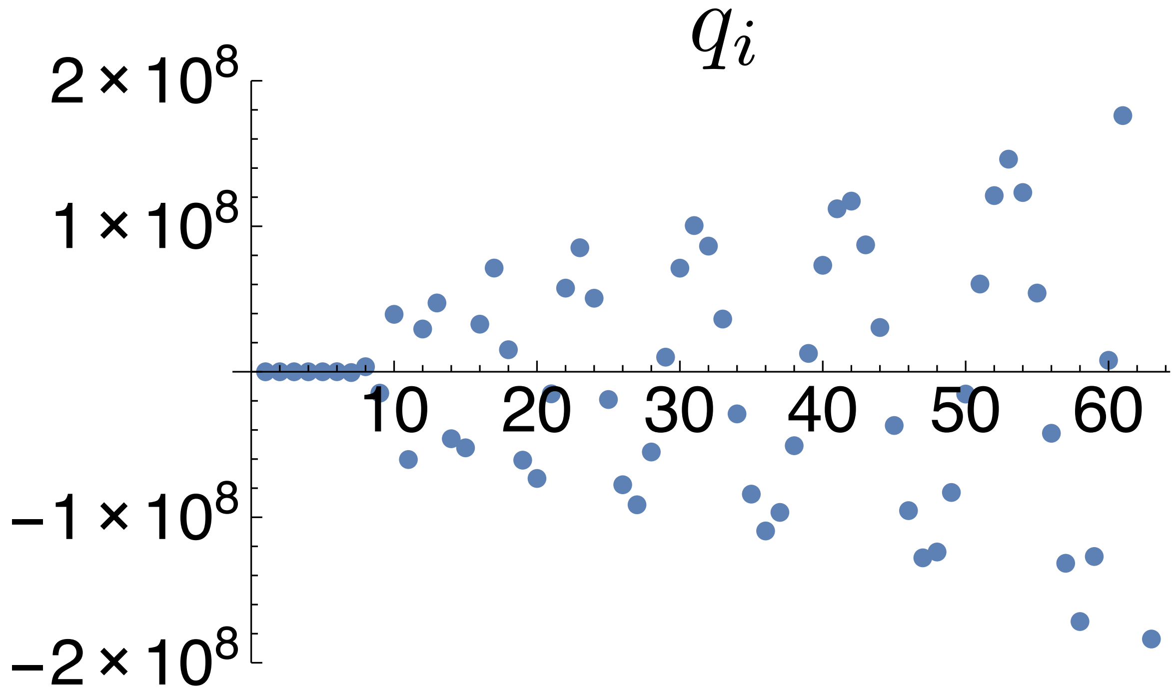
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

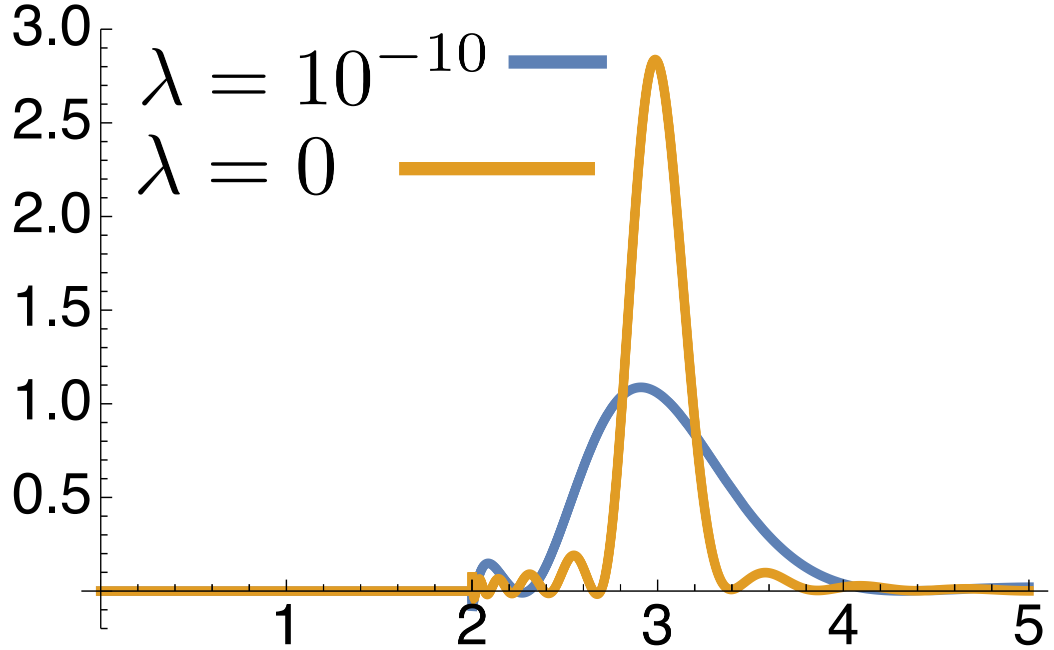
optimal coefficients

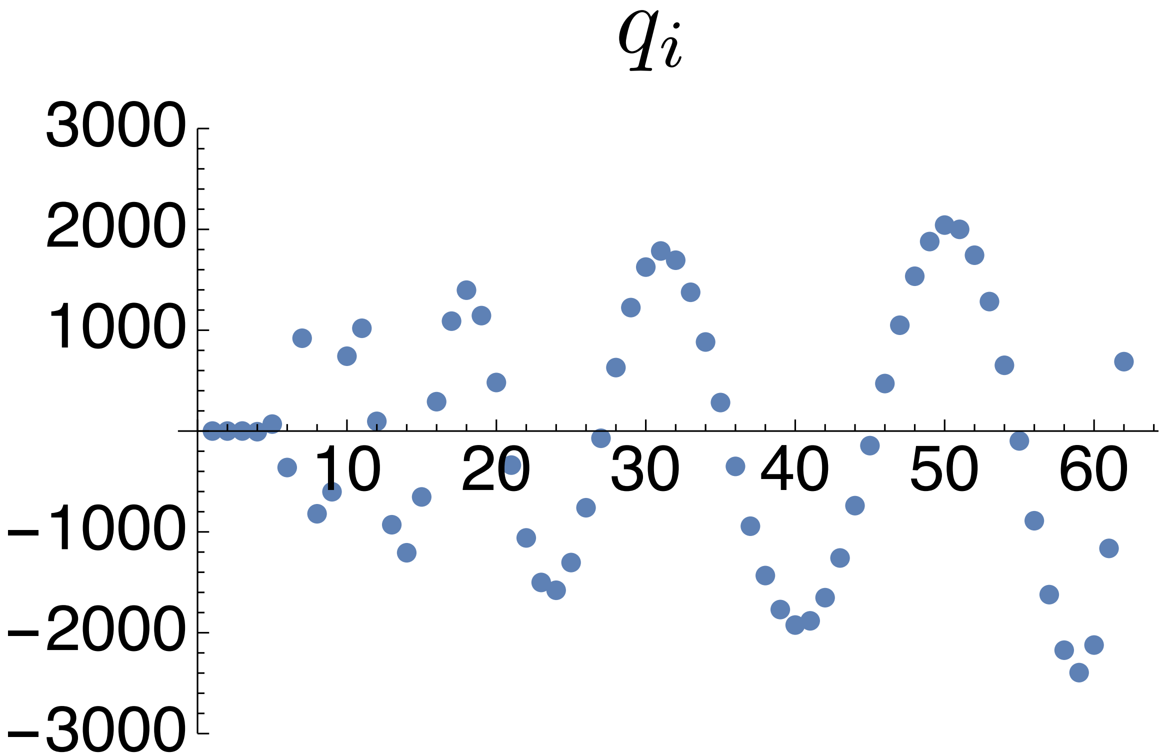
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Backus-Gilbert method

- Developed by geophysicists in 1967
- Linear, model-independent approach
- Spectral function smeared with a known resolution function
- The covariance matrix is used to stabilize the inverse

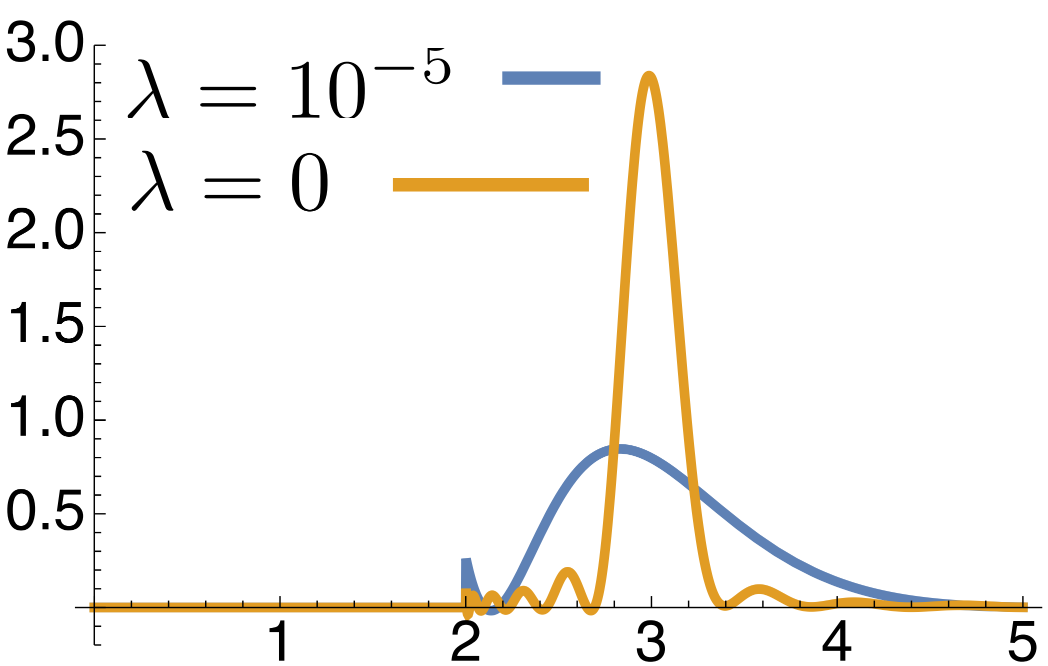
optimal coefficients

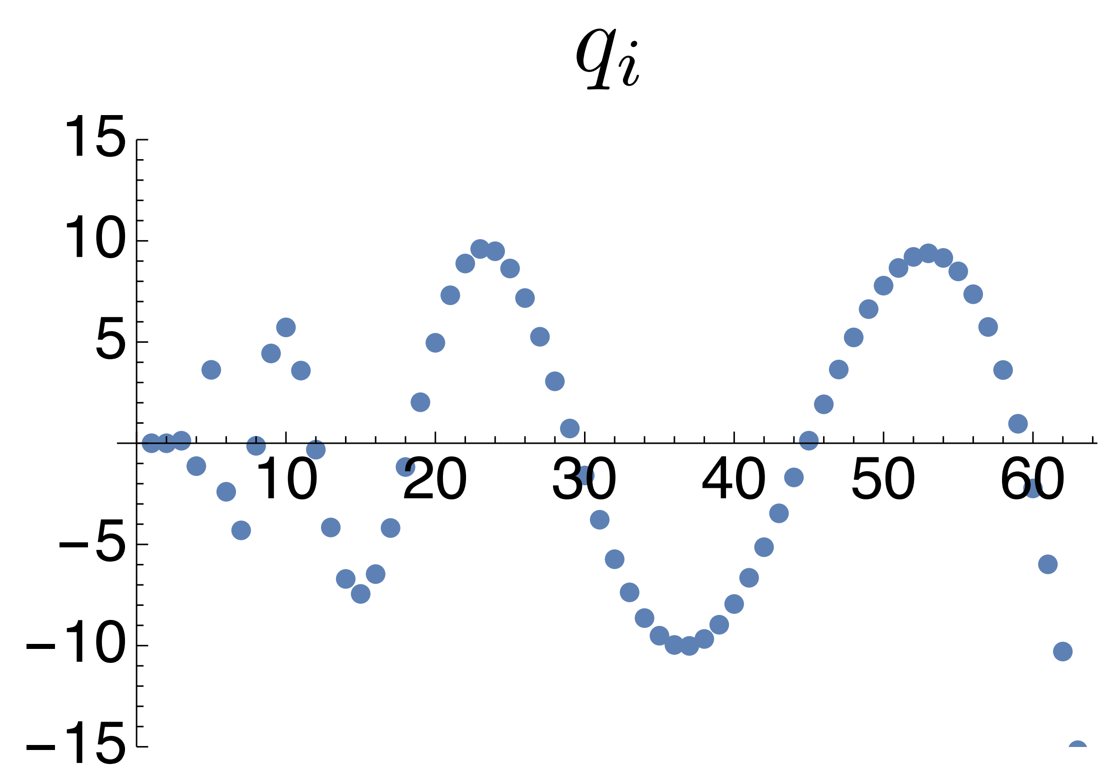
$$q_i(E) = \frac{[W(E) + \lambda S]^{-1} \cdot R}{R \cdot [W(E) + \lambda S]^{-1} \cdot R}$$

covariance

resolution function

$$\sum_i e^{-E\tau_i} q_i(E') = \widehat{\delta}_\Delta(E - E')$$





Ordered double limit: $\rho(\omega) = \lim_{\Delta\to 0} \lim_{L\to\infty} \widehat{\rho}(\omega, L, \Delta)$

- **Can be explored analytically…**

  e.g. two non-interacting pions and a Gaussian resolution function:

$$\widehat{\rho}(\omega, L, \Delta) - \widehat{\rho}(\omega, \Delta) = \sum_{\mathbf{m}\neq\mathbf{0}} \frac{2\omega}{L|\mathbf{m}|(\omega^2 + 4M_\pi^2)} \sin\left(\frac{L|\mathbf{m}|}{2\omega}\left(\omega^2 - 2M_\pi^2\right)\right) \exp\left[\frac{4M_\pi^2}{\Delta^2} - \frac{\Delta^2 L^2 \mathbf{m}^2}{8}\right]$$

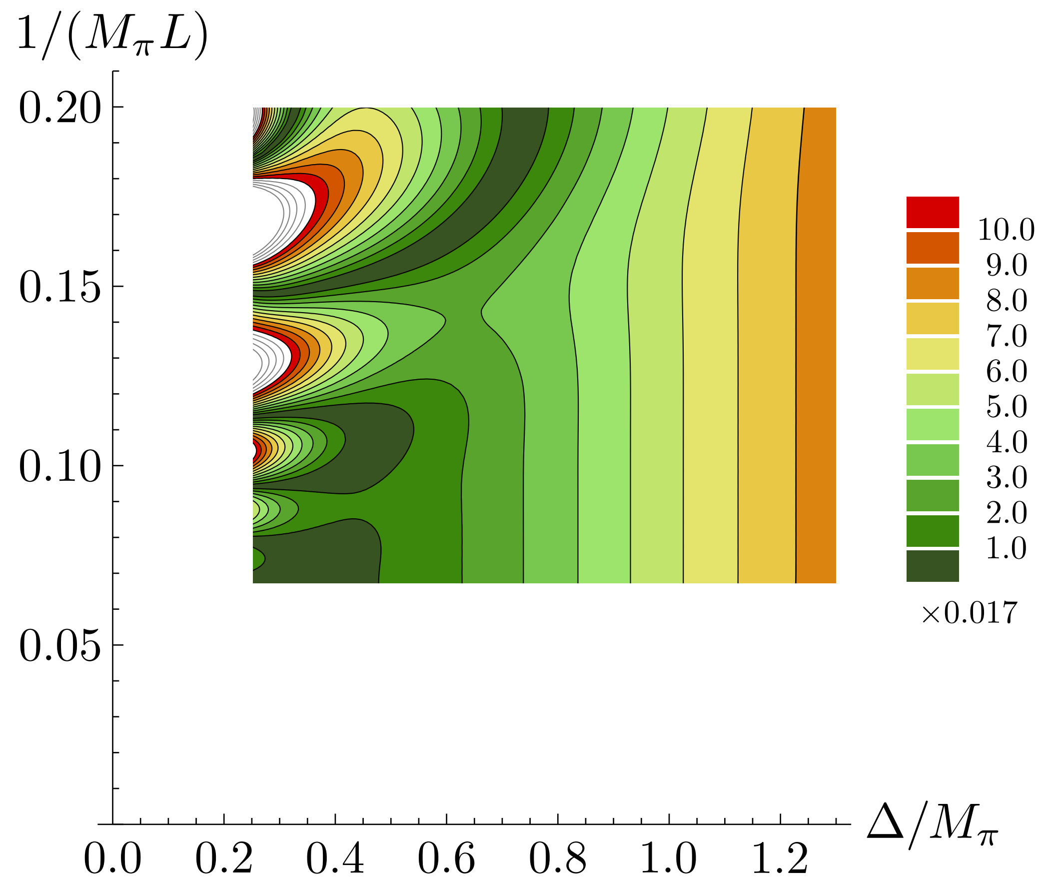
Ordered double limit: $\rho(\omega) = \lim_{\Delta\to 0} \lim_{L\to\infty} \widehat{\rho}(\omega, L, \Delta)$

- **Can be explored analytically…**

e.g. two non-interacting pions and a Gaussian resolution function:

$$\widehat{\rho}(\omega, L, \Delta) - \widehat{\rho}(\omega, \Delta) = \sum_{\mathbf{m}\neq\mathbf{0}} \frac{2\omega}{L|\mathbf{m}|(\omega^2 + 4M_\pi^2)} \sin\left(\frac{L|\mathbf{m}|}{2\omega}\,(\omega^2 - 2M_\pi^2)\right) \exp\left[\frac{4M_\pi^2}{\Delta^2} - \frac{\Delta^2 L^2 \mathbf{m}^2}{8}\right]$$

The aim is to access an optimal trajectory in the (Δ,1/L) plane



$$\frac{|\rho(\omega) - \widehat{\rho}(\omega, L, \Delta)|}{\rho(\omega)}$$

MTH, Robaina, Meyer (2017)

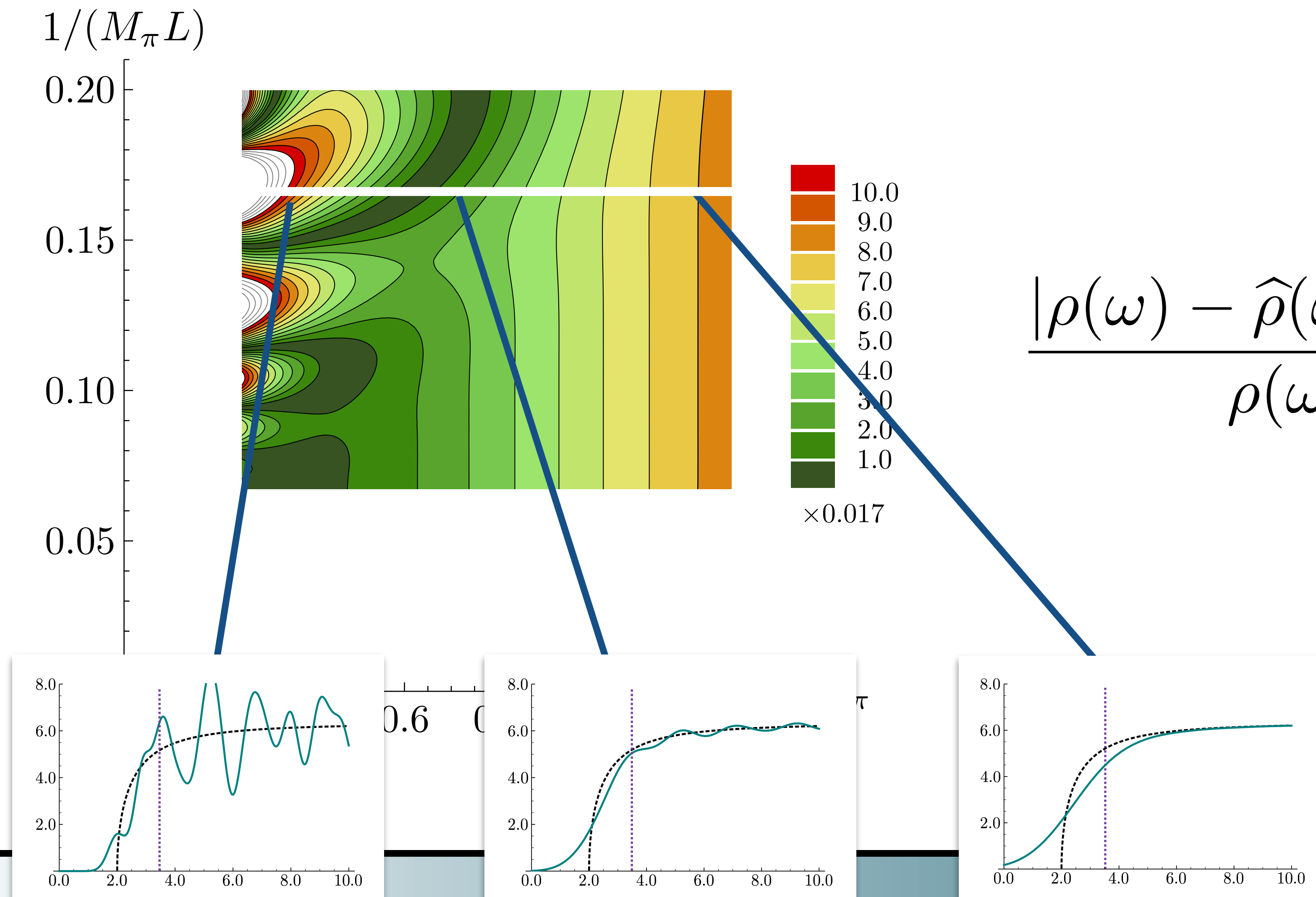
Ordered double limit: $\rho(\omega) = \lim_{\Delta \to 0} \lim_{L \to \infty} \widehat{\rho}(\omega, L, \Delta)$

- **Can be explored analytically…**

e.g. two non-interacting pions and a Gaussian resolution function:

$$\widehat{\rho}(\omega, L, \Delta) - \widehat{\rho}(\omega, \Delta) = \sum_{\mathbf{m} \neq \mathbf{0}} \frac{2\omega}{L|\mathbf{m}|(\omega^2 + 4M_\pi^2)} \sin\left(\frac{L|\mathbf{m}|}{2\omega}(\omega^2 - 2M_\pi^2)\right) \exp\left[\frac{4M_\pi^2}{\Delta^2} - \frac{\Delta^2 L^2 \mathbf{m}^2}{8}\right]$$

The aim is to access an optimal trajectory in the (Δ,1/L) plane

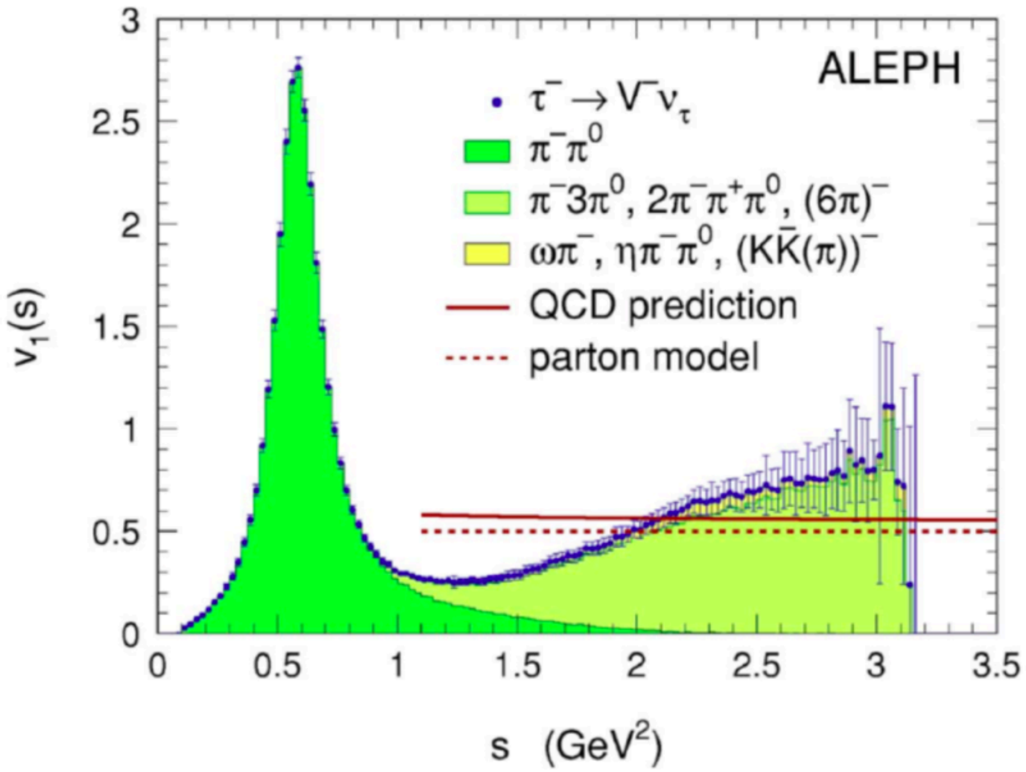


$$\frac{|\rho(\omega) - \widehat{\rho}(\omega, L, \Delta)|}{\rho(\omega)}$$

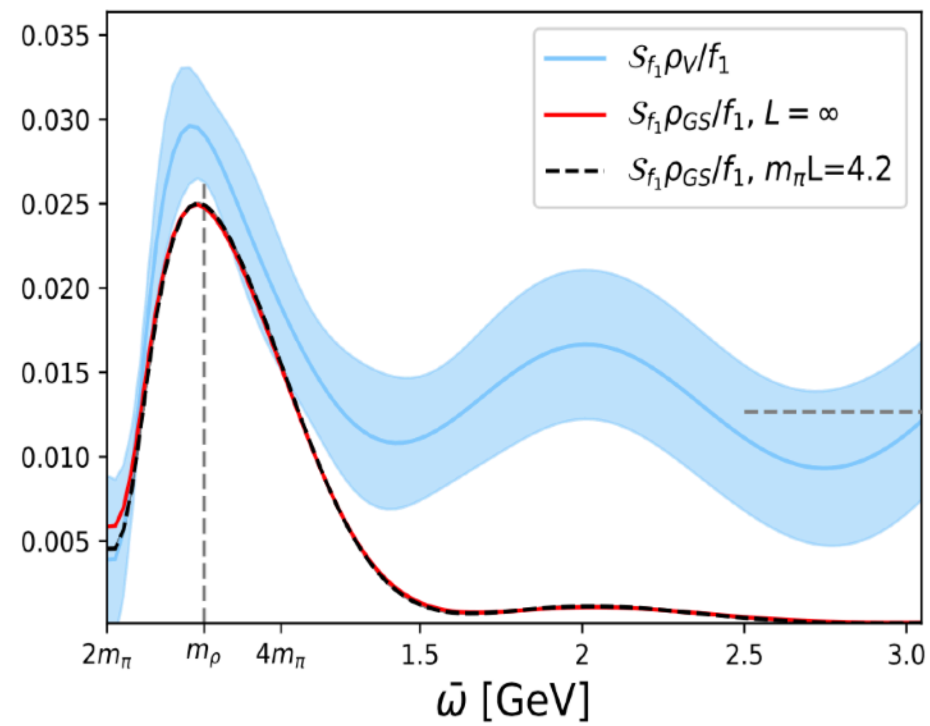
MTH, Robaina, Meyer (2017)

Combining methods?…

Harvey's Slide…



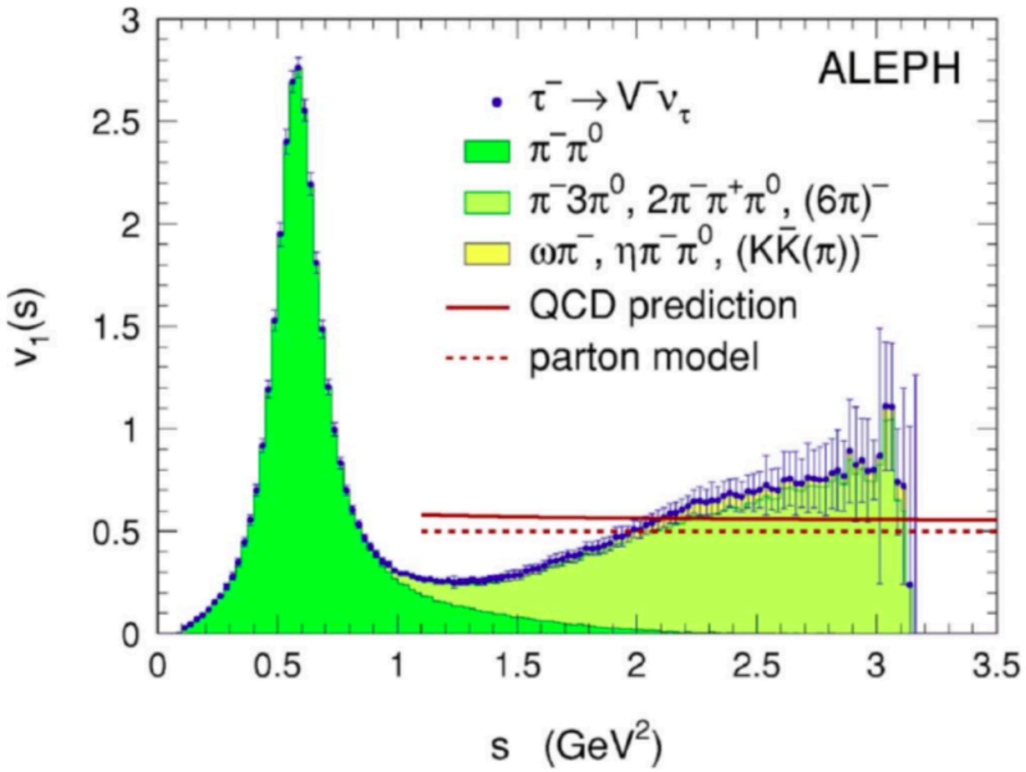
$v_1(s) = \frac{1}{3} R_1(s)$ up to isospin breaking corr.
From Davier, Höcker, Zhang
DOI:10.1103/RevModPhys.78.1043

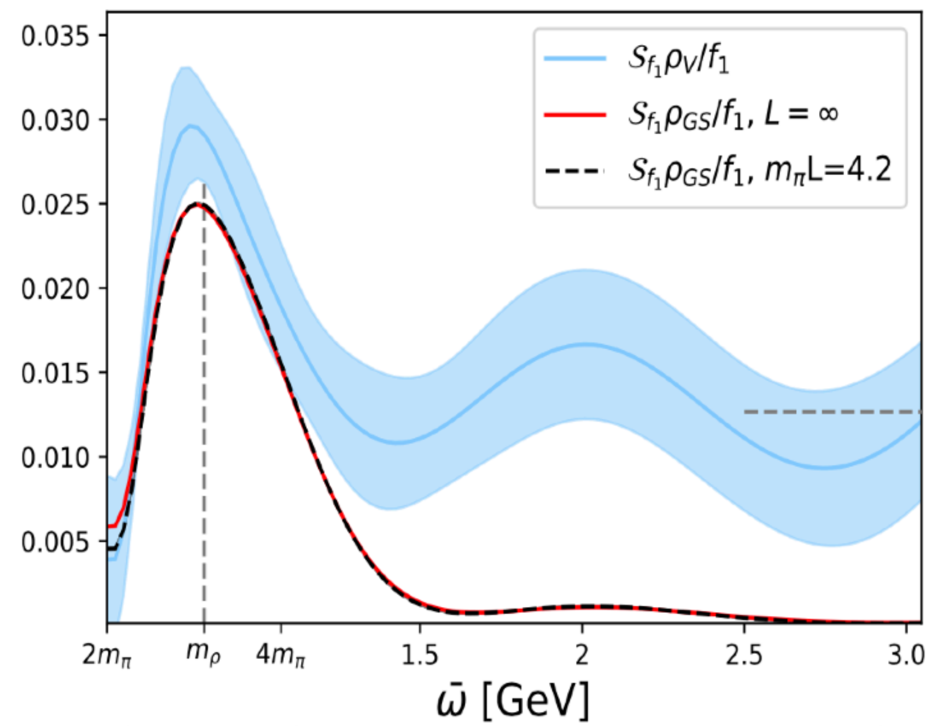


Blue curve = smoothened version of $\frac{R_1(\bar{\omega}^2)}{12\pi^2}$

Combining methods?…

Harvey's Slide…



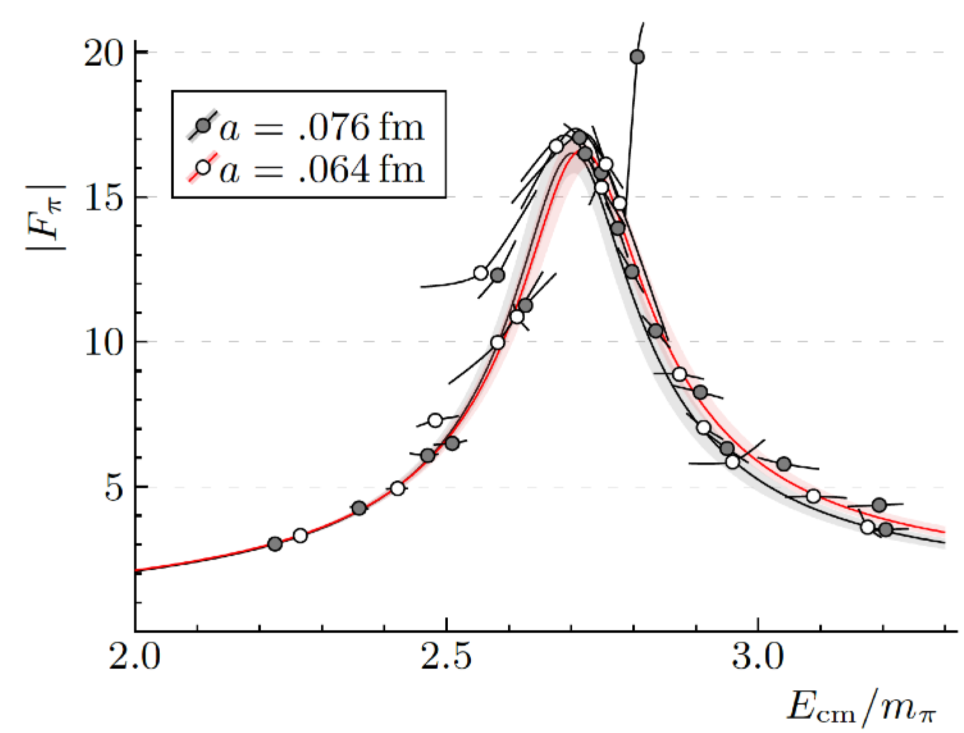


$v_1(s) = \frac{1}{3}R_1(s)$ up to isospin breaking corr.

From Davier, Höcker, Zhang

DOI:10.1103/RevModPhys.78.1043

B



Andersen, Bulava, Hörz, Morningstar
arXiv:1808.05007

Take home messages…

- **The finite volume can be a tool (interactions leave imprint in *E(L)*)**
- **In an inverse approach the volume is an unwanted artifact**

$$1/L \ll \Delta \ll M_{\mathrm{QCD}}$$

Important to better understand this dependence

- **The Backus-Gilbert method has *two outputs***

$$\widehat{\rho}_\Delta(E, L) = \int dE' \, \widehat{\delta}_\Delta(E', E) \, \rho(E', L) \qquad\qquad \widehat{\delta}_\Delta(E', E)$$

- **Perhaps we have to embrace the convolution (which observables best survive?)**

Open questions…

- What can the $T=0$ and $T\neq0$ communities learn from each other?
- e.g. Is there an analog for the importance of K-matrices as a branch-cut free observable?
- Is there an $T\neq0$ analog of a finite-volume quantization condition
- Can we systematically define/construct optimal resolution functions?
- Can we define interesting observables that do not require much resolution?