Higher order corrections to Higgs boson pair production
RADCOR 2019, Avignon, France, September 9-13, 2019
Matthias Steinhauser | in collaboration with J. Davies, M. Gerlach, F. Herren, G. Mishima, D. Wellmann
\[ gg \rightarrow HH \]

- NLO: approximations vs. exact
- NNLO: large \( m_t \)
- \( N^3\text{LO}: m_t \rightarrow \infty \)
Double Higgs production in SM
Double Higgs production in SM (2)

\[ \sigma(pp \to HH + X) \text{ [fb]} \]
\[ M_H = 125 \text{ GeV} \]

- \( gg \to HH \)
- \( q\bar{q}/gg \to t\bar{t}HH \)
- \( q\bar{q}' \to WHH \)
- \( q\bar{q}' \to HHqq' \)
- \( gg \to HH \)

\[ M_H = 125 \text{ GeV} \]
\[ \sqrt{s} \text{ [TeV]} \]

- \( K_{NLO} \approx 1.9 \)
- \( K_{NNLO} \approx 1.2 \)
\[ \lambda_{HHH} \text{ from } gg \rightarrow HH \]

\textbf{Standard Model:} \hspace{1cm} \text{gauge} \hspace{1cm} \alpha, \alpha_{\text{weak}}, \alpha_{s} \hspace{1cm} \text{Yukawa} \hspace{1cm} y_f \sim m_f \hspace{1cm} \text{Higgs} \hspace{1cm} V = \frac{1}{2} m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4 \hspace{1cm} m_H^2 = 2 \lambda v^2 \Leftrightarrow \lambda = 0.13 \ldots \hspace{1cm} \text{not yet measured}

\lambda \text{ from } H \text{ pair production (?)}
$\lambda_{HHH}$ from $gg \rightarrow HH$

**Standard Model:**
- **gauge**
  - $\alpha$, $\alpha_{\text{weak}}$, $\alpha_s$
- **Yukawa**
  - $y_f \sim m_f$
- **Higgs**
  - $V = \frac{1}{2} m_H^2 H^2 + \lambda v H^3 + \frac{\lambda}{4} H^4$
  - $m_H^2 = 2 \lambda v^2 \iff \lambda = 0.13 \ldots$

$\lambda$ from $H$ pair production (?)

very challenging; combine $b\bar{b}b\bar{b}$, $b\bar{b}\tau^+\tau^-$, $b\bar{b}\gamma\gamma$, $b\bar{b}VV$

[CMS: arXiv:1811.09689]: $-11.8 < \lambda/\lambda_{\text{SM}} < 18.8$

[ATL-PHYS-PUB-2019-009]: $-3.2 < \lambda/\lambda_{\text{SM}} < 11.9$
$gg \to HH$ at NLO: known results

**LO** [Glover, van der Bij’88; Plehn, Spira, Zerwas’96]

**NLO** for $m_t \to \infty$ [Dawson, Dittmaier, Spira’98]

**NLO** incl. $1/m_t$ terms [Grigo, Hoff, Melnikov, Steinhauser’13; Degrassi, Giardine, Gröber’16]

**NLO** exact (real rad.): [Maltoni, Vryonidou, Zaro’14]

**NLO** Padé: [Gröber, Maier, Rauh’17]

**NLO** small-$p_T$: [Bonciani, Degrassi, Giardino, Gröber’18]

**NLO** high energy: [Davies, Mishima, Steinhauser, Wellmann’18’19; Mishima’18]

**NLO** exact (numerical): [Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke’16]

[Baglio, Campanario, Glaus, Mühlleitner, Spira, Streicher’18]
LO: exact vs $s, t \gg m_t^2 > m_H^2$ expansion

\[ \frac{d\sigma}{d\theta}(s) \]

![Graph showing the differential cross-section $d\sigma_{LO}(gg \rightarrow HH)/d\theta|_{\theta=\pi/2}$ as a function of $\sqrt{s}$ [GeV] for different mass scenarios: $m_t^4$, $m_t^8$, $m_t^{16}$, $m_t^{32}$, $m_H \rightarrow 0$, $m_t \rightarrow \infty$.](image-url)
Non-planar MIs

\[ G_{33}(1,1,1,0,1,1,0,0) \]
\[ G_{33}(1,1,1,0,2,1,0,0) \]
\[ G_{51}(1,1,0,1,1,1,0,0) \]

\[ G_{47}(1,1,1,0,1,2,1,0,0) \]
\[ G_{91}(1,1,1,0,1,1,0,0) \]
\[ G_{47}(1,1,0,1,1,2,1,0,0) \]

\[ G_{33}(1,1,1,1,1,1,0,0) \]
\[ G_{59}(1,0,1,1,1,1,0,0) \]
\[ G_{47}(1,0,1,1,1,2,1,0,0) \]

\[ G_{51}(1,1,1,1,1,1,0,0) \]

\[ -(l_1 + q_4)^2 \]
\[ ((l_1 + q_4)^2)^2 \]
\[ ((l_2 + q_1)^2)^2 \]

\[ G_{33}(1,1,1,1,1,1,-1,0) \]
\[ G_{33}(1,1,1,1,1,1,-2,0) \]
\[ G_{33}(1,1,1,1,1,1,-2,0) \]

\[ G_{51}(1,1,1,1,1,1,1,0,0) \]
\[ G_{51}(1,1,1,1,1,1,1,0,-1) \]
\[ G_{51}(1,1,1,1,1,1,1,0,-1) \]

\[ G_{51}(1,1,1,1,1,1,1,1,0,0) \]

total: 161 MIs \((s, t \gg m_t^2)\)
Compute MIs

- Differentiate MIs \( (X = s, t, m_t^2) \)
  \[
  \frac{d}{dX} \mathcal{J} = M(s, t, m_t^2, \epsilon) \cdot \mathcal{J}
  \]

- Expand in \( m_t^2 \) \(\implies\) ansatz
  \[
  \mathcal{J} = \sum_i \sum_j \sum_k C_{ijk}(s, t) \epsilon^i (m_t^2)^j \log (m_t^2)^k
  \]

  \(\implies\) system of linear equations for \( C_{ijk}(s, t) \)

- Solution requires BCs for \( m_t \to 0 \)

- Compute MIs such that \( F_{\text{tri}}, F_{\text{box1}}, F_{\text{box2}} \) are available up to \( m_t^{32} \)

[Davies, Mishima, Steinhauser, Wellmann'18'19; Mishima'18]

\[
\mathcal{M} = \varepsilon_1, \mu \varepsilon_2, \nu \left( \mathcal{M}_1 A_1^{\mu \nu} + \mathcal{M}_2 A_2^{\mu \nu} \right) \\
\mathcal{M}_1 \sim \frac{3m_t^2}{s-m_\nu^2} F_{\text{tri}} + F_{\text{box1}} \\
\mathcal{M}_2 \sim F_{\text{box2}}
\]
NLO $\mathcal{V}_{\text{fin}}$ for $p_T = 250$ GeV

[Davies, Heinrich, Jones, Kerner, Mishima, Steinhauser, Wellmann’19]

$m_t$ expansion $\oplus$ Padé

exact $\oplus$ interpolation

[Borowka, Greiner, Heinrich, Jones, Kerner, Schlenk, Schubert, Zirke’16]

more details: [talk Matthias Kerner]
From NLO to NNLO

- exact: NO

- high-energy: box integrals with $s$, $t$ and $m_t$
  - 3 scales: not yet at 3 loops (NNLO)

- large-$m_t$: tadpoles + massless triangles (+ 1-loop massless box)
  - 1-scale integrals: doable at 3 loops

- threshold: some non-analytic terms [Gröber, Maier, Rauh’17]
  ($s \approx 4m_t^2$)
NLO approximation for $gg \rightarrow hh$: large-$m_t$ + threshold + Padé

[Gröber, Maier, Rauh'17]

Input for Padé: large-$m_t$: up to $1/m_t^8$

threshold: non-analytic terms ("logs")
NNLO approximation for $gg \rightarrow h$:
large-$m_t$ + threshold + Padé

\[ z = \frac{s}{4m_t^2} \]
\[ \bar{z} = 1 - z \]

(analytic $n_l$ terms: [Harlander, Prausa, Usovitsch’19])
NNLO approximation for $gg \to hh$: large-$m_t$ + threshold + Padé

- large $m_t$
  - expansion up to order $1/m_t^4$ for cross section [Grigo,Hoff,Steinhauser’15]
  - form factors $F_{\text{box}1}$, $F_{\text{box}2}$ up to order $1/m_t^8$ [Davies,Steinhauser’19]
  - $F_{\text{tri}}$ up to order $1/m_t^{14}$
  - Expansion: $\approx 10$ days wall time
    (a few $\times \geq 96$ GB RAM, 12 cores; stored expression 324 GB)
  - Projection/integration: $\approx 1$ month wall time
    (tasks are simpler: $\geq 8$ GB RAM, 4 cores; total time $\approx 1,600$ days)
  - heavy use of modern (T)FORM commands: [Ruijl,Ueda,Vermaseren’17]
    e.g. ArgToExtraSymbols

- threshold input from [Gröber, Maier, Rauh’17]

\[ F_{\text{tri}}: [\text{Davies, Gröber, Maier, Rauh, Steinhauser'19}] \]
\[ F_{\text{box}1}, F_{\text{box}2}: [\text{Davies, Gröber, Maier, Rauh, Steinhauser, WIP}] \]
NNLO real corrections for large $m_t$

- $m_t \to \infty$: use NNLO $gg \to H + \ldots + C_{HH}$
  
  [de Florian, Mazzitelli’14; Grigo, Melnikov, Steinhauser’14]

- Motivation for $1/m_t^2$ terms: input for Padé approximation (or similar)

- known numerically: $gg \to HH + gg$ and $gg \to HH + q\bar{q}$
  
  [Grazzini, Heinrich, Jones, Kallweit, Kerner, Lindert, Mazzitelli’18]
NNLO real corrections for large $m_t$

- 1-loop $2 \rightarrow 4$
- 2-loop $2 \rightarrow 3$

- use optical theorem

- asymptotic expansion for $m_t^2 \gg m_H^2, s$

  - tadpoles
  - “phase-space” integrals

1 and 2 loops

$\begin{align*}
m_t^2
\end{align*}$

3-loop 3- and 4-particle cuts [DHMS WIP]
and 2-loop 3-particle cut [DHMS'19]

[DHMS = Davies, Herren, Mishima, Steinhauser]
NNLO real corrections for large $m_t$ (2)

<table>
<thead>
<tr>
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<th>→</th>
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[Prausa’17] [Lee] [Henn’13] [Vollinga,Weinzierl’05]
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$n^3_h$: 

2 loops, 3-particle cuts
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$$\partial_x \tilde{I} = \sum_i \frac{M(i)(\epsilon)}{x-x_i} \tilde{I}$$

$$x_i = 0, 1, 1/4, -1, -1/4, e^{\pm i\pi/3}, -1/3$$

apply variable transformations, e.g., $x = \frac{y}{(1+y)^2}$
NNLO real corrections for large $m_t$ (2)

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$$\partial_x \vec{I} = \sum_i \frac{M(i)(\epsilon)}{x-x_i} \vec{I}$$

$$x_i = 0, 1, 1/4, -1, -1/4, e^{\pm i\pi/3}, -1/3$$

$$\partial_y \vec{J} = \epsilon \sum_i \frac{\tilde{M}(i)}{y-y_i} \vec{J}$$
NNLO real corrections for large $m_t$ (2)

amplitude $\rightarrow$ map to integral family $\rightarrow$ IBP reduction
FIRE $\rightarrow$ minimal set of MIs

$\rightarrow$ diff. eqs. in $x = m_H^2/s$

$\rightarrow$ Fuchsian form epsilon
$\rightarrow$ $\epsilon$ form
LIBRA $\rightarrow$ solution
Goncharov polylogs
ginac

$\rightarrow$ expand in $\delta = \sqrt{1 - 4m_H^2/s}$
PolyLogTools

[Duhr,Dulat’19]

alternative: use diff. eqs. to obtain expansion in $\delta$
($>500$ terms possible)
NNLO real corrections for large $m_t$ (2)

$\delta = 0.9 \Rightarrow \sqrt{s} = 800$ GeV
NNLO real corrections for large $m_t$ (3)

Cross section for $n_h^3$ diagrams

- expansion up to $1/m_t^8$
- combine with virtual corrections [Grigo,Hoff,Steinhauser’15; Davies,Steinhauser’19]
NNLO real corrections for large $m_t$ (3)

Cross section for $n_h^3$ diagrams

- expansion up to $1/m_t^8$
- combine with virtual corrections [Grigo, Hoff, Steinhauser’15; Davies, Steinhauser’19]

$$\rho = \frac{m_H^2}{m_t^2}$$
$N^3\text{LO:} \; m_t \rightarrow \infty$

\[
\mathcal{L}_{\text{eff}} = -\frac{H}{v} C_H O_1 + \frac{1}{2} \frac{H^2}{v^2} C_{HH} O_1
\]

\[
O_1 = -G_{\mu \nu}^2 / 4
\]

and have same operator

most parts can be taken over from $gg \rightarrow H$

$N^3\text{LO:}$ [Anastasiou,Duhr,Dulat,Furlan,Gehrmann,Herzog,Mistlberger'14]

NEW for $gg \rightarrow HH$:

$N^3\text{LO:}$ [Banerjee,Borowka,Dhani,Gehrmann,Ravindran'18]

\[
C_{HH} \neq C_H
\]

[Grigo,Melnikov,Steinhauser'14]

[de Florian,Mazzitelli'14]

[Spira'16; Gerlach,Herren,Steinhauser'18]
$C_{HH}$ to 4 loops

$$\mathcal{L}_{\text{eff}} = -\frac{H}{v} C_H O_1 + \frac{1}{2} \frac{H^2}{v^2} C_{HH} O_1$$

$$C_{HH} = C_H + \Delta_{HH}$$

$\Delta_{HH} = 0$

$\Delta_{HH} = 0$

$$\Delta_{HH} = \frac{7}{8} C_A^2 - \frac{11}{8} C_A C_F - \frac{5}{6} C_A T_F + \frac{1}{2} C_F T_F + n_l C_F T_F$$

[Grigo, Melnikov, Steinhauser’14]
$C_{HH}$ to 4 loops

\[ \mathcal{L}_{\text{eff}} = -\frac{H}{v} C_H \mathcal{O}_1 + \frac{1}{2} \frac{H^2}{v^2} C_{HH} \mathcal{O}_1 \]

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\[ \Delta_{HH} = \frac{7}{8} C_A^2 - \frac{11}{8} C_A C_F - \frac{5}{6} C_A T_F + \frac{1}{2} C_F T_F + n_i C_F T_F \]

[Grigo,Melnikov,Steinhauser'14]

\[ \Delta_{HH} = ? \]
Renormalization and matching

\[ \langle O_1 O_1 \rangle^{\text{ren}} = (Z_{O_1})^2 \langle O_1 O_1 \rangle^{\text{bare}} + Z_{11}^L \langle O_1 \rangle^{\text{bare}} \]

\[ Z_{11}^L = 1 + \mathcal{O}(\alpha_s^2) \]

[Zoller'16]

Matching

\[ (C_{HH} Z_{O_1} + C_H^2 Z_{11}^L) A_{\text{LO,1PI}}^{\text{eff}} + C_H^2 Z_{O_1} A_{\text{LO,1PR},\lambda=0}^{\text{eff}} = A_{\text{1PI}}^{h} + A_{\text{1PR},\lambda=0}^{h} \]

[Gerlach, Herren, Steinhauser'18]
\[ \Delta_{HH}^{(4)} = \frac{1993}{576} C_A^3 - \frac{1289}{144} C_A^2 C_F - \frac{3191}{864} C_A^2 T_F + \frac{165}{32} C_A C_F^2 + \frac{67}{18} C_A C_F T_F + \frac{5}{72} C_A T_F^2 \\
- \frac{3}{2} C_F^2 T_F + \frac{1}{9} C_F T_F^2 + \left[ \frac{77}{48} C_A^3 - \frac{121}{48} C_A^2 C_F - \frac{7}{12} C_A^2 T_F + \frac{11}{12} C_A C_F T_F \right] \ln \left( \frac{\mu^2}{M_t^2} \right) \\
+ n_l T_F \left[ - \frac{55}{144} C_A^2 + \frac{55}{18} C_A C_F + \frac{109}{216} C_A T_F - \frac{11}{4} C_F^2 + \frac{19}{36} C_F T_F \right] \\
+ n_l^2 T_F^2 \left[ \frac{5}{72} C_A + \frac{1}{9} C_F \right] + n_l T_F \left[ - \frac{7}{12} C_A^2 + \frac{11}{4} C_A C_F - \frac{2}{3} C_F T_F \right] \ln \left( \frac{\mu^2}{M_t^2} \right) \\
- \frac{2}{3} n_l^2 C_F T_F^2 \ln \left( \frac{\mu^2}{M_t^2} \right) \\

[\text{Spira’16}] \]  

**Low-energy theorem:**  
\[ C_{HH} = \frac{m_t^2}{\zeta_{\alpha_s}} \frac{\partial^2}{\partial m_t^2} \zeta_{\alpha_s} - 2 \left( \frac{m_t}{\zeta_{\alpha_s}} \frac{\partial}{\partial m_t} \zeta_{\alpha_s} \right)^2 \]

\[ [C_H = - \frac{m_t}{\zeta_{\alpha_s}} \frac{\partial}{\partial m_t} \zeta_{\alpha_s} \text{ [Chetyrkin,Kniehl,Steinhauser’98]}] \]

\[ \alpha_s^{(5)} = \zeta_{\alpha_s} \alpha_s^{(6)} \]
Conclusions

- **LO, NLO**: exact
- **NLO**: many approximations
  - exact result is expensive! ⇨ combine approximations and exact result
  - test bed for NNLO

- **NNLO**: large-$m_t$ (virt. + real)

- **N^3LO**: effective-theory ($m_t \rightarrow \infty$)
  - $C_{HH}$ to 4 loops
  - cross section in reach