Master integrals for all cuts of massless 4-loop propagators

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Motivation: NNLO time-like splitting functions

Time-like splitting functions:

- Scale evolution kernels for parton fragmentation distributions.
- Mostly known at NNLO. [Mitov, Moch, Vogt '06; Moch, Vogt '07; Almasy, Moch, Vogt '11]
- A leading color limit known at $\mathrm{N^3LO}$. [Moch, Ruijl, Ueda, Vermaseren, Vogt '17; '18]

NNLO corrections:

- Calculated via an analytic continuation from the space-like case.
- An inherent uncertainty in the analytic continuation leaves one of the terms in $P_{qg}^{(2)T}$ and $P_{gq}^{(2)T}$ undetermined.
- To fix the missing terms, a direct calculation is required.

Calculating time-like splitting functions

"Direct" calculation: [Gituliar, Moch '15]

- Analytically calculate the differential cross-section of

$$e^+e^- \to \gamma^* \to \text{parton } p + \dots,$$

in the energy fraction of one of the outgoing partons,

$$x = 2\, q\cdot k_p/q^2.$$

- Read off the splitting functions from the poles of this cross-section:

$$\frac{d\sigma_p}{dx} \sim \dots + \left(\frac{\alpha_s}{2\pi}\right)^1 \left\{\dots - \frac{1}{\epsilon} P_{pq}^{(0)T}\right\} + \left(\frac{\alpha_s}{2\pi}\right)^2 \left\{\dots - \frac{1}{2\epsilon} P_{pq}^{(1)T}\right\} + \left(\frac{\alpha_s}{2\pi}\right)^3 \left\{\dots - \frac{1}{3\epsilon} P_{pq}^{(2)T}\right\}.$$

- The needed $P_{gq}^{(2)T}$ shows up at α_s^3 (N^3LO) of $d\sigma_g/dx$!
- To obtain $P_{qg}^{(2)T}$ do the same, but use a Higgs instead of γ^*.

Calculating the differential cross-section

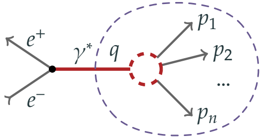
Master integrals for $d\sigma/dx$ at N^3LO are not all known!

- They can be calculated via differential equations (in x).
  - Partially done. $\mathscr{O}(500)$ of master integrals expected in total.
  - An ϵ-form can be found for most of the differential equations; automatized via `Fuchsia`. [Gituliar, V.M. '17]
  - Solvable in multiple polylogarithms, with poles at $\{-1, 0, 1, 2\}$.
- How to fix the integration constants?
  - An integral over all x values should turn a differential cross-section into a fully inclusive one, $\int dx \, \frac{d\sigma}{dx} = \sigma$. Apply for each master separately!
  - Master integrals for fully inclusive cross-sections at N^3LO are needed.

In short:

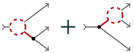
- Eventual goal: NNLO corrections to time-like splitting functions.
- Current goal: master integrals for fully inclusive cross-sections at N^3LO.

Inclusive cross-section master integrals



For a fully inclusive decay cross-section of an off-shell particle:

$$\sigma \sim \sum_n \int d\mathrm{PS}_n \left|\langle p_1, \dots, p_n|S|q\rangle\right|^2 = \sum_n \int d\mathrm{PS}_n \left| \quad + \quad + \dots \right|^2$$

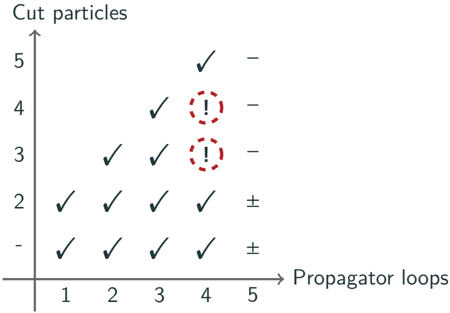


Expanding the module squared, each term becomes a cut propagator:

For N^{3}LO (α_s^3) we need cuts of 4-loop propagators:

2-particle cuts 3-particle cuts 4-particle cuts 5-particle cuts

Propagators and their cuts: state of the art

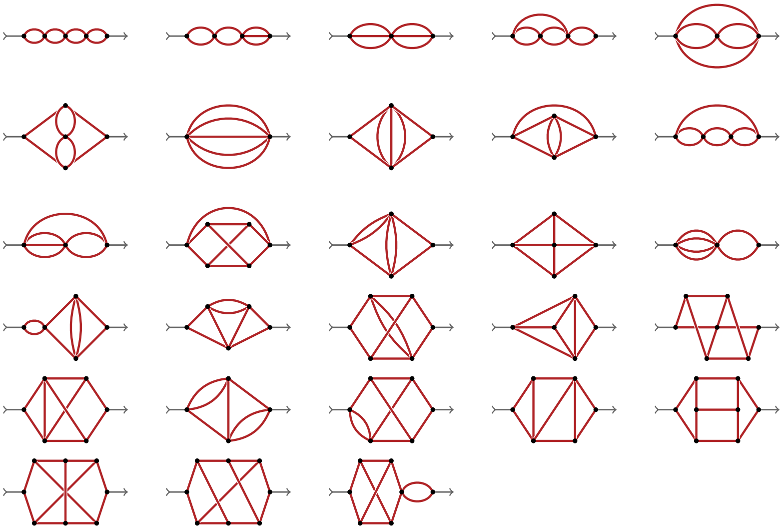


For four loop propagators we know:

- Virtual integrals. [Baikov, Chetyrkin '10]
- Two-particle cuts (3-loop form-factors).
  [Heinrich, Huber, Maitre '07; Heinrich, Huber, Kosower, Smirnov '09; Lee, Smirnov, Smirnov '10]
- Five-particle cuts (purely phase-space integrals). [Gituliar, V.M., Pikelner '18]
- Three- and four-particle cuts: completed now, this talk.

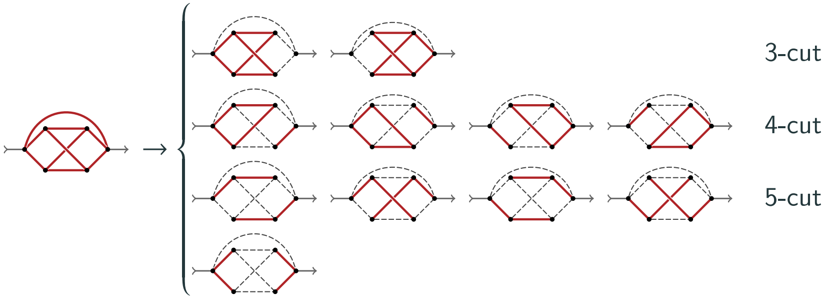
Identifying cut masters

Start with the 28 master integrals for 4-loop propagators. [Baikov, Chetyrkin '10]

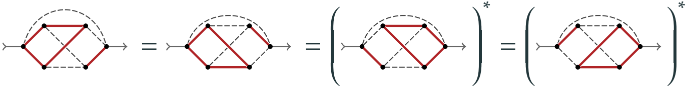


Identifying cut masters, II

Cut each virtual master in all possible ways.



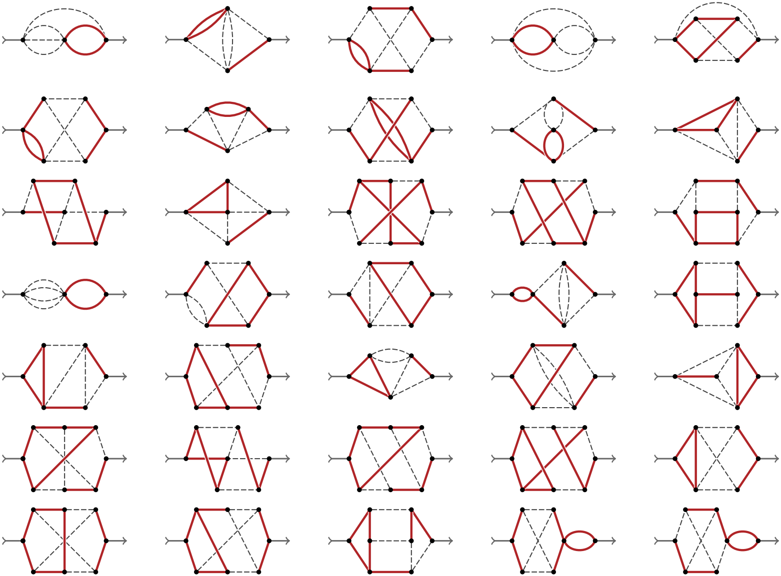
Remove symmetric duplicate integrals.



Done! There are no additional IBP relations between the remaining integrals. No additional master integrals either.

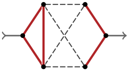
4-particle cut masters

There are 35 master integrals for 4-particle cuts.

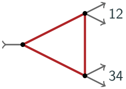


4-particle cuts: direct analytic integration?

Integrate the 1-loop amplitude over the phase space?



$$= \int$$



$$\frac{1}{s_{13}s_{24}} d\mathrm{PS}_4$$

4-particle phase-space has 5 degrees of freedom and nontrivial shape.

$$d\mathrm{PS}_4 = \left(q^2\right)^{\frac{3d-8}{2}} 2^{1-2d} \left(2\pi\right)^{4-3d} \Omega_{d-1}\Omega_{d-2}\Omega_{d-3} \left(\Delta_4\right)^{\frac{d-5}{2}} \times$$
$$\Theta(\Delta_4)\ \Theta(s_{ij})\, \delta\Big(1 - \sum s_{ij}\Big)\, ds_{12}\, ds_{13}\, ds_{14}\, ds_{23}\, ds_{24}\, ds_{34}$$

$$\Delta_4 = \det \begin{vmatrix} 0 & s_{12} & s_{13} & s_{14} \\ s_{12} & 0 & s_{23} & s_{24} \\ s_{13} & s_{23} & 0 & s_{34} \\ s_{14} & s_{24} & s_{34} & 0 \end{vmatrix}$$

4-particle cuts: direct analytic integration?

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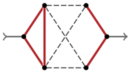
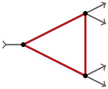


$= \int$



$\frac{1}{s_{13}s_{24}} d\mathrm{PS}_4$

4-particle phase-space has 5 degrees of freedom and nontrivial shape.

Parametrisations exist that simplify that shape to a hypercube, e.g. the "tripole parametrisation": [Gehrmann-De Ridder, Gehrmann, Heinrich '03]

$$
\begin{aligned}
s_{12} &\to (1-t)\,(1-y)\,(1-z) \\
s_{13} &\to y\left(1-t-v+tv+tvz-2\,(1-2\xi)\,\sqrt{(1-t)(1-v)tvz}\right) \\
s_{14} &\to vy\,(1-z) \\
s_{23} &\to z(1-y) \\
s_{24} &\to t(1-y)\,(1-z) \\
s_{34} &\to y\left(t-tv+vz-tvz+2\,(1-2\xi)\,\sqrt{(1-t)(1-v)tvz}\right)
\end{aligned}
$$

Such parametrisations contain roots, making analytic integration difficult.

Dimensional recurrence relations

Using a parametric representation of integrals and the IBP tables, obtain dimensional recurrence relations (DRR) for each master I_i: [Tarasov '96; Lee '09]

$$I_i(d+2) = M_{ij}(d)\, I_j(d)$$

For 4-particle cuts, the DRR matrix M_{ij} is:

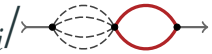
- Triangular! No coupled blocks, can be solved one by one.
- Has a factorizable diagonal of the form $M_{ii} = C \prod_k (d/2 - a_k)^{n_k}$.

The general solution is

$$I_i(d) = H_i(d)\, \omega(d) + R_i(d)\,,$$

- H_i is a homogeneous solution, $H_i = C^{d/2} \prod_k \Gamma^{n_k}(d/2 - a_k)$;
- R_i is a particular solution, $R_i = -H_i(d) \sum_{k=d}^{\infty} H_i^{-1}(k+2) \sum_{j<i} M_{ij}(k)\, I_j(k)$;
- ω is an arbitrary periodic function, $\omega(d+2) = \omega(d)$, which can't be determined from DRR alone. Finding $\omega(d)$ is the main challenge.

4-particle cut masters: fixing $\omega(d)$

1. Find d_0, such that $I_i(d)$ is finite if $\mathrm{Re}(d) \in (d_0, d_0 + 2]$.
   - 4-particle masters I_i mostly diverge at even d.
   - But $J_i \equiv I_i /$  do not! All J_i are finite if $\mathrm{Re}(d) \in [6, 8]$.
   - All homogeneous solutions $H_i(d)$ can be chosen to be finite too.

4-particle cut masters: fixing $\omega(d)$

1. Find d_0, such that $I_i(d)$ is finite if $\mathrm{Re}(d) \in (d_0, d_0 + 2]$.
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   - All homogeneous solutions $H_i(d)$ can be chosen to be finite too.
2. Find the poles of the particular solution $R_i(d)$ by evaluating it numerically for many values of d.
   - Use `DREAM` for the evaluation. [Lee, Mingulov '17]
   - All $R_i(d)$ appear to be smooth.

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3. Construct an ansatz for $\omega_i(d)$ looking at the poles of J_i, H_i, and R_i.
   - We use $C_0 + \sum C_i \cot\left(\frac{\pi}{2}(d - d_i)\right)$ or modifications.
   - For smooth J_i, H_i, and R_i only one choice is possible: a constant, C_0.

4-particle cut masters: fixing $\omega(d)$

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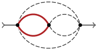
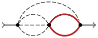
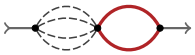
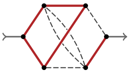
   - For smooth J_i, H_i, and R_i only one choice is possible: a constant, C_0.
4. Fix the constants in the ansatz from various considerations.
   - The leading pole of I_i can be calculated by inserting a mass in the loop, and looking at large mass expansion of the integral—this is enough to fix one constant.
   - Turns out $\omega_i(d) = 0$ for all integrals, except for three trivial integrals:

     , , and .

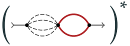
4-particle cut masters, wrap up

Once ω_i are fixed, the masters can be evaluated with SummerTime or DREAM numerically to arbitrary precision, and then restored in terms of MZVs via PSLQ.

Example result:



$$= \frac{\left(\text{[diagram]} \right)^*}{\left(q^2\right)^3} \Bigg[-6\zeta_2 \frac{1}{\epsilon^2} + \left(59\zeta_2 - 60\zeta_3\right)\frac{1}{\epsilon} + \Big(-203\zeta_2 + 590\zeta_3 - 129\zeta_2^2 \Big) + \left(288\zeta_2 - 2030\zeta_3 + \frac{2537}{2}\zeta_2^2 + 192\zeta_2\zeta_3 - 1806\zeta_5 \right)\epsilon + \mathcal{O}\left(\epsilon^2\right) \Bigg]$$



Overall, we've restored the series up to MZVs of weight 12. They have poles up to $1/\epsilon^5$. Zetas up to weight 6 in the ϵ-finite part.

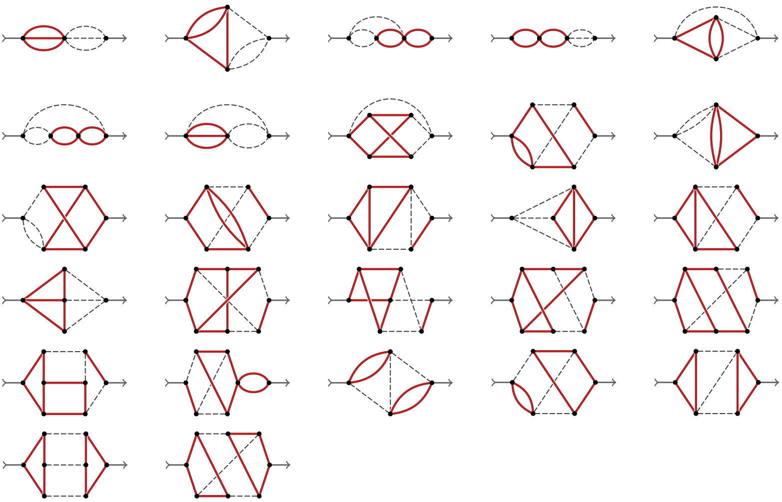
4-particle cut masters, numerical checks

Numerical evaluation:

- Feynman parameterization of the loop amplitudes.
  - 1-loop amplitude—all the UV divergence is in the prefactor of the parameterization, the integral part is finite.
- Tripole parameterization of the phase space. [Gehrmann-De Ridder et al. '03]
- Direct Monte-Carlo integration of the result.
  - Converges, because the divergent prefactor is separated!
- Checked at many different values of d.

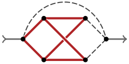
3-particle cut masters

There are 27 master integrals among the 3-particle cuts.

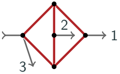


How to calculate 3-particle cuts?

Split integrals into the 2-loop 1→3 amplitude and 3-particle phase space,



$$= \int$$

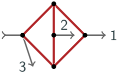


$$d\mathrm{PS}_3.$$

Parameterize the phase-space via $s_{ij} = \left(p_i + p_j\right)^2 / q^2$,

$$d\mathrm{PS}_3 = \left(q^2\right)^{d-3} \frac{2^{4-3d} \pi^{\frac{3}{2}-d}}{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{d-1}{2}\right)} \left(s_{12}s_{13}s_{23}\right)^{\frac{d-4}{2}} \delta\left(1 - \sum s_{ij}\right) \prod ds_{ij}.$$

Insert 2-loop 1→3 amplitudes as series in ϵ in terms of multiple polylogarithms depending on s_{12} and s_{13}:



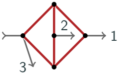
$$= \frac{\mathscr{C}}{\epsilon^4} \left(s_{12}\right)^{d-6} = \frac{\mathscr{C}}{s_{12}^2} \left(\frac{1}{\epsilon^4} - \frac{2 \ln s_{12}}{\epsilon^3} + O\left(\frac{1}{\epsilon^2}\right) \right)$$

Multiply series and integrate order by order?

How to calculate 3-particle cuts, II

Integration of the series does not converge:

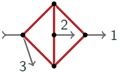
$$\int$$



$$dPS_3 = \int \frac{\mathscr{C}}{s_{12}^2} \left(\frac{1}{\epsilon^4} - \frac{2 \ln s_{12}}{\epsilon^3} + O\left(\frac{1}{\epsilon^2}\right)\right) dPS_3 = \frac{\infty}{\epsilon^4} + \frac{\infty}{\epsilon^3} + \ldots$$

Taking series after integration does:

$$\int$$



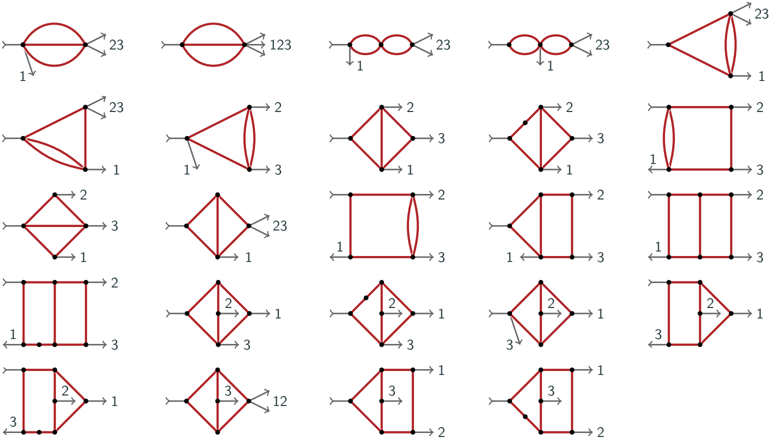
$$dPS_3 = \int \frac{\mathscr{C}}{\epsilon^4} (s_{12})^{d-6} dPS_3 = \frac{\mathscr{K}}{\epsilon^5} + \frac{\mathscr{K}'}{\epsilon^4} + \ldots$$

Problem: the integral is not infrared-finite (in 4 dimensions).

Solution: look at the same integrals in $d = 6 - 2\epsilon$, which are. Taking series and integrating order-by-order will commute there.

Aside: 2-loop 1→3 amplitudes

28 in total, not counting $\{p_1, p_2, p_3\}$ permutations.



Aside: 2-loop 1→3 amplitudes, II

First computed up to weight 4 by Gehrmann and Remiddi, recomputed to weight 8 by us.

- Higher weights are needed because the ϵ-finite part of the cut integrals contain MZVs up to weight 6.

To compute: [Gehrmann, Remiddi '00, '01]

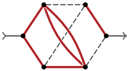
1. Write down the differential equations in s_{12} and s_{13}.
2. Find the ϵ-form in 2 variables (automatized by the new Fuchsia) and construct the general solution. [github.com/magv/fuchsia.cpp]
   - The general form is $\sum \mathscr{R}\left(s_{ij}, \mathscr{C}\right) G(\{0,1,1-s_{13},-s_{13}\};s_{12})\, G(\{0,1\};s_{13})$.
3. Fix the integration constants $\mathscr{C}$ by:
   - matching with known single-scale integrals;
   - enforcing regularity at $s_{ij} \to 1$ (because the integrals are massless);
   - enforcing regularity at $s_{ij} \to 0$ for planar integrals if i and j are not adjacent.

How to calculate 3-particle cuts, III

In summary:

1. Use dimensional recurrence for the 1→3 amplitudes to get them as series around $d = 6 - 2\epsilon$.
2. Multiply by ϵ-expansion of $d\mathrm{PS}_3$, integrate order by order.
3. Use dimensional recurrence for the cut integrals to lower the series to $4 - 2\epsilon$.

Example result:



=

$$\frac{}{\left(q^2\right)^2}\Big[2\zeta_2\frac{1}{\epsilon} + \Big(-13\zeta_2 + 16\zeta_3\Big) + \Big(27\zeta_2 - 104\zeta_3 + \frac{156}{5}\zeta_2^2\Big)\epsilon + \Big(-18\zeta_2 + 216\zeta_3 - \frac{1014}{5}\zeta_2^2 - 90\zeta_2\zeta_3 + 448\zeta_5\Big)\epsilon^2 + \mathscr{O}\left(\epsilon^3\right)\Big]$$

Overall, series up to MZVs of weight 8. Poles up to $1/\epsilon^6$. Zetas up to weight 6 in the ϵ-finite part.

3-particle cut masters, numerical checks

Numerical evaluation:

- Feynman parameterization for the loop amplitude.
- Direct parameterization of the phase space in Mandelstam variables.
- Sector decomposition with `FIESTA`. [Smirnov '15]
- Checked in $d = 6 - 2\epsilon$, and $8 - 2\epsilon$. Too slow in $4 - 2\epsilon$.

Cutkosky rules check

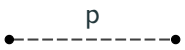
Cutkosky rules for each Feynman diagram F:

$$F + F^* = -\sum_i \mathrm{Cut_i} F$$

To use this simple form, add Feynman rules to the integrals:

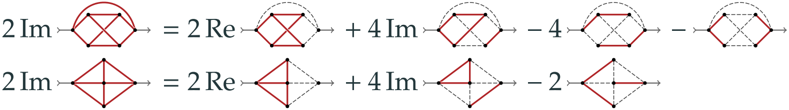
$= \cdots = i$

$= \dfrac{i}{p^2 + i0}$



$= 2\pi\delta^+\left(p^2\right)$

Then, for each of the 31 propagator master:



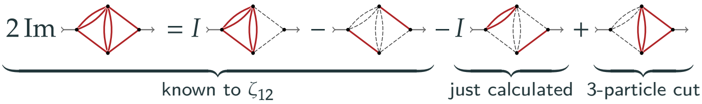
...

All cuts are now known (at least to weight 8)! Insert them here to check the result consistency.

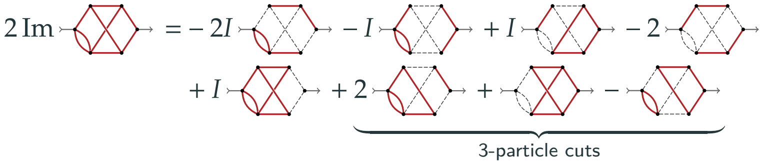
3-particle cut masters via DRR

Bonus: solve DRR for 3-particle cuts too, using everything calculated so far.

- The simple case: only one 3-particle cut in a Cutkosky relation.



- The complex case: multiple 3-particle cuts enter a Cutkosky relation.



  - Only the sum is constrained, but we have also calculated the expansion to weight 8 of these integrals—enough information in total to fix the constants in the $\omega_i(d)$ ansatz, and solve the DRR completely!

Result: all 3-particle cut masters upgraded from fixed weight 8 series to any-weight any-d DRR solutions.

Summary

- Master integrals for:
  - 3-particle (+ 2-loop) cuts of 4-loop propagators;
  - 4-particle (+ 1-loop) cuts of 4-loop propagators.
- Results as series in ϵ, up to MZVs of weight 12.
- Bonus:
  - `SummerTime` files to expand around any d with arbitrary precision.
  - For convenience, results for all other cuts included too.
  - Master integrals for 2-loop 1→3 amplitudes up to weight 8.
- To be published shortly.

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Thank you for your attention.