The R*-method: Recent Developments and Applications

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Franz Herzog
What is R*?

A generalisation of BPHZ to Euclidean IR divergences [1983 Chetyrkin, Tkachov; Smirnov]
What is $R^*$?

A generalisation of BPHZ to Euclidean IR divergences [1982 Chetyrkin, Tkachov; Smirnov]

\[ R^*(\Gamma) = \text{finite} \]

- an offshell Feynman diagram
- By adding counter-terms associated to UV- or IR- divergent subdiagrams
2-loop example

\[ R^* \left( \begin{array}{c}
\hspace{1cm} 1 \\
\hspace{1cm} 2 \\
\hspace{1cm} 3 \\
\end{array} \right) = \]
2-loop example

\[ R^*( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} ) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} + Z( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} ) \]
2-loop example

\[ R^*(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}) = \begin{array}{c} 1 \\ 2 \\ 3 \end{array} + Z(\begin{array}{c} 1 \\ 2 \\ 3 \end{array}) + Z(\begin{array}{c} 2 \\ 3 \end{array}) \]
2-loop example

\[ R^* \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) = R^* \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) + Z \left( \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \right) \]

\[ + Z \left( \begin{array}{c} 2 \\ 3 \end{array} \right) * \begin{array}{c} 1 \end{array} + \tilde{Z} \left( \begin{array}{c} 1 \end{array} \right) * \begin{array}{c} 2 \\ 3 \end{array} \]
2-loop example

\[ R^* \left( \begin{array}{cc}
1 & \\
2 & 3
\end{array} \right) = \begin{array}{c}
1 \\
2 \\
3
\end{array} + Z \left( \begin{array}{cc}
1 & \\
2 & 3
\end{array} \right) \]

\[ + Z \left( \begin{array}{cc}
2 & \\
3 & 1
\end{array} \right) * \begin{array}{c}
1 \\
2 \\
3
\end{array} + \tilde{Z} \left( \begin{array}{c}
1 \\
2 \\
3
\end{array} \right) * \begin{array}{c}
1 \\
2 \\
3
\end{array} \]

\[ + \tilde{Z} \left( \begin{array}{c}
1 \\
2 \\
3
\end{array} \right) * Z \left( \begin{array}{cc}
2 & \\
3 & 1
\end{array} \right) * 1 \]
General structure at L loops

\[ R^*(\Gamma) = \sum_{S \subseteq \Gamma, \tilde{S} \subseteq \Gamma \atop S \cap \tilde{S} = \emptyset} \tilde{Z}(\tilde{S}) \ast Z(S) \ast \Gamma/S \setminus \tilde{S} \]
General structure at L loops

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UV counterterm

\[ Z(\Gamma) = -K \left( \sum_{s \subseteq \Gamma, \tilde{s} \subseteq \Gamma \atop s \cap \tilde{s} = \emptyset} \tilde{Z}(\tilde{s}) \ast Z(S) \ast \Gamma / S \setminus \tilde{s} \right) \]
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IR counterterm

\[ \tilde{Z}(\Gamma_0) = -K \left( \sum_{S \subseteq \Gamma_0, \tilde{S} \subseteq \Gamma_0} \tilde{Z}(\tilde{S}) \ast Z(S) \ast \Gamma_0 / S \setminus \tilde{S} \right) \]
Hopf algebraic formulation of R*

with M Borinsky and R Beekveldt, to be published!
Hopf algebraic formulation of $R^*$

with M Borinsky and R Beekveldt, to be published!

- $R^*$ can be embedded into Francis Browns motic Hopf Algebra

\[
R^* (\Gamma) = \mu_A \circ (id_A \otimes \mu_A) \circ (Z \otimes \phi \otimes \tilde{Z}) \circ (id_M \otimes \Delta^I_R) \circ \Delta^U_M (\Gamma)
\]
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\[ R^*(\Gamma) = \mu_A \circ (id_A \otimes \mu_A) \circ (Z \otimes \phi \otimes \tilde{Z}) \circ (id_M \otimes \Delta^{IR}_M) \circ \Delta^{UV}_M (\Gamma) \]

- IR and UV subgraph combinatorics is organised via IR and UV coactions, which are coassociative:

\[ (\Delta^{UV} \otimes id) \circ \Delta^{IR} = (id \otimes \Delta^{IR}) \circ \Delta^{UV} \]
Hopf algebraic formulation of \( R^* \)

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- \( R^* \) can be embedded into Francis Browns *motic* Hopf Algebra

\[
R^*(\Gamma) = \mu_A \circ (id_A \otimes \mu_A) \circ (Z \otimes \phi \otimes \tilde{Z}) \circ (id_\mathcal{M} \otimes \Delta^{IR}_\mathcal{M}) \circ \Delta^{UV}_\mathcal{M}(\Gamma)
\]

- IR and UV subgraph combinatorics is organised via IR and UV coactions, which are coassociative:

\[
(\Delta_{UV} \otimes id) \circ \Delta_{IR} = (id \otimes \Delta_{IR}) \circ \Delta_{UV}
\]

- UV and IR counterterm operations or "homomorphisms" are related via the antipode "endomorphism":

\[
\tilde{Z} = Z \circ S
\]
How is $R^*$ useful?

The UV counterterm is a homogeneous polynomial in the external momenta whose degree equals the graphs superficial degree of divergence.
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→ IR rearrangement [Vladimirov]

\[ Z(\quad) = Z(\quad) = Z(\quad) \]
How is R* useful?

IR rearrangement with R* can be used to extract the UV counterterm of an L-loop n-point diagram from at most (L-1)-loop 2-point diagrams. [Chetyrkin, Tkachov]
How is $R^*$ useful?

IR rearrangement with $R^*$ can be used to extract the UV counterterm of an $L$-loop $n$-point diagram from at most $(L-1)$-loop 2-point diagrams.

[Chetyrkin, Tkachov]

\[
\begin{align*}
\text{L=3} & \quad \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{((k + P)^2)^2} \\
\text{L=2} & \quad P \bigg|_{P^2=1} \cdot \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{((k + P)^2)^2(k^2)^{2\epsilon}} \\
\text{L=1 (with non integer power)} & \quad
\end{align*}
\]
How has R* been used?  
(Apologies for incompleteness)

**Local approach 1982**  
[Chetyrkin, Tkachov; Smirnov]  
(applied diagram by diagram)

Chetyrkin, Kataev, Tkachov 1980,  
Larin, Schulte, Neu Frohlinde 1990

Renormalisation of $\phi^4$ at 5 loops  
Kompaniets, Panzer 2016  
Renormalisation of $\phi^4$ at 6 loops
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Global approach [Chetyrkin]
(relies on constructing global IR counterterms via heavy mass expansion; not automated but highly efficient) 90s?

Chetyrkin, Baikov, Kuehn:
- e+e-→Hadrons, Z decay & H→bb up to N4LO in QCD, tau decays 2005-2009
- QCD renormalisation at 5 loops for SU(3) 2016 .. more

Chetyrkin, Falcioni, FH, Vermaseren
- QCD renormalisation at 5 loops for general gauge group 2017

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DeVries, Falcioni, FH, Ruijl 2019
- AD of Weinberg's operator at 2&3 loop
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Local for generic numerators 2017 [FH, Ruijl]
(applicable for poles of 2point functions only; automated)

FH, Ruijl, Vermaseren, Vogt, Ueda:
- QCD beta function at 5 loops for general gauge group 2017
- e+e→Hadrons, & Higgs width up to N4LO in QCD 2017
- 5-loop Mellin moments of splitting functions 2019
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Local approach for general QFTs 2018 [FH, Ruijl] to be published
(completely general for all local QFTs and fully automisable!)

DeVries, Falcioni, FH, Ruijl 2019
- AD of Weinberg’s operator at 2&3 loop
First moments of N4LO Splitting functions

• Splitting functions are the (non-local) anomalous dimensions of the parton distribution functions:

\[ \frac{d}{d \log \mu^2} f_i(x, \mu) = \sum_j \int_x^1 \frac{dy}{y} P_{ij}(y) f_j(x/y, \mu) \]
First moments of N4LO
Splitting functions

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\]

- However, \( R^* \) is applicable for Mellin moments via the OPE

\[
O_{g^\mu_1...\mu_n} = \frac{1}{2} F_{a\mu_1}^{\alpha} D_{\mu_2}^{\mu_2} ... D_{\mu_{n-1}}^{\mu_{n-1}} F_{a\alpha \mu_n}^{\mu_n}
\]
First moments of N4LO non-singlet splitting functions

- Used $R^*$ to compute $N=2,3$ moments of the N4LO non-singlet splitting functions [arxiv:1812.11818]
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Parton distribution functions

$$q_{ij}^\pm(x) = q_i(x) \pm \bar{q}_i(x) - (q_j(x) \pm \bar{q}_j(x))$$

$$q_V(x) = \sum_{i=1}^{n_f} (q_i(x) - \bar{q}_i(x))$$

Computed Mellin moments of anomalous dimensions:

$$\gamma_{ns}^+(N = 2) \quad \gamma_{ns}^-(N = 3)$$

$$\gamma_{ns}^v(N = 3)$$
Approximate Splitting functions at N4LO for leading $N_c$

$\gamma_{ns}^+ = \gamma_{ns}^- + O(1/N_c)$

Fit use known large-$x$ and small-$x$ endpoint constraints

$(1-x) P_{ns}^{(4)}(x)/10^5$

$n_f = 3, \ln c$
Cusp anomalous dimension at N4LO

The cusp is related to the (Mellin) $N \to \infty$ limit of the splitting function:

$$\gamma_{kk}(N) = A_k \ln \tilde{N} - B_k + C_k N^{-1} \ln \tilde{N} - (D_k - \frac{1}{2} A_k) N^{-1} + O(N^{-2} \ln^n \tilde{N})$$

$$A_q = A_1 a_s + A_2 a_s^2 + \ldots$$
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$$A_q = A_1 a_s + A_2 a_s^2 + \ldots$$

**QCD result:**

$$A_5 = (1.7 \pm 0.5 , 1.1 \pm 0.5 , 0.7 \pm 0.5) \cdot 10^5 \quad \text{for} \quad n_f = 3, 4, 5.$$  

**QCD result at leading color:**

$$A_{5, L} = (1.5 \pm 0.25 , 0.8 \pm 0.2 , 0.4 \pm 0.1) \cdot 10^5 \quad \text{for} \quad n_f = 3, 4, 5.$$
Automating $R^*$ for general QFTs

- Input: Lagrangian, 1PI correlator
- Output: UV counterterm
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- Output: UV counterterm

\[
\Gamma = \langle \Phi_1^{s_1}(x_1) ... \Phi_n^{s_n}(x_n) \rangle_{1\text{PI}}
\]

\[
\mathcal{L} = \sum_i c_i O_i
\]

\[
Z(\Gamma) = \sum_i Z_i O_i^{s_1 ... s_n}
\]
Renormalising the SM EFT at two loops

\[ \mathcal{L}_{\text{SMEFT}} = \mathcal{L}_{\text{SM}} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \ldots \]
\[ \mathcal{L}^{(d)} = \sum_{i=1}^{n_d} \frac{c_i^{(d)}}{\Lambda^{d-4}} O_i^{(d)} \]

- Many operators and they mix!
  already 84(59) operators at D=6, 993 at D=8,...
  [Henning, Lu, Melia, Murayama]
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- Full AD mixing matrix so far only known at 1-loop
  [Jenkins, Manohar, Trott; ... ]
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- 2-loop corrections to ADs:
  - Improve accuracy of SM EFT predictions
  - Allow to run Wilson coefficients up to higher scales to address impact on: Vacuum stability, Inflation, ..
SMEFT constraints from EDMs

- Loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as Neutron EDM measurements.
SMEFT constraints from EDMs

- Loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as Neutron EDM measurements.
- The dominant operator relevant for Neutron EDMs is the operator

\[ O_W = \frac{d_W}{6} f^{abc} \varepsilon_{\mu \nu \alpha \beta} G^{a}_{\alpha \beta} G^{b}_{\mu \rho} G^{c}_{\nu \rho} \]

[Weinberg 1989]
Calculation of 2&3-loop AD of Weinberg’s operator

- **Method:**
  - Local R*-method in background field gauge

- Pros:
  - Operator does not mix for \( nf = 0 \)

- Cons:
  - 2pt function vanishes
  - Computed 3-pt at 2&3 loops (3 derivatives) and 4-pt at 2 loops
  - Not well defined in dimreg.

We used both \( \text{T Hooft-Veltman} \) and Larin scheme check.
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  - Computed **3-pt** at 2&3 loops (3 derivatives) and **4-pt** at 2 loops
  - $\varepsilon^{\mu\nu\rho\sigma}$ **not** well defined in dimreg. We used both `T Hooft-Veltman and Larin scheme check.`
Calculation of 2&3-loop AD of Weinberg’s operator

\[ \gamma_W = \frac{\alpha_s(\mu^2)}{4\pi} \left[ \frac{C_A}{2} + n_f \right] + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^2 \frac{119}{18} C_A^2 
+ \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^3 \left[ C_A^3 \left( -\frac{3203}{648} + 44\zeta_3 \right) 
+ \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A} \left( 40 - 1056\zeta_3 \right) \right], \]

Result: \( (n_f=0, L>1) \)
Calculation of 2&3-loop AD of Weinberg’s operator

Result:
\[ \gamma_W = \frac{\alpha_s(\mu^2)}{4\pi} \left[ \frac{C_A}{2} + n_f \right] + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^2 \frac{119}{18} C_A^2 \]
\[ + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^3 \left[ C_A^3 \left( -\frac{3203}{648} + 44\zeta_3 \right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A} \left( 40 - 1056\zeta_3 \right) \right], \]

Numerical:
\[ \frac{8\pi \gamma_W}{\alpha_s(\mu^2) C_A} = 1 + 3.15657 \alpha_s - 23.72872 \alpha_s^2 \]
\[ = 5.46537 - \frac{29.19409}{\text{Quartic!}} \]
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Result: \( \gamma_W = \frac{\alpha_s(\mu^2)}{4\pi} \left[ \frac{C_A}{2} + n_f \right] + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^2 \frac{119}{18} C_A^2 \)

\[ + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^3 \left[ C_A^3 \left( - \frac{3203}{648} + 44\zeta_3 \right) \right. \]
\[ + \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A} \left( 40 - 1056\zeta_3 \right) \]

Interesting to compare with QCD beta function at 3 loops:

\[ \beta_2 = \frac{2857}{54} C_A^3 \]

But this has no quartic? Why?
Answer: quartic can not appear in 3-loop 2-point function
Calculation of 2&3-loop AD of Weinberg’s operator

Result: \( nf=0, L>1 \)

\[
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\[
+ \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^3 \left[ C_A^3 \left( -\frac{3203}{648} + 44\zeta_3 \right) \right.
\]

\[
+ \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A} \left( 40 - 1056\zeta_3 \right)
\],

What about 4-loops?:

\[
\beta_3 = C_A^4 \left( \frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{80}{9} + \frac{704}{3} \zeta_3 \right)
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Calculation of 2&3-loop AD of Weinberg’s operator

Result:  
\( \gamma_W = \frac{\alpha_s(\mu^2)}{4\pi} \left[ \frac{C_A}{2} + n_f \right] + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^2 \frac{119}{18} C_A^2 \\ + \left( \frac{\alpha_s(\mu^2)}{4\pi} \right)^3 \left[ C_A^3 \left( -\frac{3203}{648} + 44\zeta_3 \right) \right. \\
\left. + \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A} \left( 40 - 1056\zeta_3 \right) \right] \),

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\( \beta_3 = C_A^4 \left( \frac{150653}{486} - \frac{44}{9} \zeta_3 \right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A} \left( -\frac{80}{9} + \frac{704}{3} \zeta_3 \right) \)

Coincidence?

\( 40 - 1056\zeta_3 = -\frac{9}{2} \left( -\frac{80}{9} + \frac{704}{3} \zeta_3 \right) \)
Conclusions

- Summarised developments and applications of R*
- Reported on status towards automation of R* for general local QFTs
- Computed first 3 Moments of non-singlet splitting functions at N4LO (5 loops)
  - Also extracted corresponding Cusp anomalous dimension
- Computed 2- and 3-loop anomalous dimensions of Weinberg’s operator