# The R*-method: Recent Developments and Applications

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Franz Herzog

# What is R*?

A generalisation of BPHZ to Euclidean IR divergences [1983 Chetyrkin,Tkachov; Smirnov]

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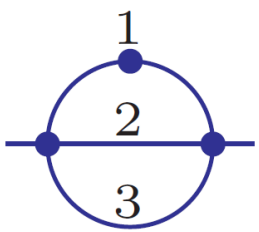
A generalisation of BPHZ to Euclidean IR divergences [1982 Chetyrkin,Tkachov; Smirnov]

$$R^*(\Gamma) = \text{finite}$$

an offshell Feynman diagram

By adding counter-terms associated to UV- or IR- divergent subdiagrams

# 2-loop example

$R^*($  $) \quad =$

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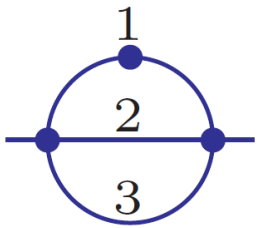
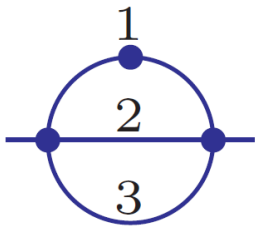
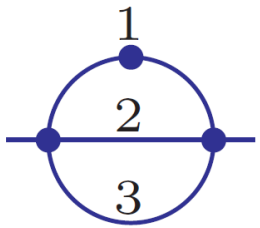
$R^*($  $) \quad = \quad$  $+ \, Z($  $)$

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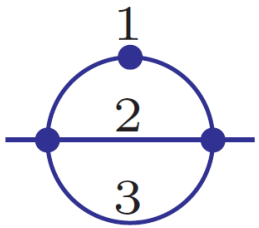
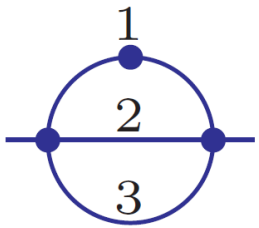
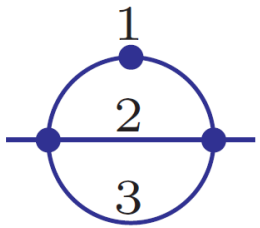
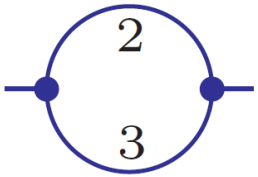
$R^*($  $) \quad = \quad$  $+ Z($  $)$

$+ Z($  $) * \quad$ 

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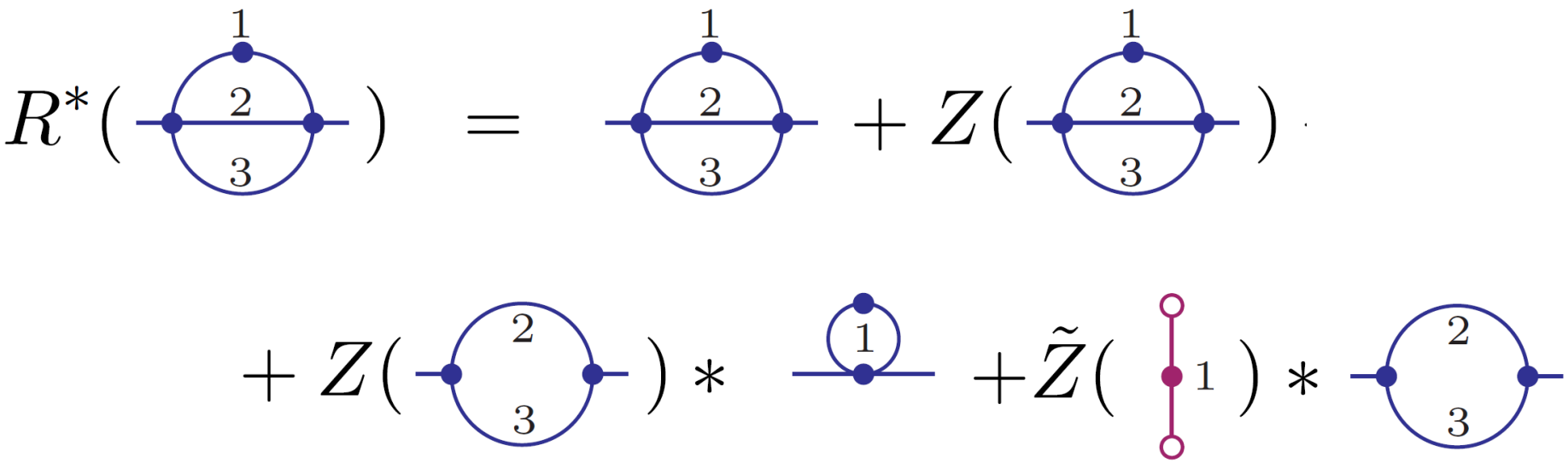

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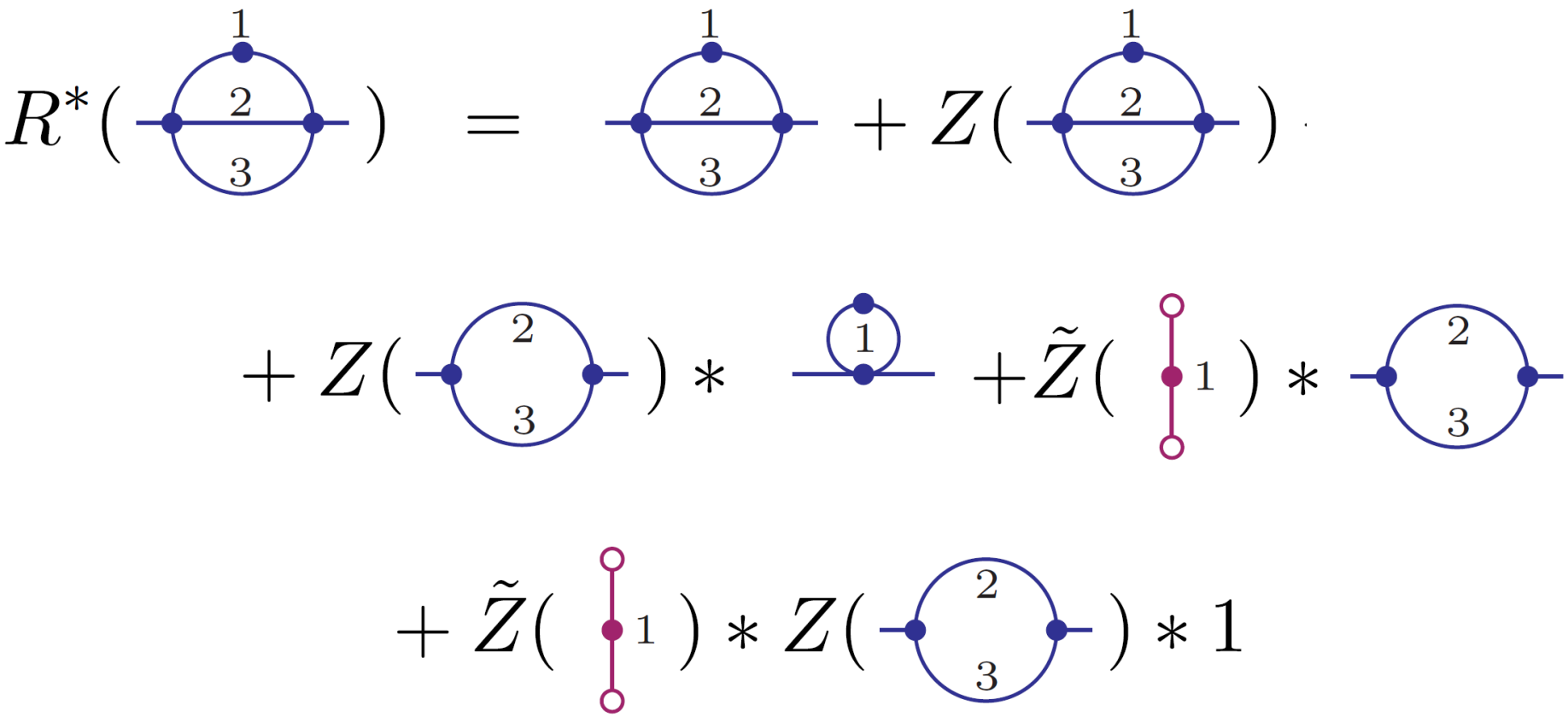

# General structure at L loops

$$R^*(\Gamma) = \sum_{\substack{S \subseteq \Gamma, \tilde{S} \subseteq \Gamma \\ S \cap \tilde{S} = \emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma / S \setminus \tilde{S}$$

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UV counterterm

$$Z(\Gamma) = -K\Big( \sum_{\substack{S\subsetneq\Gamma,\tilde{S}\subseteq\Gamma \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma/S \setminus \tilde{S}\Big)$$

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IR counterterm

$$\tilde{Z}(\Gamma_0) = -K\Big(\sum_{\substack{S\subseteq\Gamma_0,\tilde{S}\subsetneq\Gamma_0 \\ S\cap\tilde{S}=\emptyset}} \widetilde{Z}(\tilde{S}) * Z(S) * \Gamma_0/S\setminus\tilde{S}\Big)$$

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with M Borinsky and R Beekveldt, *to be published!*

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$$R^*(\Gamma) = \mu_A \circ (id_A \otimes \mu_A) \circ (Z \otimes \phi \otimes \widetilde{Z}) \circ (id_{\mathcal{M}} \otimes \Delta^{IR}_{\mathcal{M}}) \circ \Delta^{UV}_{\mathcal{M}}(\Gamma)$$

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- IR and UV subgraph combinatorics is organised via IR and UV coactions, which are coassociative:

$$(\Delta_{\mathrm{UV}} \otimes \mathrm{id}) \circ \Delta_{\mathrm{IR}} = (\mathrm{id} \otimes \Delta_{\mathrm{IR}}) \circ \Delta_{\mathrm{UV}}$$

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- UV and IR counterterm operations or "homomorphisms" are related via the antipode "endomorphism":

$$\tilde{Z} = Z \circ S$$

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S\otimes\mathrm{id}"] & & H \otimes H \arrow[dr, "\nabla"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & K \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}\otimes S"'] & & H \otimes H \arrow[ur, "\nabla"'] &
\end{tikzcd}$$

# How is R* useful?

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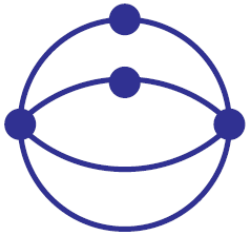
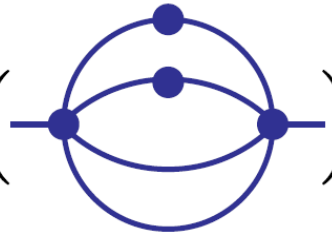
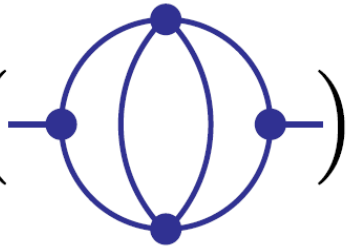
→ IR rearrangement [Vladimirov]

$Z($  $) = Z($  $) = Z($  $)$

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IR rearrangement with R* can be used to extract the UV counterterm of an L-loop n-point diagram from at most (L-1)-loop 2-point diagrams.
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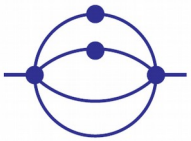
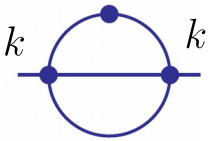
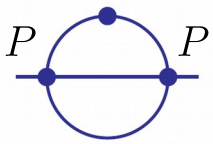
L=3



$$= \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{((k+P)^2)^2}$$




$$= \left.$$




$$\right|_{P^2=1} \cdot \int \frac{d^D k}{i\pi^{D/2}} \frac{1}{((k+P)^2)^2 (k^2)^{2\epsilon}}$$

L=2

L=1 (with non integer power)

# How has R* been used?

**(Apologies for incompleteness)**

Local approach 1982
[Chetyrkin,Tkachov; Smirnov]
(applied diagram by diagram)

Chetyrkin, Kataev, Tkachov 1980,
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*Renormalisation of* $\phi^4$ *at 5 loops*
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(relies on constructing global IR counterterms via heavy mass expansion; not automated but highly efficient) 90s?

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*- e+e-→Hadrons, Z decay & H→bb up to N4LO in QCD, tau decays 2005-2009*
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## Local approach for general QFTs 2018 [FH, Ruijl] *to be published*

(completely general for all local QFTs and fully automisable!)

DeVries, Falcioni, FH, Ruijl 2019
*- AD of Weinberg's operator at 2&3 loop*

# First moments of N4LO Splitting functions

- Splitting functions are the (non-local) anomalous dimensions of the parton distribution functions:

$$\frac{d}{d\log\mu^2} f_i(x,\mu) = \sum_j \int_x^1 \frac{dy}{y} P_{ij}(y) f_j(x/y,\mu)$$

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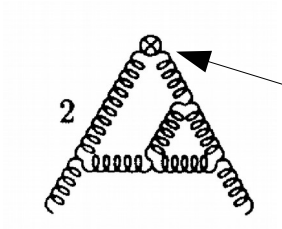
- However, R* is applicable for Mellin moments via the OPE




$$O_g^{\mu_1 \cdot \mu_n} = \frac{1}{2} F_\alpha^{a\mu_1} D^{\mu_2} ... D^{\mu_{n-1}} F^{a\alpha\mu_n}$$

# First moments of N4LO non-singlet splitting functions

- Used R* to compute N=2,3 moments of the N4LO non-singlet splitting functions [arxiv:1812.11818]

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Parton distribution functions

$$q_{ij}^{\pm}(x) = q_i(x) \pm \bar{q}_i(x) - (q_j(x) \pm \bar{q}_j(x))$$

$$q_{\mathrm{v}}(x) = \sum_{i=1}^{n_f} (q_i(x) - \bar{q}_i(x))$$

Computed Mellin moments of anomalous dimensions:

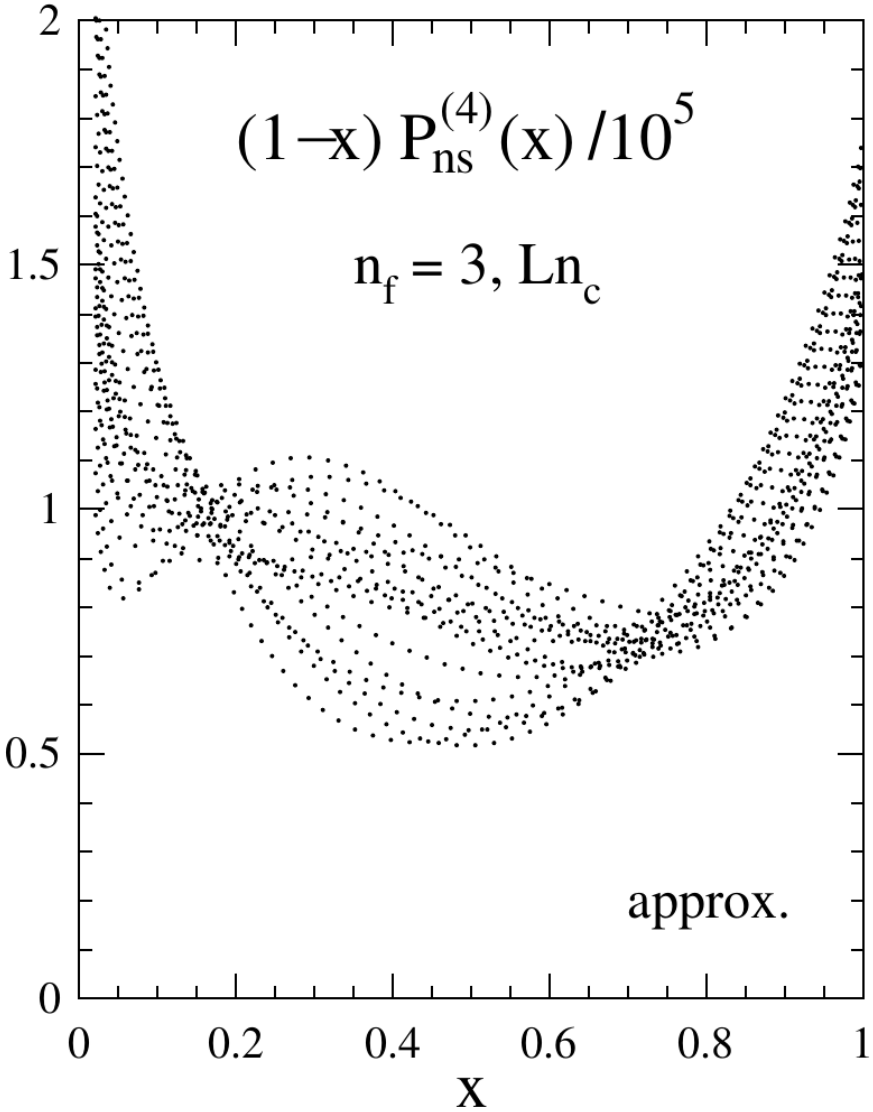
$$\gamma_{\mathrm{ns}}^{+}(N=2) \quad \gamma_{\mathrm{ns}}^{-}(N=3)$$

$$\gamma_{\mathrm{ns}}^{\mathrm{v}}(N=3)$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)+}(N=2) \;=\; & \\
& C_F^5\left[\frac{9306376}{19683}-\frac{802784}{729}\zeta_3-\frac{557440}{81}\zeta_5+\frac{12544}{9}\zeta_3^2+8512\zeta_7\right] \\
& -C_AC_F^4\left[\frac{81862744}{19683}-\frac{1600592}{243}\zeta_3+\frac{59840}{81}\zeta_4-\frac{142240}{27}\zeta_5+3072\zeta_3^2-\frac{35200}{9}\zeta_6+19936\zeta_7\right] \\
& +C_A^2C_F^3\left[\frac{63340406}{6561}-\frac{1003192}{243}\zeta_3-\frac{229472}{81}\zeta_4+\frac{61696}{27}\zeta_5+\frac{30976}{9}\zeta_3^2-\frac{35200}{9}\zeta_6+15680\zeta_7\right] \\
& -C_A^3C_F^2\left[\frac{220224724}{19683}+\frac{4115536}{729}\zeta_3-\frac{170968}{27}\zeta_4-\frac{3640624}{243}\zeta_5+\frac{70400}{27}\zeta_3^2+\frac{123200}{27}\zeta_6+\frac{331856}{27}\right. \\
& +C_A^4C_F\left[\frac{266532611}{39366}+\frac{2588144}{729}\zeta_3-\frac{221920}{81}\zeta_4-\frac{3102208}{243}\zeta_5+\frac{74912}{81}\zeta_3^2+\frac{334400}{81}\zeta_6+\frac{178976}{27}\right. \\
& -\frac{d_{AA}^{(4)}}{N_A}C_F\left[\frac{15344}{81}-\frac{12064}{27}\zeta_3-\frac{704}{3}\zeta_4+\frac{58400}{27}\zeta_5-\frac{6016}{3}\zeta_3^2-\frac{19040}{9}\zeta_7\right] \\
& +\frac{d_{FA}^{(4)}}{N_F}C_F\left[\frac{23968}{81}-\frac{733504}{81}\zeta_3+\frac{176320}{81}\zeta_5+\frac{6400}{3}\zeta_3^2+\frac{77056}{9}\zeta_7\right] \\
& -\frac{d_{FA}^{(4)}}{N_F}C_A\left[\frac{82768}{81}-\frac{555520}{81}\zeta_3+\frac{10912}{9}\zeta_4-\frac{1292960}{81}\zeta_5+\frac{84352}{27}\zeta_3^2+\frac{140800}{27}\zeta_6+12768\zeta_7\right] \\
& +n_fC_F^4\left[\frac{1824964}{19683}-\frac{463520}{243}\zeta_3+\frac{21248}{81}\zeta_4-\frac{16480}{81}\zeta_5+\frac{6656}{9}\zeta_3^2-\frac{6400}{9}\zeta_6+\frac{8960}{3}\zeta_7\right] \\
& -n_fC_AC_F^3\left[\frac{3375082}{6561}-\frac{420068}{243}\zeta_3-\frac{48256}{81}\zeta_4+\frac{458032}{81}\zeta_5+\frac{3968}{3}\zeta_3^2-\frac{8000}{3}\zeta_6+\frac{4480}{3}\zeta_7\right] \\
& +n_fC_A^2C_F^2\left[\frac{15291499}{13122}+\frac{1561600}{243}\zeta_3-\frac{114536}{27}\zeta_4-\frac{252544}{243}\zeta_5+\frac{24896}{27}\zeta_3^2+\frac{13600}{27}\zeta_6+\frac{11200}{27}\zeta\right. \\
& -n_fC_A^3C_F\left[\frac{48846580}{19683}+\frac{4314308}{729}\zeta_3-\frac{274768}{81}\zeta_4-\frac{1389080}{243}\zeta_5+\frac{27808}{81}\zeta_3^2+\frac{184000}{81}\zeta_6+\frac{39088}{27}\right. \\
& +n_f\frac{d_{FA}^{(4)}}{N_F}\left[\frac{22096}{27}+\frac{43712}{81}\zeta_3-\frac{512}{9}\zeta_4-\frac{217280}{81}\zeta_5-\frac{25088}{27}\zeta_3^2+\frac{25600}{27}\zeta_6-2464\zeta_7\right] \\
& -n_fC_F\frac{d_{FF}^{(4)}}{N_F}\left[\frac{170752}{81}-\frac{328832}{81}\zeta_3+\frac{650240}{81}\zeta_5-\frac{8192}{9}\zeta_3^2-\frac{35840}{9}\zeta_7\right] \\
& +n_fC_A\frac{d_{FF}^{(4)}}{N_F}\left[\frac{207824}{81}+\frac{251392}{81}\zeta_3-\frac{5632}{9}\zeta_4-\frac{522880}{81}\zeta_5+\frac{15872}{27}\zeta_3^2+\frac{70400}{27}\zeta_6-\frac{29120}{9}\zeta_7\right] \\
& +n_f^2C_F^3\left[\frac{1082297}{6561}-\frac{145792}{243}\zeta_3+\frac{1072}{81}\zeta_4+\frac{55552}{81}\zeta_5+\frac{1792}{9}\zeta_3^2-\frac{3200}{9}\zeta_6\right] \\
& +n_f^2C_AC_F^2\left[\frac{332254}{2187}-\frac{85016}{243}\zeta_3+\frac{20752}{27}\zeta_4-\frac{28544}{81}\zeta_5-\frac{13952}{27}\zeta_3^2+\frac{1600}{27}\zeta_6\right] \\
& +n_f^2C_A^2C_F\left[\frac{631400}{6561}+\frac{214268}{243}\zeta_3-784\zeta_4-\frac{53344}{243}\zeta_5+\frac{25472}{81}\zeta_3^2+\frac{22400}{81}\zeta_6\right] \\
& -n_f^2\frac{d_{FF}^{(4)}}{N_F}\left[\frac{43744}{81}-\frac{35648}{81}\zeta_3-\frac{1792}{9}\zeta_4-\frac{52480}{81}\zeta_5+\frac{2048}{27}\zeta_3^2+\frac{12800}{27}\zeta_6\right] \\
& +n_f^3C_F^2\left[\frac{265510}{19683}+\frac{11872}{729}\zeta_3-\frac{128}{3}\zeta_4+\frac{512}{27}\zeta_5\right] \\
& +n_f^3C_AC_F\left[\frac{168677}{19683}+\frac{11872}{729}\zeta_3+\frac{2752}{81}\zeta_4-\frac{4096}{81}\zeta_5\right]-n_f^4C_F\left[\frac{5504}{19683}+\frac{1024}{729}\zeta_3-\frac{128}{81}\zeta_4\right],
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{\rm ns}^{(4)-}(N=3) \;=\; & \\
& C_F^5\left[\frac{81472935625}{80621568}+\frac{99382175}{23328}\zeta_3-\frac{3395975}{162}\zeta_5-\frac{9650}{9}\zeta_3^2+\frac{34685}{2}\zeta_7\right] \\
& -C_AC_F^4\left[\frac{286028134219}{80621568}-\frac{23916529}{7776}\zeta_3-4490\zeta_3^2+\frac{134090}{81}\zeta_4-\frac{2468075}{108}\zeta_5-\frac{55000}{9}\zeta_6+\frac{155155}{4}\zeta_7\right] \\
& +C_A^2C_F^3\left[\frac{20173099267}{3359232}-\frac{15401281}{864}\zeta_3+\frac{732787}{1296}\zeta_4+\frac{1972075}{216}\zeta_5-\frac{63830}{9}\zeta_3^2-\frac{79750}{9}\zeta_6+\frac{139895}{4}\zeta_7\right] \\
& -C_A^3C_F^2\left[\frac{166662991819}{20155392}-\frac{36397493}{2916}\zeta_3-\frac{103763}{54}\zeta_4+\frac{30994565}{3888}\zeta_5-\frac{133990}{27}\zeta_3^2-\frac{72875}{54}\zeta_6+\frac{2127335}{108}\zeta_7\right] \\
& +C_A^4C_F\left[\frac{75932079965}{10077696}-\frac{27693563}{23328}\zeta_3-\frac{1791229}{1296}\zeta_4-\frac{9417425}{1944}\zeta_5-\frac{96700}{81}\zeta_3^2+\frac{163625}{81}\zeta_6+\frac{199640}{27}\zeta_7\right] \\
& -\frac{d_{AA}^{(4)}}{N_A}C_F\left[\frac{81725}{162}-\frac{33505}{18}\zeta_3-\frac{1100}{3}\zeta_4+\frac{52025}{18}\zeta_5-\frac{7000}{3}\zeta_3^2-\frac{48125}{36}\zeta_7\right] \\
& -\frac{d_{FA}^{(4)}}{N_F}C_F\left[\frac{231575}{36}+\frac{6351445}{324}\zeta_3-\frac{2927225}{162}\zeta_5+\frac{23210}{3}\zeta_3^2-\frac{200410}{9}\zeta_7\right] \\
& +\frac{d_{FA}^{(4)}}{N_F}C_A\left[\frac{165871}{54}+\frac{1816625}{162}\zeta_3-\frac{41800}{9}\zeta_4-\frac{4456145}{162}\zeta_5+\frac{196880}{27}\zeta_3^2+\frac{200750}{27}\zeta_6-\frac{7525}{4}\zeta_7\right] \\
& +n_fC_F^4\left[\frac{1776521549}{40310784}-\frac{1332919}{486}\zeta_3+\frac{5000}{9}\zeta_3^2+\frac{33290}{81}\zeta_4-\frac{30325}{81}\zeta_5-\frac{10000}{9}\zeta_6+\frac{14000}{3}\zeta_7\right] \\
& -n_fC_AC_F^3\left[\frac{3737356319}{3359232}-\frac{2327111}{432}\zeta_3+\frac{1280}{3}\zeta_3^2+\frac{262069}{648}\zeta_4+\frac{1693715}{162}\zeta_5-\frac{14000}{3}\zeta_6+\frac{7000}{3}\zeta_7\right] \\
& +n_fC_A^2C_F^2\left[\frac{5637513931}{3359232}+\frac{2711207}{486}\zeta_3-\frac{5020}{27}\zeta_3^2-\frac{457499}{108}\zeta_4+\frac{508820}{243}\zeta_5-\frac{20375}{27}\zeta_6+\frac{50155}{108}\zeta_7\right] \\
& -n_fC_A^3C_F\left[\frac{8766012215}{2519424}+\frac{45697231}{5832}\zeta_3+\frac{1195}{81}\zeta_3^2-\frac{2848403}{648}\zeta_4-\frac{1808870}{243}\zeta_5+\frac{222250}{81}\zeta_6+\frac{250915}{108}\zeta_7\right] \\
& -n_fC_F\frac{d_{FF}^{(4)}}{N_F}\left[\frac{24385}{27}-\frac{334010}{81}\zeta_3-\frac{8480}{9}\zeta_3^2+\frac{1622600}{81}\zeta_5-\frac{135380}{9}\zeta_7\right] \\
& +n_f\frac{d_{FA}^{(4)}}{N_F}\left[\frac{297889}{162}+\frac{154970}{81}\zeta_3-\frac{62600}{27}\zeta_3^2+\frac{3700}{9}\zeta_4-\frac{122780}{81}\zeta_5-\frac{36500}{27}\zeta_6-910\zeta_7\right] \\
& +n_fC_A\frac{d_{FF}^{(4)}}{N_F}\left[\frac{241835}{162}+\frac{333487}{81}\zeta_3+\frac{30560}{27}\zeta_3^2-10780/9\,\zeta_4-\frac{316900}{81}\zeta_5+\frac{110000}{27}\zeta_6-\frac{71960}{9}\zeta_7\right] \\
& +n_f^2C_F^3\left[\frac{512848319}{1679616}-\frac{57109}{54}\zeta_3+\frac{2800}{9}\zeta_3^2+\frac{9118}{81}\zeta_4+\frac{86440}{81}\zeta_5-\frac{5000}{9}\zeta_6\right] \\
& +n_f^2C_AC_F^2\left[\frac{1080083}{5832}-\frac{296729}{972}\zeta_3-\frac{21800}{27}\zeta_3^2+\frac{56327}{54}\zeta_4-\frac{42860}{81}\zeta_5+\frac{2500}{27}\zeta_6\right] \\
& +n_f^2C_A^2C_F\left[\frac{61747877}{419904}+\frac{2496811}{1944}\zeta_3+\frac{39800}{81}\zeta_3^2-\frac{3503}{3}\zeta_4-\frac{88990}{243}\zeta_5+\frac{35000}{81}\zeta_6\right] \\
& -n_f^2\frac{d_{FF}^{(4)}}{N_F}\left[\frac{19435}{27}-\frac{53366}{81}\zeta_3+\frac{3200}{27}\zeta_3^2-\frac{3160}{9}\zeta_4-\frac{70000}{81}\zeta_5+\frac{20000}{27}\zeta_6\right] \\
& +n_f^3C_F^2\left[\frac{28758139}{1259712}+\frac{21673}{729}\zeta_3-\frac{610}{9}\zeta_4+\frac{800}{27}\zeta_5\right] \\
& +n_f^3C_AC_F\left[\frac{13729181}{1259712}+\frac{14947}{729}\zeta_3+\frac{4390}{81}\zeta_4-\frac{6400}{81}\zeta_5\right]-n_f^4C_F\left[\frac{259993}{629856}+\frac{1660}{729}\zeta_3-\frac{200}{81}\zeta_4\right] \qquad (5)
\end{aligned}
$$

# Approximate Splitting functions at N4LO for leading Nc




$$\gamma_{\rm ns}^{+} = \gamma_{\rm ns}^{-} + O(1/N_c)$$

Fit use known large-x and small-x endpoint constraints

# Cusp anomalous dimension at N4LO

The cusp is related to the (Mellin) N→∞ limit of the splitting function:

$$\gamma_{\mathrm{kk}}(N) = \boxed{A_{\mathrm{k}}} \ln \widetilde{N} - B_{\mathrm{k}} + C_{\mathrm{k}} N^{-1} \ln \widetilde{N} - (D_{\mathrm{k}} - \frac{1}{2} \boxed{A_{\mathrm{k}}}) N^{-1} + O(N^{-2} \ln^n \widetilde{N})$$

$$A_{\mathrm{q}} = A_1 a_{\mathrm{s}} + A_2 a_{\mathrm{s}}^2 + \ldots$$

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QCD result:

$$A_5 = (1.7 \pm 0.5\,,\ 1.1 \pm 0.5\,,\ 0.7 \pm 0.5) \cdot 10^5 \quad \text{for} \quad n_f = 3\,,4\,,5\,.$$

QCD result at leading color:

$$A_{5,\rm L} = (1.5 \pm 0.25\,,\ 0.8 \pm 0.2\,,\ 0.4 \pm 0.1) \cdot 10^5 \quad \text{for} \quad n_f = 3\,,4\,,5$$

# Automating $R^*$ for general QFTs

- Input: Lagrangian, 1PI correlator
- Output: UV counterterm

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$$\Gamma = \langle \Phi_1^{s_1}(x_1) ... \Phi_n^{s_n}(x_n) \rangle_{1\mathrm{PI}}$$

$$\longrightarrow$$

$$\mathcal{L} = \sum_i c_i O_i$$

local $R^*$

$$\longrightarrow$$

$$Z(\Gamma) = \sum_i Z_i O_i^{s_1..s_n}$$

# Renormalising the SM EFT at two loops

$$\mathcal{L}_{\rm SMEFT} = \mathcal{L}_{\rm SM} + \mathcal{L}^{(5)} + \mathcal{L}^{(6)} + \mathcal{L}^{(7)} + \ldots \qquad \mathcal{L}^{(d)} = \sum_{i=1}^{n_d} \frac{c_i^{(d)}}{\Lambda^{d-4}} O_i^{(d)}$$

- Many operators and they mix!
  already 84(59) operators at D=6, 993 at D=8,...
  [Henning, Lu, Melia, Murayama]

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- Full AD mixing matrix so far only known at 1-loop
  [Jenkins, Manohar, Trott; ... ]
- 2-loop corrections to ADs:
  - Improve accuracy of SM EFT predictions
  - Allow to run Wilson coefficients up to higher scales to address impact on: Vacuum stability, Inflation, ..

# SMEFT constraints from EDMs

- loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as **Neutron EDM measurements**

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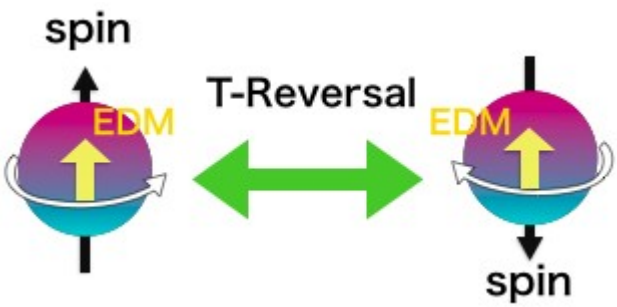
- loop QCD effects are important for extracting Wilson coefficients from low energy experiments, such as Neutron EDM measurements
- The dominant operator relevant for Neutron EDMs is the operator




$$O_W = \frac{d_W}{6} f^{abc} \varepsilon^{\mu\nu\alpha\beta} G^a_{\alpha\beta} G^b_{\mu\rho} G^{c\rho}_{\nu}$$

[Weinberg 1989]

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- Method:
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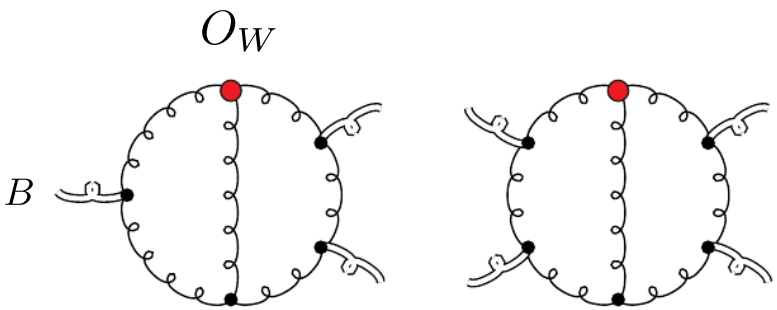
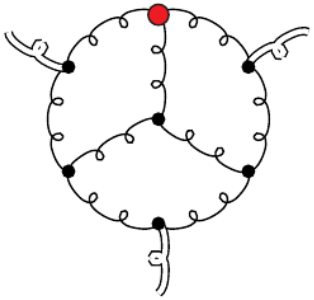
  - Local R*-method in background field gauge
- Pros:
  - Operator does not mix for nf=0
- Cons:
  - 2pt function vanishes
  - Computed 3-pt at 2&3 loops (3 derivatives) and 4-pt at 2 loops
  - $\varepsilon^{\mu\nu\rho\sigma}$ not well defined in dimreg. We used both `T Hooft-Veltman and Larin scheme check.



# Calculation of 2&3-loop AD of Weinberg's operator

**Result:**
(nf=0,L>1)

$$\gamma_W = \frac{\alpha_s(\mu^2)}{4\pi}\left[\frac{C_A}{2} + n_f\right] + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \frac{119}{18} C_A^2$$
$$+ \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 \left[C_A^3\Big(-\frac{3203}{648} + 44\zeta_3\Big)\right.$$
$$\left. + \frac{d_A^{abcd} d_A^{abcd}}{N_A C_A}\Big(40 - 1056\zeta_3\Big)\right],$$

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**Numerical:**

$$\frac{8\pi\gamma_W}{\alpha_s(\mu^2)C_A} = 1 + 3.15657\alpha_s - 23.72872\alpha_s^2$$

$$= 5.46537 - \underline{29.19409}$$

Quartic!

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Interesting to compare with QCD beta function at 3 loops:

$$\beta_2 = \frac{2857}{54} C_A^3$$

But this has no quartic? Why?
Answer: quartic can not appear in 3-loop 2-point function

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**What about 4-loops?:**

$$\beta_3 = C_A^4\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right) + \frac{d_A^{abcd} d_A^{abcd}}{N_A}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right)$$

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$$\boxed{40 - 1056\zeta_3 = -\frac{9}{2}\left(-\frac{80}{9} + \frac{704}{3}\zeta_3\right)}$$

**Coincidence?**

# Conclusions

- Summarised developments and applications of R*
- Reported on status towards automation of R* for general local QFTs
- Computed first 3 Moments of non-singlet splitting functions at N4LO (5 loops)
  - Also extracted corresponding Cusp anomalous dimension
- Computed 2- and 3-loop anomalous dimensions of Weinberg's operator